A Fixed point approach to the stability of a mixed quadratic-quartic(QQ) functional equation in quasi—β—normed spaces

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Abstract: In this paper we prove the generalized Hyers-Ulam stability of the following mixed quadratic-quartic functional equation

\[
f(3x + 2y + z) + f(3x + 2y - z) + f(3x - 2y + z) + f(3x - 2y - z)
\]

\[
= 72[f(x + y) + f(x - y)] + 18[f(x + z) + f(x - z)] + 8[f(y + z) + f(y - z)]
\]

\[+ 24f(2x) + 4f(2y) - 240f(x) - 160f(y) - 48f(z)
\]

in the quasi—β—normed spaces via fixed point method. Counterexamples for non-stability cases are also discussed.

Keywords: Hyers-Ulam stability; quadratic mapping; quartic mapping, mixed type functional equation, quasi - Banach space, fixed point

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1 Introduction


\[
f(x + y + z) + f(x + y - z) + f(x - y + z) + f(x - y - z) + 4f(x) + 4f(y) + 4f(z)
\]

\[= 2f(x + y) + 2f(x - y) + 2f(x + z) + 2f(x - z) + 2f(y + z) + 2f(y - z). \quad (1.1)
\]

Eshaghi Gordji et al.[6] introduced another mixed type of quartic and quadratic functional equation:

\[
f(nx + y) + f(nx - y) = n^4f(x + y) + n^2f(x - y) + 2n^2(n^2 - 1)f(x) - 2(n^2 - 1)f(y)
\]

for all fixed integer \(n\) with \(n \neq 0, \pm 1\). They established the general solutions and proved the Hyer-Ulam-Rassias stability of this equation in quasi-Banach spaces. Also, for the case \(n = 2\), they established the general solution and investigated generalized Hyers - Ulam stability for the following equation:

\[
f(2x + y) + f(2x - y) = 4f(x + y) + 4f(x - y) + 2f(2x) - 8f(x) - 6f(y)
\]

with \(f(0) = 0\) in RN-spaces(see [7]).

Arunkumar and Agilan [14] introduced and investigated the generalized Hyers - Ulam stability for the following mixed type of quadratic and additive functional equation via fixed point method:

\[
f(x + 2y + 3z) + f(x - 2y + 3z) + f(x + 2y - 3z) + f(x - 2y - 3z)
\]

\[= 4f(x) + 8f(y) + 4f(-y) + 18[f(z) + f(-z)]
\]

Balamurugan et al.[4][see also [3]] introduced and investigated the generalized Hyers-Ulam stability for the following mixed additive-cubic functional equation:

\[
f(3x + 2y + z) + f(3x + 2y - z) + f(3x - 2y + z) + f(3x - 2y - z)
\]

\[= 24[f(x + y) + f(x - y)] + 6[f(x + z) + f(x - z)] + 16f(2x) - 80f(x)
\]
Zhou Xu et al. [22] achieved the general solution and proved the stability of the following quintic functional equation

\[ f(x + 3y) - 5f(x + 2y) + 10f(x + y) - 10f(x) + 5f(x - y) - f(x - 2y) = 120f(y) \]

and the sextic functional equation

\[ f(x + 3y) - 6f(x + 2y) + 15f(x + y) - 20f(x) + 15f(x - y) - 6f(x - 2y) + f(x - 3y) = 720f(y) \]

in the quasi-\(\beta\)-normed spaces via fixed point method. The stability of several functional equations have been extensively investigated by a number of authors and there are many interesting results concerning this problem (see [1], [5], [10], [12], [15], [18], [20]).

In this paper, we prove the generalized Hyers-Ulam stability for the following mixed quadratic-quartic functional equation:

\[
\begin{align*}
&f(3x + 2y + z) + f(3x + 2y - z) + f(3x - 2y + z) + f(3x - 2y - z) + 240f(x) + 160f(y) + 48f(z) \\
&= 72[f(x + y) + f(x - y)] + 240[f(x + z) + f(x - z)] + 18[f(y + z) + f(y - z)] + 24f(2x) + 4f(2y)
\end{align*}
\]

(1.4)

It is easy to see that the mapping \( f(x) = ax^2 + bx^4 \) is a solution of the functional equation (1.4).

2 Preliminary results on quasi-\(\beta\)-normed spaces

In this section, we present some preliminary results concerning to quasi-\(\beta\)-normed spaces.

We fix a real number \( \beta \) with \( 0 < \beta \leq 1 \) and let \( \mathbb{K} \) denote either \( \mathbb{R} \) or \( \mathbb{C} \).

**Definition 2.1.** Let \( X \) be a linear space over \( \mathbb{K} \). A quasi-\(\beta\)-norm \( \| \cdot \| \) is a real-valued function on \( X \) satisfying the following:

(i) \( \| x \| \geq 0 \) for all \( x \in X \) and \( \| x \| = 0 \) if and only if \( x = 0 \).

(ii) \( \| \lambda x \| = |\lambda| \| x \| \) for all \( \lambda \in \mathbb{K} \) and all \( x \in X \).

(iii) There is a constant \( K \geq 1 \) such that \( \| x + y \| \leq K (\| x \| + \| y \|) \) for all \( x, y \in X \).

The pair \((X, \| \cdot \|)\) is called quasi-\(\beta\)-normed space if \( \| \cdot \| \) is a quasi-\(\beta\)-norm on \( X \). The smallest possible \( K \) is called the modulus of concavity of \( \| \cdot \| \).

