Characteristics of P-Semi pseudo Symmetric Ideals in Ternary Semiring

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Abstract: In this paper we introduce and study about P-semipseudo symmetric ideals in ternary semirings and characterized p-semipseudo symmetric ideals in ternary semirings.

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Key Words: Pseudo Symmetric ideal, P-pseudo symmetric ideal, P-Prime, Completely P-Prime, P-Semiprime, Completely P-Semiprime, P-semipseudo symmetric ideal.

1.INTRODUCTION:


2.PRELIMINARIES:

Definition 2.1[6]: A nonempty set T together with a binary operation called addition and a ternary multiplication denoted by [ ] is said to be a ternary semiring if T is an additive commutative semigroup satisfying the following conditions:

i) \([abc]de = [a(bcd)e] = [ab[cd]e]\),

ii) \([a+b]cd = [acd] +[bcd]\),

iii) \([ab+c]d = [abd] +[acd]\),

iv) \([abc + d] = [abc] +[abd]\) for all \(a; b; c; d; e \in T\).

Throughout Twill denote a ternary semiring unless otherwise stated.

Note 2.2: For the convenience we write \(x_1x_2x_3\) instead of \([x_1,x_2,x_3]\).

Note 2.3: Let T be a ternary semiring. If A,B and C are three subsets of T , we shall denote the set \(ABC = \{\Sigma abc : a \in A, b \in B, c \in C\}\).

Note 2.4: Let T be a ternary semiring. If A,B are two subsets of T , we shall denote the set \(A + B = \{a+b : a \in A, b \in B\}\).

Note 2.5: Any semiring can be reduced to a ternary semiring.
Example 2.6 [6]: Let T be an semigroup of all $m \times n$ matrices over the set of all non negative rational numbers. Then T is a ternary semiring with matrix multiplication as the ternary operation.

Example 2.7 [6]: Let $S = \{, -2i, -i, 0, i, 2i, ...\}$ be a ternary semiring with respect to addition and complex triple multiplication.

Definition 2.8 [6]: A ternary semiring T is said to be **commutative ternary semiring** provided $abc = bca = cab = cba = acbf$ for all $a, b, c \in T$.

Definition 2.9 [8]: A nonempty subset A of a ternary semiring T is said to be **ternary ideal** or simply an **ideal** of T if

1. $a, b \in A$ implies $a + b \in A$
2. $b, c \in T$, $a \in A$ implies $bca \in A$, $bac \in A$, $abc \in A$.

Definition 2.10 [9]: An ideal A of a ternary semiring T is said to be a **completely prime ideal** of T provided $x, y, z \in T$ and $xyz \in A$ implies either $x \in A$ or $y \in A$ or $z \in A$.

Definition 2.11 [11]: An ideal A of a ternary semiring T is said to be a **completely P-prime ideal** of T provided $x, y, z \in T$ and $xyz + P \subseteq A \Rightarrow X \subseteq A$ or $Y \subseteq A$ or $Z \subseteq A$ for any ideal P.

Theorem 2.13 [11]: Every completely P-prime ideal of a ternary semiring T is a P-prime ideal of T.

Theorem 2.14 [11]: Every completely P-semiprime ideal of a ternary semiring T is a P-semiprime ideal of T.

Theorem 2.15 [11]: If T is a globally idempotent ternary semiring then every maximal ideal of T is a P-prime ideal of T.

Definition 2.16 [11]: An ideal A of a ternary semiring T is said to be a **completely P-semiprime ideal** provided $x \in T$, $x^n + p \in A$ for some odd natural number $n \geq 1$ and $p \in P$ implies $x \in A$.

Definition 2.17 [11]: An ideal A of a ternary semiring T is said to be **semiprime ideal** provided $X$ is an ideal of T and $X^n \subseteq A$ for some odd natural number $n$ implies $X \subseteq A$.

Definition 2.18 [11]: An ideal A of a ternary semiring T is said to be **P-semiprime ideal** provided $X$ is an ideal of T and $X^n + P \subseteq A$ for some odd natural number $n$ implies $X \subseteq A$.

Theorem 2.19 [11]: Let A be a P-prime ideal of a ternary semiring T. If A is completely P-Semiprime ideal of T then A is completely P-prime.

Theorem 2.20 [11]: Every completely P-Semiprime ideal of a ternary semiring T is a P-Semiprime ideal of T.

Theorem 2.21 [11]: Every P-prime ideal of a ternary semiring T is P-Semiprime.