**Definition 2.2.** A quasi-\(\beta\)-Banach space is a complete quasi-\(\beta\)-normed space.

**Definition 2.3.** A quasi-\(\beta\)-norm \( \| \cdot \| \) is called a \( (\beta, p) \)-norm \((0 < p \leq 1)\) if

\[ \| x + y \|^p \leq \| x \|^p + \| y \|^p \]

for all \( x, y \in X \). In this case, a quasi-\(\beta\)-Banach space is called a \( (\beta, p) \)-Banach space.

More details, one can refer [16, 17, 21, 22] for the concepts of quasi-normed spaces and \( p \)-Banach space.

Now, we present the following theorem due to Margolis and Diaz [23] for fixed point Theory.

**Theorem 2.1.** [23] Suppose that for a complete generalized metric space \((\Omega, d)\) and a strictly contractive mapping \( T : \Omega \to \Omega \) with Lipschitz constant \( L \). Then, for each given \( x \in \Omega \), either

\[ d(T^n x, T^{n+1} x) = \infty \quad \forall \quad n \geq n_0, \]

or there exists a natural number \( n_0 \) such that

(i) \( d(T^n x, T^{n+1} x) < \infty \) for all \( n \geq n_0 \);

(ii) The sequence \((T^n x)\) is convergent to a fixed point \( y^* \) of \( T \);

(iii) \( y^* \) is the unique fixed point of \( T \) in the set \( \Delta = \{ y \in \Omega : d(T^n x, y) < \infty \} \);

(iv) \( d(y^*, y) \leq \frac{1}{1-L}d(y, Ty) \) for all \( y \in \Delta \).

Throughout this paper, unless otherwise explicitly stated, we will assume that \( X \) is a linear space, and \( Y \) is a \((\beta, p)\)-Banach space with \((\beta, p)\) norm \( \| \cdot \|_Y \). Let \( K \) be the modulus of concavity of \( \| \cdot \|_Y \). For notational convenience, given a mapping \( f : X \to Y \), we define the difference operator

\[ Df(x, y, z) = f(3x + 2y + z) + f(3x + 2y - z) + f(3x - 2y + z) + f(3x - 2y - z) \]

\begin{align*}
&-72[f(x + y) + f(x - y)] - 18[f(x + z) + f(x - z)] - 8[f(y + z) + f(y - z)] \\
&-24f(2x) - 4f(2y) + 240f(x) + 160f(y) + 48f(z)
\end{align*}

for all \( x, y, z \in X \).
3 Stability of Eq. (1.4): Quadratic case

we will use the following lemma in this paper.

Lemma 3.1. [22] Let \( j \in \{-1,1\} \) be fixed, \( s, a \in \mathbb{N} \) with \( a \geq 2 \) and \( \psi : X \to [0, \infty) \) a function such that there exists an \( L < 1 \) with \( \psi(a^{n}x) \leq a^{\alpha s} \psi(x) \) for all \( x \in X \). Let \( f : X \to Y \) be a mapping satisfying

\[
\|f(ax) - a^{s}f(x)\| \leq \psi(x)
\]

for all \( x \in X \). Then there exists a uniquely determined mapping \( F : X \to Y \) such that \( F(ax) = a^{s}F(x) \) and

\[
\|f(x) - F(x)\| \leq \frac{1}{a^{\alpha s} | 1 - L^{j} |} \psi(x)
\]

for all \( x \in X \).

Proof. Consider the set

\[ \Omega = \{ g \mid g : X \to Y \} \]

and introduce the generalized metric on \( \Omega \),

\[
d(g,h) = \inf \{ \mu > 0 \mid \|g(x) - h(x)\| \leq \mu \psi(x) \quad \forall x \in X \}.
\]

It is easy to show that \((\Omega, d)\) is a complete generalized metric space (see [24, 25, 26]). Define a function \( J : \Omega \to \Omega \) by \( Jg(x) = a^{-j}g(a^{j}x) \) for all \( x \in X \). Let \( g, h \in \Omega \) be given such that \( d(g,h) < \epsilon \), by the definition,

\[
\|g(x) - h(x)\| \leq \epsilon \psi(x), \quad \forall x \in X.
\]

Hence

\[
\|Jg(x) - Jh(x)\| = a^{-j}s\|g(a^{j}x) - h(a^{j}x)\| \leq a^{-j}s \epsilon \psi(a^{j}x) \leq L_{j} \psi(x)
\]

for all \( x \in X \). By definition, \( d(Jg, Jh) < \epsilon \). Therefore,

\[
d(Jg, Jh) \leq d(g,h), \quad \forall g, h \in \Omega.
\]

This means that \( J \) is a strictly contractive self-mapping of \( \Omega \) with Lipschitz constant \( L \). It follows from (3.1) that

\[
d(f,Jf) \leq \left\{ \begin{array}{ll}
\frac{1}{a^{\alpha s} | 1 - L^{j} |} \psi(x), & \text{if } j=1, \\
\frac{a^{\alpha s}}{a^{\alpha s} | 1 - L^{j} |} \psi(x), & \text{if } j=-1,
\end{array} \right.
\]

for all \( x \in X \). Therefore, by Theorem 1.3 of [25], \( J \) has a unique fixed point \( F : X \to Y \) in the set \( \Delta = \{ g \in \Omega : d(g,f) < \infty \} \). This implies that \( F(ax) = a^{s}F(x) \) and

\[
F(x) = \lim_{n \to \infty} J^{n}f(x) = \lim_{n \to \infty} a^{-jn}s f(a^{jn}x)
\]

for all \( x \in X \). Moreover,

\[
d(f,F) \leq \frac{1}{1 - L} d(f,Jf) \leq \frac{1}{a^{\alpha s} | 1 - L^{j} |}.
\]

This implies that the inequality (3.2) holds.

To prove the uniqueness of the mapping \( F \), assume that there exists another mapping \( G : X \to Y \) which satisfies (3.2) and \( G(ax) = a^{s}G(x) \) for all \( x \in X \). Fix \( x \in X \). Clearly, \( F(a^{jn}x) = a^{jn}F(x) \) and \( G(a^{jn}x) = a^{jn}G(x) \) for all \( n \in \mathbb{N} \). Thus

\[
\|F(x) - G(x)\| \leq K \|F(a^{jn}x) - G(a^{jn}x)\| \leq K \|F(a^{jn}x) - F(a^{jn}x)\| \leq \frac{2KL^{j}}{a^{\alpha s} | 1 - L^{j} |} \psi(x).
\]

Since, for every \( x \in X \), \( \lim_{n \to \infty} (2KL^{j}a^{jn} | 1 - L^{j} |) \psi(x) = 0 \), we get \( G = F \). This completes the proof. \( \square \)

Theorem 3.2. Let \( j \in \{-1,1\} \) be fixed, and let \( \psi_{b} : X^{3} \to [0, \infty) \) be a mapping such that there exists an \( L < 1 \) with \( \psi_{b}(a^{j}x,a^{j}y,a^{j}z) \leq 16^{j}L \psi_{b}(x,y,z) \) for all \( x,y,z \in X \). Let a mapping \( f : X \to Y \) with \( f(0) = 0 \) satisfies the inequality

\[
\|f(x,y,z)\| \leq \psi_{b}(x,y,z)
\]

for all \( x,y,z \in X \). Then there exists a unique quadratic mapping \( B : X \to Y \) such that

\[
\|f(2x) - 16f(x) - B(x)\| \leq \frac{1}{16^{j} | 1 - L^{j} |} \psi_{b}(x)
\]
for all $x \in X$, where

$$
\tilde{\psi}_b(x) = K[M_b(x, 2x, x) + K M_b(x, x, x) + \left(\frac{11}{2}\right)^\beta K^2 M_b(x, 0, x) \\
+ 20^\beta K^3 M_b(x, 0, 0) + K^4 \left(\frac{1}{2}\right)^\beta M_b(0, 0, x)] \quad \text{for all } x \in X.
$$

**Proof.** Replacing $(x, y, z)$ by $(x, 2x, x), (x, x, x), (x, 0, x), (x, 0, 0)$, and $(0, 0, x)$ in (3.12), respectively, we get the following inequalities

$$
\|f(8x) + f(6x) - 4f(4x) - 80f(3x) + 118f(2x) + 280f(x) - 72f(-x) + f(-2x)\|_Y \\
\leq \psi_b(x, 2x, x), \quad \forall x \in X.
$$

(3.14)

$$
\|f(6x) + f(4x) - 125f(2x) + 448f(x)\|_Y \leq \psi_b(x, x, x), \quad \forall x \in X.
$$

(3.15)

$$
\|2f(4x) - 40f(2x) + 136f(x) - 8f(-x)\|_Y \leq \psi_b(x, 0, x), \quad \forall x \in X.
$$

(3.16)

$$
\|4f(3x) - 24f(2x) + 60f(x)\|_Y \leq \psi_b(x, 0, 0), \quad \forall x \in X.
$$

(3.17)

$$
\|-2f(2x) + 72f(x) - 72f(-x) + 2f(-2x)\|_Y \leq \psi_b(0, 0, 0), \quad \forall x \in X.
$$

(3.18)

$$
\|24f(x) - 24f(-x)\|_Y \leq \psi_b(0, 0, x), \quad \forall x \in X.
$$

(3.19)

Let $g, \tilde{\psi}_b : X \to Y$ be mappings defined by $g(x) = f(2x) - 16f(x)$ for all $x \in X$ and

$$
\tilde{\psi}_b(x) = K[M_b(x, 2x, x) + K M_b(x, x, x) + \left(\frac{11}{2}\right)^\beta K^2 M_b(x, 0, x) \\
+ 20^\beta K^3 M_b(x, 0, 0) + K^4 \left(\frac{1}{2}\right)^\beta M_b(0, 0, x)]
$$

(3.20)

for all $x \in X$. It follows from (3.14) – (3.20) that

$$
\|f(8x) - 16f(4x) - 16f(2x) + 256f(x)\|_Y \leq \xi_b(x), \quad \forall x \in X.
$$

(3.21)

Therefore (3.21) means

$$
\|g(4x) - 4g(x)\|_Y \leq \tilde{\psi}_b(x)
$$

(3.22)

for all $x \in X$. By Lemma 3.1, there exists a unique mapping $B : X \to Y$ such that $B(4x) = 16B(x)$ and

$$
\|f(2x) - 16f(x) - B(x)\|_Y \leq \frac{1}{16^\beta} \tilde{\psi}_b(x)
$$

(3.23)

for all $x \in X$. It remains to show that $B$ is a quadratic map. By (3.12), we have

$$
\|\frac{1}{16^\beta} D f(4^n x, 4^n y, 4^n z)\|_Y \leq \frac{1}{16^\beta} \psi_b(4^n x, 4^n y, 4^n z) \\
\leq \frac{1}{16^\beta} (16^\beta L)^n \psi_b(x, y, z) = L^n \psi_b(x, y, z)
$$

(3.24)

for all $x, y, z \in X$ and all positive integers $n$. so

$$
\|DB(x, y, z)\|_Y = 0
$$

(3.25)

for all $x, y, z \in X$. Thus the mapping $B : X \to Y$ is quadratic, as desired.