Corollary 2.22 [11]: If an ideal A of a ternary semiring T is completely P-semiprime then $x, y, z \in T$, $p \in P$ and $xyz + p \in A \Rightarrow < x > < y > < z > + P \subseteq A$.

Theorem 2.23 [11]: Every completely P-prime ideal of a ternary semiring T is a completely P-Semiprime ideal of T.

Notation 2.24 [11]: If A is an ideal of a ternary semiring T, then we associate the following four types of sets.

$A_i = \text{The intersection of all completely prime ideals of T containing } A$. 

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A_2 = \{x \in T: x^n \in A \text{ for some odd natural numbers } n\}

A_3 = \text{The intersection of all prime ideals of } T \text{ containing } A.

A_4 = \{x \in T: <x>^n \subseteq A \text{ for some odd natural number } n\}

**Theorem 2.25** [11]: If A is an ideal of a ternary semiring T, then \( A \subseteq A_4 \subseteq A_1 \subseteq A_2 \subseteq A_3 \).

**Corollary 2.26**: If \( a \in \sqrt{A} \), then there exist a positive integer \( n \) such that \( a^n \in A \) for some odd natural number \( n \in N \).

**Theorem 2.27**: An ideal Q of ternary semiring T is a semiprime ideal of T if and only if \( \sqrt{Q} = Q \).

**Definition 2.28**: An ideal A of a ternary semiring T is said to be *pseudo symmetric* provided \( x, y, z \in T, xyz \in A \) implies \( xystz \in A \) for all \( s, t \in T \).

**Definition 2.29**: A pseudo symmetric ideal A of a ternary semiring T is said to be *P-pseudo symmetric ideal* provided \( x, y, z \in T \) and P is an ideal of T, \( xyz + p \in A \) implies \( xystz + p \in A \) for all \( s, t \in T \) and \( p \in P \).

**Note 2.30**: A pseudo symmetric ideal A of a ternary semiring T is said to be *P-pseudo symmetric ideal* provided \( x, y, z \in T \) and P is an ideal of T, \( xyz + P \subseteq A \) implies \( xystz + P \subseteq A \) for all \( s, t \in T \).

**Corollary 2.31**: Let A be a P-pseudo symmetric ideal in a ternary semiring T, then for any odd natural number \( n \), \( a^n + p \in A \) implies \(<a>^n + P \subseteq A \).

**Theorem 2.32**: Let A be an ideal of a ternary semiring T. Then A is completely P-prime iff A is P-prime and P-pseudo symmetric.

**Corollary 2.33**: Let A be an ideal of a ternary semiring T. Then A is completely P-semiprime iff A is P-semiprime and P-pseudo symmetric.

3. **P-SEMPSEUDO SYMMETRIC TERNARY IDEALS**

We now introduce the notion of P-semipseudo symmetric ideal of a ternary semiring

**Definition 3.1**: An ideal A in a ternary semiring T is said to be *P-semipseudo symmetric* provided for any odd natural number \( n \), \( x \in T \) and P is an ideal of T, \( x^n + P \subseteq A \) \( \Rightarrow \ <x>^n + P \subseteq A \).

**Theorem 3.2**: Every P-pseudo symmetric ideal of a ternary semiring is a P-semipseudo symmetric ideal.

**Proof**: Let A be a P-pseudo symmetric ideal of a ternary semiring T. Let \( x \in T \) and \( x^n + P \subseteq A \) for some odd natural number \( n \). Since A is P-pseudo symmetric, by corollary 2.31, \( x^n + P \subseteq A \Rightarrow <x>^n + P \subseteq A \). Therefore A is a P-semipseudo symmetric ideal.

**Note 3.3**: The converse of theorem 3.2, is not true, i.e. a P-semipseudo symmetric ideal of a ternary semiring need not be a P-pseudo symmetric ideal.

**Example 3.4**: Let T be a free ternary semiring over the alphabet \( \{a, b, c, d, e\} \) and P is any ideals of T. Let \( A = <abc> + <bca> + <cab> + P \). Since \( abc + P \subseteq A \) and \( adbec + P \not\subseteq A \), A is not P-pseudo symmetric. Suppose \( x^n + P \in A \) for some odd natural number \( n \). Now the word x contains \( abc \) or \( bca \) or \( cab \) and hence \( <x>^n + P \subseteq A \). Therefore \( x^n + P \subseteq A \) for some odd natural number \( n \Rightarrow <x>^n + P \subseteq A \). Therefore A is a P-semipseudo symmetric ideal.