Throughout this paper, we will assume that $X$ is a quasi-$\alpha$-normed space with quasi-$\alpha$-norm $\|\cdot\|_X$ in all the corollaries. The following corollaries are immediate consequence of Theorem 3.2.
Corollary 3.3. Let $\nu \geq 0$ and $r, s, t$ which are all $> 0$ be real numbers such that $r, s$ and $t$ are all $\neq \frac{2\beta}{\alpha}$. Suppose that a mapping $f : X \rightarrow Y$ with $f(0) = 0$ satisfies the inequality

$$
\|Df(x, y, z)\|_Y \leq \begin{cases} 
\nu||x||^r_X, & r > 0, s = 0, t = 0; \\
\nu||y||^r_X, & r = 0, s > 0, t = 0; \\
\nu||z||^r_X, & r = 0, s = 0, t > 0; \\
\nu\{||x||^r_X + ||y||^r_X + ||z||^r_X\}, & r > 0, s > 0, t > 0; 
\end{cases}
$$

(3.26)

for all $x, y, z \in X$. Then there exists a unique quadratic mapping $B : X \rightarrow Y$ such that

$$
\|f(2x) - 16f(x) - B(x)\|_Y \leq \begin{cases} 
\alpha_b, & r > 0, s = 0, t = 0; \\
\beta_b(x), & r = 0, s > 0, t = 0; \\
\gamma_b(x), & r = 0, s = 0, t > 0; \\
\delta_b(x), & r > 0, s > 0, t > 0; \\
\zeta_b(x), & r > 0, s > 0, t > 0; 
\end{cases}
$$

(3.27)

for all $x \in X$, where

$$
\alpha_b = K\nu \left\{ 1 + K \left( \frac{4}{3} \right)^\beta K^2 + 20^\beta K^3 + \left[ \left( \frac{1}{2} \right)^\beta + \left( \frac{1}{3} \right)^\beta \right] K^4 \right\},
$$

$$
\beta_b(x) = K\nu \left\{ \frac{4^\alpha}{16^\beta} \right\} \left\{ 1 + K \left( \frac{4}{3} \right)^\beta K^2 + 20^\beta K^3 \right\} ||x||^r_X,
$$

$$
\gamma_b(x) = K\nu \left\{ \frac{4^\alpha}{16^\beta} \right\} \left\{ 2^\alpha + K \left( \frac{1}{2} \right)^\beta K_4 \right\} ||x||^r_X,
$$

$$
\delta_b(x) = K\nu \left\{ \frac{4^\alpha}{16^\beta} \right\} \left\{ \frac{1}{2^\beta} + K \left( \frac{1}{2} \right)^\beta K^2 + \left( \frac{1}{3} \right)^\beta K^4 \right\} ||x||^r_X
$$

and

$$
\zeta_b(x) = \beta_b(x) + \gamma_b(x) + \delta_b(x) \quad \text{for all} \ x \in X.
$$

Corollary 3.4. Let $\nu \geq 0$ and $r, s, t$ which are all $> 0$ be real numbers such that $\lambda = r + s + t \neq \frac{2\beta}{\alpha}$. Suppose that a mapping $f : X \rightarrow Y$ with $f(0) = 0$ satisfies the inequality

$$
\|Df(x, y, z)\|_Y \leq \begin{cases} 
\nu \left\{ ||x||^r_X ||y||^r_X ||z||^r_X \right\}, & \\
\nu \left\{ ||x||^r_X ||y||^r_X ||z||^r_X + ||x||^r_X + ||y||^r_X + ||z||^r_X \right\}
\end{cases}
$$

(3.28)

for all $x, y, z \in X$. Then there exists a unique quadratic mapping $B : X \rightarrow Y$ such that

$$
\|f(2x) - 16f(x) - B(x)\|_Y \leq \begin{cases} 
\rho_b(x), & \\
\tau_b(x)
\end{cases}
$$

(3.29)

for all $x \in X$, where

$$
\rho_b(x) = K\nu \left\{ \frac{4^\alpha}{16^\beta} \right\} \left\{ 2^\alpha + K \left( \frac{1}{2} \right)^\beta K_4 \right\} ||x||^r_X \quad \text{and}
$$

$$
\tau_b(x) = K\nu \left\{ \frac{4^\alpha}{16^\beta} \right\} \left\{ \frac{2 + 2^\alpha + 4^\alpha + 4K + 2 \left( \frac{1}{2} \right)^\beta K^2 + 20^\beta K^3 + \left[ \left( \frac{1}{2} \right)^\beta + \left( \frac{1}{3} \right)^\beta \right] K^4}{16^\beta - 4^\alpha \lambda} \right\} ||x||^r_X
$$

for all $x \in X$.

A counter example to illustrate the non stability in Condition (v) of Corollary 3.3.

Example 3.5. Let $\alpha : \mathbb{R} \rightarrow \mathbb{R}$ be a function defined by

$$
\alpha(x) = \begin{cases} 
\mu x^2, & \text{if} \ |x| < 1 \\
\mu, & \text{otherwise}
\end{cases}
$$

where $\mu > 0$ is a constant, and define a function $f_{\alpha} : \mathbb{R} \rightarrow \mathbb{R}$ by

$$
f_{\alpha}(x) = f(2x) - 16f(x) = \sum_{n=0}^{\infty} \frac{\alpha(4^n x)}{16^n} \quad \text{for all} \ \ x \in \mathbb{R}.
$$
Then $f_b$ satisfies the functional inequality
\[ |Df_b(x, y, z)| \leq \frac{676 \cdot 16^3}{15} \mu \left( |x|^2 + |y|^2 + |z|^2 \right) \tag{3.30} \]
for all $x, y, z \in \mathbb{R}$. Then there do not exist a quadratic mapping $B : \mathbb{R} \to \mathbb{R}$ and a constant $c > 0$ such that
\[ |f_b(x) - B(x)| \leq c|x| \quad \text{for all} \quad x \in \mathbb{R}. \tag{3.31} \]

Proof. Now
\[ |f_b(x)| \leq \sum_{n=0}^{\infty} \frac{|\alpha(4^n x)|}{|16^n|} = \sum_{n=0}^{\infty} \frac{\mu}{16^n} = \frac{16}{15} \mu. \]
Therefore we see that $f_b$ is bounded. We are going to prove that $f_b$ satisfies (3.30).

If $|x|^2 + |y|^2 + |z|^2 = 0$ or $|x|^2 + |y|^2 + |z|^2 \geq \frac{1}{16}$ then
\[ |Df_b(x, y, z)| \leq \frac{676 \cdot 16}{15} \left( |x|^2 + |y|^2 + |z|^2 \right). \tag{3.32} \]

Now suppose that $0 < |x|^2 + |y|^2 + |z|^2 < \frac{1}{16}$. Then there exists a non-negative integer $k$ such that
\[ \frac{1}{16^{k+2}} \leq |x|^2 + |y|^2 + |z|^2 < \frac{1}{16^{k+1}}, \tag{3.33} \]
so that $4^k x < \frac{1}{4}$, $4^k y < \frac{1}{4}$, $4^k z < \frac{1}{4}$, and consequently $4^{k-1}(3x \pm 2y \pm z), 4^{k-1}(x \pm y),
4^{k-1}(x \pm z), 4^{k-1}(2x), 4^{k-1}(x), 4^{k-1}(y), 4^{k-1}(z), 4^n(x), 4^n(y), 4^n(z) \in (-1, 1)$.

Therefore for each $n = 0, 1, \ldots, k - 1$, we have
\[ 4^n(3x \pm 2y \pm z), 4^n(x \pm y), 4^n(x \pm z), 4^n(2x), 4^n(x), 4^n(y), 4^n(z) \in (-1, 1) \]
and $Df_b(4^n x, 4^n y, 4^n z) = 0$. From the definition of $f$ and (3.33), we obtain that
\[
\begin{align*}
\|Df_b(x, y, z)\| &\leq \sum_{n=0}^{\infty} \frac{1}{16^n} \left| \alpha(4^n(3x + 2y + z)) + \alpha(4^n(3x + 2y - z)) + \alpha(4^n(3x - 2y - z)) + 72\alpha(4^n(x + y)) + 72\alpha(4^n(x - y)) - 18\alpha(4^n(x + z)) + 18\alpha(4^n(x - z)) \right| \\
&\leq \sum_{n=0}^{\infty} \frac{1}{16^n} \left| \alpha(4^n(3x + 2y + z)) + \alpha(4^n(3x + 2y - z)) + \alpha(4^n(3x - 2y - z)) + 72\alpha(4^n(x + y)) + 72\alpha(4^n(x - y)) - 18\alpha(4^n(x + z)) + 18\alpha(4^n(x - z)) \right| \\
&\leq \sum_{n=0}^{\infty} \frac{1}{16^n} \left| \alpha(4^n(3x + 2y + z)) + \alpha(4^n(3x + 2y - z)) + \alpha(4^n(3x - 2y - z)) + 72\alpha(4^n(x + y)) + 72\alpha(4^n(x - y)) - 18\alpha(4^n(x + z)) + 18\alpha(4^n(x - z)) \right| \\
&\leq \sum_{n=0}^{\infty} \frac{1}{16^n} 676\mu = 676\mu \times \frac{16}{15 \cdot 16^k} = \frac{676 \cdot 16^3}{15} \mu \left( |x|^2 + |y|^2 + |z|^2 \right). \end{align*}
\]
Thus $f_b$ satisfies (3.30) for all $x, y, z \in \mathbb{R}$ with $0 < |x|^2 + |y|^2 + |z|^2 < \frac{1}{16}$.

We claim that the functional equation (1.4) is not stable for $r = s = t = 2$ in condition (v) Corollary 3.3 ($\alpha = \beta = p = 1$). Suppose on the contrary that there exist a quadratic mapping $B : \mathbb{R} \to \mathbb{R}$ and a constant $\eta > 0$ satisfying (3.31). Since $f_b$ is bounded and continuous for all $x \in \mathbb{R}$, $B$ is bounded on any open interval containing the origin and continuous at the origin. In view of Theorem 3.2, $B$ must have the form $B(x) = cx^2$ for any $x \in \mathbb{R}$. Thus we obtain that
\[ |f_b(x)| \leq (\eta + |c|) |x|^2. \tag{3.34} \]
But we can choose a positive integer $m$ with $m\mu > \eta + |c|$. If $x \in (0, \frac{1}{\sqrt{m\mu}})$, then $4^n x \in (0, 1)$ for all $n = 0, 1, \ldots, m - 1$. For this $x$, we get
\[ f_b(x) = \sum_{n=0}^{\infty} \frac{\alpha(4^n x)}{16^n} \geq \sum_{n=0}^{m-1} \frac{\mu(4^n x)^2}{16^n} = m\mu x^2 > (\eta + |c|) x^2 \]
which contradicts (3.34). \qed
4 Stability of Eq. (1.4): Quartic Case