**Theorem 3.5**: Every P-semiprime ideal Q minimal relative to containing a P-semipseudo symmetric ideal A in a ternary semiring T is completely P-semiprime.
Proof: Write $S=\{x^n : x \in TQ\}$ for any odd natural number $n$. First we show that $A \cap S = \emptyset$.

If $A \cap S \neq \emptyset$, then there exists an element $x \in TQ$ such that $x^n + P \subseteq A$ where $n$ is odd natural number. Since $A$ is a P-semipseudo symmetric ideal, $< x >^n + P \subseteq A \subseteq Q$ \Rightarrow $< x >^n + P \subseteq Q \Rightarrow x \in Q$. It is a contradiction. Thus $A \cap S = \emptyset$. Consider the set $\Sigma = \{B : B$ is an ideal in $T$ containing $A$ such that $B \cap S = \emptyset\}$. Since $A \in \Sigma$, $\Sigma$ is nonempty. Now $\Sigma$ is a poset under set inclusion and satisfies the hypothesis of Zorn’s lemma. Thus by Zorn’s lemma, $\Sigma$ contains a maximal element, say $M$. Suppose $< a >^3 + P \subseteq M$ and $a \notin M$. Then $M \cup a >$ is an ideal containing $A$. Since $M$ is maximal in $\Sigma$, we have $(M \cup a >) \cap S = \emptyset$.

Then there exists $x \in TQ$ such that $x^n \in < a > \cap S$ for some odd natural number $n$.

Therefore $x^{3n} \in < a >^3 \cap S \subseteq M \cap S \Rightarrow x^{3n} \in M \cap S$. It is a contradiction.

Therefore $M$ is a P-semiprime ideal containing $A$.

Now, $A \subseteq M \subseteq TS \subseteq Q$. Since $Q$ is a minimal P-semiprime ideal relative to containing $A$, we have $M = TS = Q$. Let $x \in S$, $x^n + P \subseteq Q$. Suppose if possible $x \notin Q$.

Now $x \notin Q \Rightarrow x \in S \Rightarrow x^{3n} \in S$. It is a contradiction. Therefore $x \in Q$.

Hence $Q$ is a completely P-semiprime ideal.

Corollary 3.6: Every P-prime ideal $Q$ in a ternary semiring $T$ minimal relative to containing a P-semipseudo symmetric ideal $A$ is completely P-prime.

Proof: Since every P-prime ideal is a P-semiprime ideal, by theorem 3.5, we have $Q$ is a completely P-semiprime ideal and by theorem 2.19, $P$ is a completely P-prime ideal.

Corollary 3.7: Every P-prime ideal minimal relative to containing a P-semipseudo symmetric ideal $A$ in a ternary semiring $T$ is completely P-prime.

Proof: Let $P$ be a P-prime ideal containing a P-semipseudo symmetric ideal $A$ of a ternary semiring $T$. By theorem 3.2, every P-semipseudo symmetric ideal is a P-semipseudo symmetric ideal, by corollary 3.6, $P$ is a completely P-prime ideal of $T$.

Theorem 3.8: If $A$ is an ideal in a ternary semiring $T$, then the following are equivalent.

1) $A$ is completely semiprime.
2) $A$ is P-semiprime and P-semipseudo symmetric.
3) $A$ is P-semiprime and P-semipseudo symmetric.

Proof: (1) $\Rightarrow$ (2): Suppose $A$ is a completely P-semiprime ideal of $T$. By theorem 2.20, $A$ is a P-semiprime ideal of $T$ and by theorem 2.33, $A$ is a P-semipseudo symmetric ideal of $T$.

(2) $\Rightarrow$ (3): Suppose $A$ is P-semiprime and P-semipseudo symmetric. By theorem 3.2, $A$ is a P-semipseudo symmetric ideal. Hence $A$ is P-semiprime and P-semipseudo symmetric.

(3) $\Rightarrow$ (1): Suppose $A$ is P-semiprime and P-semipseudo symmetric.

Let $x \in T$, $x^3 + P \subseteq A$. Since $A$ is P-semipseudo symmetric, $x^3 + P \subseteq A$ \Rightarrow $< x >^3 + P \subseteq A$. Since $A$ is P-semiprime, by definition 2.17, $< x >^3 + P \subseteq A \Rightarrow < x > \subseteq A$. \therefore $A$ is completely P-semiprime.