Theorem 4.1. Let \( j \in \{-1, 1\} \) be fixed, and let \( \psi_d : X^3 \to [0, \infty) \) be a mapping such that there exists an \( L < 1 \) with \( \psi_d(4^j x, 4^j y, 4^j z) \leq 256^\beta L \psi_d(x, y, z) \) for all \( x, y, z \in X \). Let a mapping \( f : X \to Y \) with \( f(0) = 0 \) satisfies the inequality
\[
\|Df(x, y, z)\|_Y \leq \psi_d(x, y, z)
\]
for all \( x, y, z \in X \). Then there exists a unique quartic mapping \( D : X \to Y \) such that
\[
\|f(2x) - 4f(x) - D(x)\|_Y \leq \frac{1}{256^\beta |1 - L^j|} \psi_d(x)
\]
for all \( x \in X \), where
\[
\tilde{\psi}_d(x) = \psi_d(x, 2x, x) + K \psi_d(x, x, x) + \left(\frac{1}{2}\right)^h K^2 \psi_d(x, 0, x)
+ 20^h K^3 \psi_d(x, 0, 0) + \left(\frac{1}{2}\right)^b K^4 \psi_d(0, x, 0) + \left(\frac{5}{3}\right)^b K^4 \psi_d(0, 0, x)
\]
for all \( x \in X \).

Proof. Similar to the proof of Theorem 3.2, we have
\[
\|f(8x) - 4f(4x) - 256f(2x) + 1024f(x)\|_Y \leq \tilde{\psi}_d(x)
\]
for all \( x \in X \), where
\[
\tilde{\psi}_d(x) = \psi_d(x, 2x, x) + K \psi_d(x, x, x) + \left(\frac{1}{2}\right)^h K^2 \psi_d(x, 0, x)
+ 20^h K^3 \psi_d(x, 0, 0) + \left(\frac{1}{2}\right)^b K^4 \psi_d(0, x, 0) + \left(\frac{5}{3}\right)^b K^4 \psi_d(0, 0, x)
\]
for all \( x \in X \). Let \( h : X \to Y \) be a mapping defined by \( h(x) = f(2x) - 4f(x) \), then (4.3) means
\[
\|h(4x) - 256h(x)\|_Y \leq \tilde{\psi}_d(x)
\]
for all \( x \in X \). By Lemma 3.1, there exists a unique mapping \( D : X \to Y \) such that
\[
D(4x) = 256D(x)
\]
and
\[
\|f(2x) - 4f(x) - D(x)\|_Y \leq \frac{1}{256^\beta |1 - L^j|} \rho_d(x)
\]
for all \( x \in X \). It remains to show that \( D \) is a quartic map. By (4.1), we have
\[
\left\| \frac{1}{256^n} Df(4^n x, 4^n y, 4^n z) \right\|_Y \leq \frac{1}{256^n |1 - L^j|} \rho_d(x)
\leq \frac{1}{256^n |1 - L^j|} (256^\beta L)^n \rho_d(x, y, z) = L^n \rho_d(x, y, z)
\]
for all \( x, y, z \in X \) and all positive integers \( n \). so
\[
\|DD(x, y, z)\|_Y = 0
\]
for all \( x, y, z \in X \). Thus the mapping \( D : X \to Y \) is quartic, as desired \( \Box \)

The following corollaries are immediate consequence of Theorem 4.1.

Corollary 4.2. Let \( \nu \geq 0 \) and \( r, s, t \) which are all \( > 0 \) be real numbers such that \( r, s, t \) are all \( \neq \frac{4\beta}{\alpha} \). Suppose that a mapping \( f : X \to Y \) with \( f(0) = 0 \) satisfies the inequality (3.26) for all \( x, y, z \in X \). Then there exists a unique quartic mapping \( D : X \to Y \) such that
\[
\|f(2x) - 4f(x) - D(x)\|_Y \leq \begin{cases} 
\alpha_d(x), & r > 0, s > 0, t = 0; \\
\beta_d(x), & r = 0, s > 0, t = 0; \\
\gamma_d(x), & r = 0, s = 0, t > 0; \\
\delta_d(x), & r > 0, s = 0, t > 0; \\
\zeta_d(x), & r > 0, s > 0, t > 0;
\end{cases}
\]

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Suppose that a mapping $f : X \to Y$, where
\[
\alpha_d = K \nu \left\{ \frac{1 + K + \left( \frac{1}{\gamma} \right)^\beta K^2 + 20\beta K^3 + \left[ \left( \frac{1}{\gamma} \right)^\beta + \left( \frac{3}{\gamma} \right)^\beta \right] K^4}{|256^d - 1|} \right\},
\]
\[
\beta_d(x) = K \nu \left\{ \frac{4^{\nu \gamma}}{256^d} \left\{ \frac{1 + K + \left( \frac{1}{\gamma} \right)^\beta K^2 + 20\beta K^3}{|256^d - 4^{\nu \gamma}|} \right\} \right\} \|x\|^\gamma_X,
\]
\[
\gamma_d(x) = K \nu \left\{ \frac{4^{\nu \gamma}}{256^d} \left\{ \frac{2^{\nu \gamma} + K + \left( \frac{1}{\gamma} \right)^\beta K^2}{|256^d - 4^{\nu \gamma}|} \right\} \right\} \|x\|^\gamma_X,
\]
\[
\delta_d(x) = K \nu \left\{ \frac{4^{\nu \gamma}}{256^d} \left\{ \frac{1 + K + \left( \frac{1}{\gamma} \right)^\beta K^2 + \left( \frac{3}{\gamma} \right)^\beta K^3}{|256^d - 4^{\nu \gamma}|} \right\} \right\} \|x\|^\gamma_X
\]
and
\[
\zeta_d(x) = \beta_d(x) + \gamma_d(x) \quad \text{for all } x \in X.
\]