Corollary 3.9: If $A$ is an ideal in a ternary semiring $T$, then the following are equivalent.

1) $A$ is completely semiprime.
2) $A$ is semiprime and pseudo symmetric.
3) $A$ is semiprime and semipseudo symmetric.

Theorem 3.10: If $A$ is an ideal of a semi simple ternary semiring $T$, then the following are equivalent.

1) $A$ is completely P-semiprime.
2) $A$ is P-semipseudo symmetric.
3) $A$ is P-semipseudo symmetric.

Proof: (1) $\Rightarrow$ (2): Suppose that $A$ is completely P-semi prime. By corollary 2.20, $A$ is P-semipseudo symmetric.
(2) $\Rightarrow$ (3) : Suppose that $A$ is $P$-pseudo symmetric. By theorem 3.2, $A$ is $P$-semi pseudo symmetric.

(3) $\Rightarrow$ (1) : Suppose that $A$ is $P$-semipseudo symmetric. Let $x \in T, x^3 \in A$. Since $A$ is $P$-semipseudo symmetric, $x^3 + P \subseteq A \Rightarrow <x^3> + P \subseteq A$. Since $T$ is semi simple, $x$ is a semi simple element. Therefore $x \in <x^3> \subseteq A$. Thus $A$ is completely $P$-semiprime.

Theorem 3.11 : If $A$ is an ideal of a ternary semiring $T$, then the following are equivalent.

1) $A$ is completely $P$-prime.
2) $A$ is $P$-prime and $P$-pseudo symmetric.
3) $A$ is $P$-prime and $P$-semipseudo symmetric.

Proof : (1) $\Rightarrow$ (2) : Suppose that $A$ is completely $P$-prime. By theorem 2.32, $A$ is $P$-prime and $P$-pseudo symmetric.

(2) $\Rightarrow$ (3) : Suppose $A$ is prime and pseudo symmetric. Since $A$ is $P$-pseudo symmetric by theorem 3.2, $A$ is $P$-semipseudo symmetric.

(3) $\Rightarrow$ (1) : Suppose $A$ is $P$-prime and $P$-semipseudo symmetric. Since $A$ is $P$-prime by theorem 2.21, $A$ is $P$-semiprime. Since $A$ is $P$-semiprime and $P$-semipseudo symmetric, by theorem 3.8, $A$ is completely $P$-semiprime. Since $A$ is $P$-prime and completely $P$-semiprime by theorem 2.19, $A$ is completely $P$-prime.

The following theorem is an analogue of KRULL’s Theorem.

THEOREM 3.12 : Let $A$ be a $P$-semipseudo symmetric ideal of a ternary semiring $T$. Then the following are equivalent.

1) $A_1$ = The intersection of all completely prime ideals of $T$ containing $A$.
2) $A_1^i$ = The intersection of all minimal completely prime ideals of $T$ containing $A$.
3) $A_1^i$ = The minimal completely semi prime ideal of $T$ relative to containing $A$.
4) $A_2 = \{ x \in T : x^n \in A \text{ for some odd natural number } n \}$
5) $A_2$ = The intersection of all prime ideals of $T$ containing $A$.
6) $A_3^1$ = The intersection of all minimal prime ideals of $T$ containing $A$.
7) $A_3^{11}$ = The minimal semiprime ideal of $T$ relative to containing $A$.
8) $A_4 = \{ x \in T : <x^3> \subseteq A \text{ for some odd natural number } n \}$

Proof: Since completely $P$-prime ideals containing $A$ and minimal completely $P$-prime ideals containing $A$ and hence completely prime ideals containing $A$ and minimal completely prime ideals containing $A$ and minimal completely semiprime ideal relative to containing $A$ are coincide, it follows that $A_1 = A_1^i = A_3^{11}$. Since $P$-prime ideals containing $A$ and minimal $P$-prime ideals containing $A$ and hence prime ideals containing $A$ and minimal prime ideals containing $A$ and minimal semiprime ideal relative to containing $A$ are coincide, it follows that $A_3 = A_3^1 = A_3^{11}$. Since $A$ is $P$-semipseudo symmetric ideal. Therefore $A$ is semipseudo symmetric ideal, we have $A_3 = A_3^1 = A_4$. Now by theorem 3.15, we have $A_3^{11} = A_4^{11}$. Therefore $A_1 = A_1^i = A_3^1 = A_3^{11} = A_4$ and $A_2 = A_4$. Hence the given conditions are equivalent.