**Corollary 4.3.** Let $\nu \geq 0$ and $r, s, t$ which are all $> 0$ be real numbers such that $\lambda = r + s + t < \frac{4\beta}{\alpha}$. Suppose that a mapping $f : X \to Y$ with $f(0) = 0$ satisfies the inequality (3.28) for all $x, y, z \in X$. Then there exists a unique quartic mapping $D : X \to Y$ such that
\[
\|f(2x) - 4f(x) - D(x)\|_Y \leq \begin{cases} \rho_d(x), & \tau_d(x), \\ \end{cases}
\]
for all $x \in X$, where
\[
\rho_d(x) = K \nu \left\{ \frac{4^{\nu \gamma}}{256^d} \left\{ \frac{2^{\nu \gamma} + K + \left( \frac{1}{\gamma} \right)^\beta K^2}{|256^d - 4^{\nu \gamma}|} \right\} \right\} \|x\|^\gamma_X \quad \text{and}
\]
\[
\tau_d(x) = K \nu \left\{ \frac{4^{\nu \gamma}}{256^d} \left\{ \frac{2 + 2^{\nu \gamma} + 2^{\nu \gamma} + 2 \left( \frac{1}{\gamma} \right)^\beta K^2 + 20\beta K^3 + \left[ \left( \frac{1}{\gamma} \right)^\beta + \left( \frac{3}{\gamma} \right)^\beta \right] K^4}{|256^d - 4^{\nu \gamma}|} \right\} \right\} \|x\|^\gamma_X
\]
for all $x \in X$.

A counter example to illustrate the non stability in Condition (v) of Corollary 4.2.

**Example 4.4.** Let $\alpha : \mathbb{R} \to \mathbb{R}$ be a function defined by
\[
\alpha(x) = \begin{cases} \mu x^4, & \text{if } |x| < 1 \\ \mu, & \text{otherwise} \end{cases}
\]
where $\mu > 0$ is a constant, and define a function $f_d : \mathbb{R}^2 \to \mathbb{R}$ by
\[
f_d(x) = \sum_{n=0}^{\infty} \frac{\alpha(4^n x)}{|256^n|} \quad \text{for all } x \in \mathbb{R}.
\]

Then $f_d$ satisfies the functional inequality
\[
|Df_d(x, y, z)| \leq \frac{676 \cdot 256^3}{255} \mu \left( |x|^4 + |y|^4 + |z|^4 \right)
\]
for all $x, y, z, w \in \mathbb{R}$. Then there do not exist a quartic mapping $D : \mathbb{R} \to \mathbb{R}$ and a constant $c > 0$ such that
\[
|f_d(x) - D(x)| \leq c|x|^4 \quad \text{for all } x \in \mathbb{R}.
\]

**Proof.** Now
\[
|f_d(x)| \leq \sum_{n=0}^{\infty} \frac{|\alpha(4^n x)|}{|256^n|} = \sum_{n=0}^{\infty} \frac{\mu}{256^n} = \frac{256}{255} \mu.
\]

Therefore we see that $f_d$ is bounded. We are going to prove that $f_d$ satisfies (4.11)). If $|x|^4 + |y|^4 + |z|^4 = 0$ or $|x|^4 + |y|^4 + |z|^4 \geq \frac{1}{256}$ then
\[
|Df_d(x, y, z)| \leq \frac{676 \cdot 256}{255} \leq \frac{160 \cdot 256^2}{63} (|x|^4 + |y|^4 + |z|^4).
\]

Now suppose that $0 < |x|^4 + |y|^4 + |z|^4 < \frac{1}{256}$. Then there exists a non-negative integer $k$ such that
\[
\frac{1}{256^{d+2}} \leq |x|^4 + |y|^4 + |z|^4 < \frac{1}{256^{d+1}},
\]
so that $4^k|x| < \frac{1}{4}$, $4^k|y| < \frac{1}{4}$, $4^k|z| < \frac{1}{4}$ and consequently $4^{k-1}(3x \pm 2y \pm z), 4^{k-1}(x \pm y), 4^{k-1}(x \pm z), 4^{k-1}(2x), 4^{k-1}(x), 4^{k-1}(y), 4^{k-1}(z), 4^n(x), 4^n(y), 4^n(z) \in (-1, 1)$.

Therefore for each $n = 0, 1, \ldots, k - 1$, we have

$$4^n(3x \pm 2y \pm z), 4^n(x \pm y), 4^n(x \pm z), 4^n(2x), 4^n(x), 4^n(y), 4^n(z) \in (-1, 1)$$

and $D_{fa}(x, y, 4^n z) = 0$. From the definition of $fa$ and (4.14), we obtain that

$$\|D_{fa}(x, y, z)\| \leq \sum_{n=0}^{\infty} \frac{1}{256^n} \left| \alpha(4^n(3x + 2y + z)) + \alpha(4^n(3x + 2y - z)) + \alpha(4^n(3x - 2y + z)) + \alpha(4^n(3x - 2y - z)) - 72[\alpha(4^n(x + y)) + \alpha(4^n(x - y))] - 18\alpha(4^n(x + z)) - 18\alpha(4^n(x - z)) - 8[\alpha(4^n(y + z)) + \alpha(4^n(y - z))] - 24\alpha(4^n(2x)) - 24\alpha(4^n(2z)) - 4\alpha(4^n(2y)) + 240\alpha(4^n(x)) + 160\alpha(4^n(y)) + 48\alpha(4^n(z)) \right|$$