Corollary 3.13 : Let $A$ be a $P$-pseudo symmetric ideal of a ternary semiring $T$. Then the following are equivalent.

1) $A_1$ = The intersection of all completely prime ideals of $T$ containing $A$.
2) $A_1^i$ = The intersection of all minimal completely prime ideals of $T$ containing $A$.
3) $A_1^{11}$ = The minimal completely semiprime ideal of $T$ relative to containing $A$.
4) $A_2 = \{ x \in T : x^n \in A \text{ for some odd natural number } n \}$
5) $A_3 = \text{The intersection of all prime ideals of } T \text{ containing } A.$
6) $A_4^p = \text{The intersection of all minimal prime ideals of } T \text{ containing } A.$
7) $A_4 = \text{The minimal semiprime ideal of } T \text{ relative to containing } A.$
8) $A_4 = \{x \in T : x^n \subseteq A \text{ for some odd natural number } n\}$

**Proof:** By theorem 3.2, every P-pseudo symmetric ideal is a P-semi pseudo symmetric ideal of $T$. Hence the proof follows from theorem 3.12.

**Theorem 3.14:** If $M$ is a maximal ideal of a ternary semiring $T$ with $M_4 \neq T$, then the following are equivalent.
1) $M$ is completely P-prime.
2) $M$ is completely P-semiprime.
3) $M$ is P-pseudo symmetric.
4) $M$ is P-semipseudo symmetric.

**Proof:** (1) $\Rightarrow$ (2): Suppose that $M$ is completely P-prime. By theorem 2.22, $M$ is completely semiprime.
(2) $\Rightarrow$ (3): Suppose that $M$ is completely P-semiprime. By theorem 2.33, $M$ is P-pseudo symmetric.
(3) $\Rightarrow$ (4): Suppose that $M$ is P-pseudo symmetric. By theorem 3.2, $M$ is P-semi pseudo symmetric.
(4) $\Rightarrow$ (1): Suppose $M$ is P-semi pseudo symmetric. By the theorem 3.12, $M \subseteq M_4 \subseteq T$.
Since $M$ is maximal ideal and $M_4 \neq T$, it implies that $M = M_4$. Let $x \in T$. $x^3 \subseteq M$. Since $M$ is P-semi pseudo symmetric, $x^3 \subseteq M$. Then $x \in M_4 = M$. $\therefore M$ is completely P-semiprime.

Let $x, y, z \in T$, $xyz + P \subseteq M$. Since $M$ is completely P-semiprime, by corollary 2.21, $xyz + P \subseteq M \Rightarrow x < y < z + P \subseteq M \Rightarrow x < y < z \subseteq M$. Suppose if possible $x \notin M$, $y \notin M$, $z \notin M$. Then $M \cup x \neq M$, $M \cup y \neq M$, $M \cup z \neq M$ are ideals of $T$ and $M \cup x = M \cup y = M \cup z = T$. Since $M$ is maximal, $y, z \in M \cup x \Rightarrow x < y = \neq y < z \neq z$. Now $x < y < z \subseteq M \Rightarrow x < y < z \Rightarrow x^3 \subseteq M \Rightarrow x^3 \subseteq M \Rightarrow x \in M$. It is a contradiction. $\therefore$ either $x \in M$ or $y \in M$ or $z \in M$. $\therefore M$ is completely P-prime.

We now introduce the notion of a P-semi pseudo symmetric ternary semiring.

**Definition 3.15:** A ternary semiring $T$ is said to be a **P-semi pseudo symmetric ternary semiring** provided every ideal of $T$ is P-semi pseudo symmetric.

**Theorem 3.16:** A ternary semiring $T$ is P-semi pseudo symmetric iff every principal ideal is P-semi pseudo symmetric.

**Proof:** Suppose a ternary semiring $T$ is P-semi pseudo symmetric. Then every ideal of $T$ is P-semi pseudo symmetric. Hence every principal ideal of $T$ is P-semi pseudo symmetric.

Conversely suppose that every principal ideal of $T$ is P-semi pseudo symmetric. Let $A$ be any ideal of $T$. For $x \in T$, $x^n + P \subseteq A$ for an odd natural number $n$. Since $x^n$ is a P-semi pseudo symmetric ideal, $x^n + P \subseteq A$. Now $x^n + P \subseteq A$ for an odd natural number $n$. $\therefore a \neq A$ is a P-semi pseudo symmetric ideal. Hence $T$ is a P-semi pseudo symmetric semiring.

**Theorem 3.17:** In a P-semipseudo symmetric ternary semiring $T$, an element $a$ is semi simple iff $a$ is lateral regular.

**Proof:** Let $T$ be a P-semipseudo symmetric ternary semiring. Suppose an element $a \in T$ is semi simple. Then $a \in a^3$. Since $T$ is P-semipseudo symmetric, $a^3$ is a P-semipseudo symmetric ideal. Thus $a^3 + P \subseteq a^3$ and $P \subseteq a^3 \Rightarrow a \in a^3$. Therefore $a = sa^t$ for some $s, t \in T$ and hence $a$ is lateral regular.
Conversely suppose that \( a \in T \) is lateral regular. Then \( a = xa^3y \) for some \( x, y \in T \) and hence \( a \in < a^3 > \). Therefore \( a \) is semi simple.

**Definition 3.18:** A ternary semiring \( T \) is said to be an Archimedean ternary semiring provided for any \( a, b \in T \) there exists an odd natural number \( n \) such that \( a^n \in TbT \).

**Definition 3.19:** A ternary semiring \( T \) is said to be a strongly Archimedean ternary semiring provided for any \( a, b \in T \) there exists an odd natural number \( n \) such that \( < a >^n \subseteq < b > \).

**Theorem 3.20:** Every strongly Archimedean ternary semiring is an Archimedean ternary semiring.

**Proof:** Suppose that \( T \) is strongly Archimedean ternary semiring. Let \( a, b \in T \). Since \( T \) is strongly Archimedean ternary semiring, there is an odd natural number \( n \) such that \( < a >^n \subseteq < b > \). Now \( a^n \in < a >^n \subseteq < b > \Rightarrow a^n + 2 \in T < b > T \subseteq TbT \). Therefore \( T \) is an Archimedean ternary semiring.

**Theorem 3.21:** If \( T \) is a \( P \)-semipseudo symmetric ternary semiring, then the following are equivalent.

1) \( S = \{ a \in T : \sqrt{< a >}, T \} \) is either empty or a \( P \)-prime ideal.
2) \( TS \) is either empty or an Archimedean subsemiring of \( T \).

**Proof:** (1) If \( S \) is an empty set, then there is nothing to prove. If \( S \) is nonempty, then clearly \( S \) is an ideal of \( T \). Let \( a, b, c \in T, P \) is any ideals of \( T \) and \( abc + P \subseteq S \).

Suppose if possible \( a \in S, b \notin S, c \notin S \). Then \( < a > \subseteq T, \sqrt{< b >} \subseteq T \) and \( \sqrt{< c >} \subseteq T \).

Since \( abc + P \subseteq S \) and hence \( abc \in S \). \( P \subseteq S \). Since \( abc \in S \), then \( \sqrt{< abc >} = < a > \neq T \).

Now \( T = \sqrt{< a >} \cap \sqrt{< b >} \cap \sqrt{< c >} = \sqrt{< abc >} \neq T \). It is a contradiction.

Thus \( a \in S \) or \( b \in S \) or \( c \in S \). \( \therefore S \) is a \( P \)-prime ideal.

(2) Since \( S \) is a completely \( P \)-prime ideal, \( TS \) is either empty or a ternary subsemigroup of \( T \).

Let \( a, b, c \in TS \). Then \( \sqrt{< a >} = \sqrt{< b >} = \sqrt{< c >} \subseteq T \). Now \( b, c \in \sqrt{< a >}, c, a \in \sqrt{< b >}, \sqrt{< c >} \) by the corollary 2.26. \( b^n \in < a > \) for some odd natural number \( n \).

So \( b^{n+2} \in TaT \Rightarrow b^{n+2} = sat \) for some \( s, t \in T \).

If either \( s \) or \( t \in S \), then \( b^{n+2} \in S \) and hence \( b \in S \). It is a contradiction. Hence \( s, t \in TS \).

Now \( b^{n+2} = sat \in (TS) (a) (TS) \). Hence \( TS \) is an archimedean ternary subsemiring of \( T \).

**Theorem 3.22:** If \( T \) is a \( P \)-semipseudo symmetric ternary semiring, then the following are equivalent.

1) \( T \) is a strongly Archimedean semiring.
2) \( T \) is an Archimedean semiring.
3) \( T \) has no proper completely \( P \)-prime ideals.
4) \( T \) has no proper completely \( P \)-semi prime ideals.
5) \( T \) has no proper \( P \)-prime ideals.
6) \( T \) has no proper \( P \)-semi prime ideals.

**Proof:** (1) \( \Rightarrow \) (2) : Suppose that \( T \) is a strongly Archimedean ternary semiring. By theorem 3.20, \( T \) is an Archimedean ternary semiring.

(2) \( \Rightarrow \) (3) : Suppose that \( T \) is an Archimedean ternary semiring. Let \( Q \) be any completely \( P \)-prime ideal of \( T \). Let \( a \in T, b \in Q \). Since \( T \) is an Archimedean ternary semiring, there exists a odd natural number \( n \) such that \( a^n \in TbT \subseteq Q \Rightarrow a^n \in Q \Rightarrow a \in Q \). \( \therefore T \subseteq Q \). Clearly \( Q \subseteq T \). Thus \( Q = T \). \( \therefore T \) has no proper completely \( P \)-prime ideals.

By theorem 2.24, corollary 2.25, and theorem 2.27; (3), (4), (5) and (6) are equivalent.

(5) \( \Rightarrow \) (1) : \( T \) has no proper \( P \)-prime ideals. Let \( a, b \in T \). Since \( T \) has no proper \( P \)-prime ideals, \( < b > = T \). Now \( a \in T = < b > \Rightarrow a^n \in < b > \) for some odd natural number \( n \). Since \( T \) is a \( P \)-semipseudo symmetric semiring, \( < b > \) is a \( P \)-semipseudo symmetric ideal and hence \( a^n \in < b > \Rightarrow a > a^n \subseteq < b > \). Thus \( T \) is a strongly Archimedean ternary semiring. Hence the given conditions are equivalent.
Corollary 3.23: If T is a P-pseudo symmetric ternary semiring, then the following are equivalent.
1. T is strongly Archimedean ternary semiring
2. T is an Archimedean ternary semiring
3. T has no proper completely P-prime ideals.
4. T has no proper completely P-semiprime ideals.
5. T has no proper P-prime ideals.
6. T has no proper P-semiprime ideals.

Proof: Every P-pseudo symmetric ternary semiring is a P-semipseudo symmetric ternary semiring. Therefore by theorem 3.22, (1) to (6) are equivalent.

Corollary 3.24: A commutative ternary semiring T is Archimedean iff T has no proper P-prime ideals.

Proof: Since T is a commutative ternary semiring, T is a P-semipseudo symmetric ternary semiring. By theorem 3.22, T is Archimedean iff T has no proper P-prime ideals.

Theorem 3.25: If M is a nontrivial maximal ideal of a P-semipseudo symmetric ternary semiring T then M is P-prime ideal of T.

Proof: Suppose if possible M is not P-prime. Then there exists a, b, c ∈ T\{M} such that <a> <b> <c> + P ≅ M where P is any ideal of T. Then <a> <b> <c> ⊆ M and P ⊆ M. Now <a> <b> <c> ⊆ M, then for any x ∈ T\{M}, we have T = M + <b> = M + <c> = M + <x>. Since b, c, x ∈ T\{M}, we have b, c ∈ <x> and x ∈ <b>, x ∈ <c>. So <b> = <c> = <x>. Therefore <b>³ ⊆ M, <c>³ ⊆ M.

If a ≠ b. Then a = \[\sum_{i=1}^{n} p_i q_i a + \sum_{j=1}^{n} ar_js_j + \sum_{k=1}^{n} t_k au_k + \sum_{l=1}^{n} v_l w_l ax_l y_l + na\] for some p_i, q_i, r_j, s_j, t_k, u_k, v_l, w_l, x_l, y_l, n ∈ T^e.

So a ∈ <s > <b > <t >. If either s ∈ M or t ∈ M then a ∈ M. It is a contradiction.
If s M and t M, then <s > <b > <t > <<b >³ M. : a ∈ <s > <b > <t > ⊆ M
a M. It is a contradiction. Thus a = b and hence M is trivial, which is not true. So M is P-prime.

Theorem 3.26: If T is a P-semipseudo symmetric ternary semiring and contains a nontrivial maximal ideal then T contains semisimple elements.

Proof: Let M be a nontrivial maximal ideal of T. By theorem 3.25, M is P-prime. Let a ∈ T\{M}. Then <a> M. Since M is maximal, M ∪ <a> = T.
If <a>³ M then <a> M which is not true. So <a>³ M.

Since M is maximal, M ∪ <a>³ = T. Now M ∪ <a> = M ∪ <a>³.
Therefore a ∈ <a>³ and hence a is semisimple.

Theorem 3.27: Let T be a semipseudo symmetric archimedean ternary semiring. Then an ideal M is maximal iff it is trivial, and T has no maximal ideals if T = T³.

Proof: If M is trivial, then clearly M is maximal ideal. Conversely suppose that M is maximal. Suppose if possible M is nontrivial. By theorem 3.25, M is P-prime. Since T is an Archimedean semiring, by theorem 3.22, S has no P-prime ideals. It is a contradiction. So M is trivial. If T = T³, then by theorem 2.15, every maximal ideal is P-prime and hence T has no maximal ideals.

Theorem 3.28: Let T be a P-semipseudo symmetric ternary semiring containing maximal ideals. If either T has no semisimple elements or T is an Archimedean ternary semiring, then T ≠ T³ and T³ = M° where M° denotes the intersection of all maximal ideals.
Proof: Suppose that $T$ has no semisimple elements. Then by theorem 3.25, every maximal ideal is trivial. So if $M$ is maximal, then $T = M + \{ a \}, a \notin M$.

Suppose $a \in T^3$. Then $a \in T^3 \Rightarrow a = \sum_{finite} bcd$ for some $b, c, d \in T$.

If $b \neq a$ then $b \in M$ and hence $\sum_{finite} bcd \in M$ (Since $M$ is Maximal) $\Rightarrow a \in M$. It is a contradiction. $\therefore b = a$. Similarly we can prove $c = a$ and $d = a$. $\therefore a = \sum_{finite} bcd = a^3$. $\therefore a$ is semisimple. It is a contradiction. $\therefore a \notin T^3$. $\therefore T \neq T^3$ and $T^3 \subseteq M$. Let $t \in M^*$ and $t \notin T^3$.

Let $a \in T \{ t \} \Rightarrow \sum_{finite} ars \neq t, \sum_{finite} ras \neq t, \sum_{finite} rsa \neq t$ for all $r, s \in T$

$\Rightarrow \sum_{finite} ars, \sum_{finite} ras, \sum_{finite} rsa \in T \{ t \} \Rightarrow T \{ t \}$ is an ideal. Then $T \{ t \}$ is a maximal ideal.

Hence $t \in T \{ t \}$, it is a contradiction. $\therefore M^* \subseteq T^3 \Rightarrow T^3 = M^*$. Now suppose that $T$ is an archimedean ternary semiring. Since $T$ has maximal ideals, by theorem 3.27, $T \neq T^3$. Suppose if possible $x \in T^3 \setminus M^*$. Then there exists a maximal ideal $M$, such that $x \notin M$. So by theorem 3.27, $M = T \{ x \}$. Since $x \in T^3, x = \sum_{finite} rst$ for some $s, t \in T$.

If either $r$ or $s$ or $t \in M$, then $x \in M$. It is a contradiction. Therefore $r = s = t = x$ and hence $x = x^3$. Let $a, b, c \in T, abc \in M$. Suppose if possible $a \notin M, b \notin M, c \notin M$. Then $a = x, b = x, c = x$. Therefore $abc = xxx = x \notin M$. It is a contradiction. Thus $M$ is P-prime.

By theorem 3.22, $S$ has no proper P-prime ideals. It is a contradiction. Thus $T^3 \subseteq M^*$. As above, we can show that $M^* \subseteq T^3$. Therefore $T^3 = M^*$.

Corollary 3.29: Let $T$ be a commutative ternary semiring containing maximal ideals. If either $T$ has no idempotent or $T$ is an Archimedean ternary semiring, then $T \neq T^3$ and $T^3 = M^*$ where $M^*$ denotes the intersection of all maximal ideals of $T$.

Proof: Suppose that $T$ has no idempotent. If $T$ contains a semi simple element $a$ then $a$ is regular. Hence there exists $x, y \in T$ such that $axaya = a$. Now $a$ is an idempotent in $T$. It is a contradiction. So $S$ has no semi simple elements. Then by theorem 3.28, we have $T \neq T^3$ and $T^3 = M^*$.

CONCLUSION:
In this paper mainly we studied about the P-semipseudo symmetric ideals in ternary semirings.

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