$$\leq \sum_{n=0}^{\infty} \frac{1}{256^n} \left[ \alpha(4^n(3x + 2y + z)) + \alpha(4^n(3x + 2y - z)) + \alpha(4^n(3x - 2y + z)) + \alpha(4^n(3x - 2y - z)) - 72[\alpha(4^n(x + y)) + \alpha(4^n(x - y))] - 18\alpha(4^n(x + z)) - 18\alpha(4^n(x - z)) - 8[\alpha(4^n(y + z)) + \alpha(4^n(y - z))] - 24\alpha(4^n(2x)) - 24\alpha(4^n(2z)) - 4\alpha(4^n(2y)) + 240\alpha(4^n(x)) + 160\alpha(4^n(y)) + 48\alpha(4^n(z)) \right]$$

$$\leq \frac{1}{256^n} 676\mu = \frac{1}{15} \cdot \frac{16}{15} \cdot \frac{1}{16} \cdot \frac{255}{\mu} \cdot \left( |x|^4 + |y|^4 + |z|^4 \right).$$

Thus $fa$ satisfies (4.11) for all $x, y, z \in \mathbb{R}$ with $0 < |x|^4 + |y|^4 + |z|^4 < \frac{1}{4}$.

We claim that the functional equation (1.4) is not stable for $r = s = t = 4$ in condition (v) Corollary 4.2 ($\alpha = \beta = p = 1$). Suppose on the contrary that there exist a quartic mapping $D : \mathbb{R} \to \mathbb{R}$ and a constant $\eta > 0$ satisfying (4.12). Since $fa$ is bounded and continuous for all $x \in \mathbb{R}$, $D$ is bounded on any open interval containing the origin and continuous at the origin. In view of Theorem 4.1, $D$ must have the form $D(x) = cx^4$ for any $x \in \mathbb{R}$. Thus we obtain that

$$|fa(x)| \leq (\eta + |c|) |x|^4.$$ (4.15)

But we can choose a positive integer $m$ with $m\mu > \eta + |c|$. If $x \in (0, \frac{1}{m\mu})$, then $4^n x \in (0, 1)$ for all $n = 0, 1, \ldots, m - 1$. For this $x$, we get

$$fa(x) = \sum_{n=0}^{\infty} \alpha(4^n x) 256^n \geq \sum_{n=0}^{m-1} \frac{\mu(4^n x)^4}{256^n} = m\mu x^4 > (\eta + |c|) x^4$$

which contradicts (4.15).

\section{Stability of Eq. (1.4): Mixed Case}

\textbf{Theorem 5.1.} Let $j \in \{-1, 1\}$ be fixed, and let $\psi : X^3 \to [0, \infty)$ be a mapping such that there exists an $L < 1$ with $\psi(4^j x, 4^j y, 4^j z) \leq 16^{3/2} L \psi(x, y, z)$ and $\psi(4^j x, 4^j y, 4^j z) \leq 256^{3/2} L \psi(x, y, z)$ for all $x, y, z \in X$. Let a mapping $f : X \to Y$ with $f(0) = 0$ satisfies the inequality

$$\|Df(x, y, z)\|_Y \leq \psi(x, y, z)$$ (5.1)

for all $x, y, z \in X$. Then there exists a quadratic mapping $B : X \to Y$ and a quartic mapping $D : X \to Y$ such that

$$\|f(x) - B(x) - D(x)\|_Y \leq \frac{K}{12 \cdot 256^3 |1 - L^j|} \left[ 16^3 \psi_0(x) + \psi_d(x) \right]$$ (5.2)

for all $x \in X$, where $\psi_0(x)$ and $\psi_d(x)$ are defined as in Theorems 3.2 and 4.1 respectively.
Proof. Since $\psi(4^j x, 4^j y, 4^j z) \leq 16^j L \psi(x, y, z)$ and $\psi(4^j x, 4^j y, 4^j z) \leq 256^j L \psi(x, y, z)$ for all $x, y, z \in X$, by Theorems 3.2 and 4.1, there exist a quadratic mapping $D_0 : X \to Y$ and a quartic mapping $D_0 : X \to Y$ such that

$$
\|f(2x) - 16f(x) - B_0(x)\|_Y \leq \frac{1}{16^j |1 - L^j|} \tilde{\psi}_b(x) \quad \text{and}
$$

$$
\|f(2x) - 4f(x) - D_0(x)\|_Y \leq \frac{1}{256^j |1 - L^j|} \tilde{\psi}_b(x)
$$

for all $x \in X$. Therefore the result follows from the last two inequalities.

**Theorem 5.2.** Let $j \in \{-1, 1\}$ be fixed, and let $\psi : X^3 \to [0, \infty)$ be a mapping such that there exists an $L < 1$ with $\psi(4^j x, 4^j y, 4^j z) \leq 16^j L \psi(x, y, z)$ and $\psi(4^j x, 4^j y, 4^j z) \leq 256^j L \psi(x, y, z)$ for all $x, y, z \in X$. Let a mapping $f : X \to Y$ with $f(0) = 0$ satisfies the inequality (5.1) for all $x, y, z \in X$. Then there exists a quadratic mapping $B : X \to Y$ and a quartic mapping $D : X \to Y$ such that

$$
\|f(x) - B(x) - D(x)\|_Y \leq \frac{K}{12} \cdot \frac{1}{256^j |1 - L^j|} [16^j \tilde{\psi}_b(x) + \tilde{\psi}_a(x)] \quad (5.3)
$$

for all $x \in X$, where $\tilde{\psi}_b(x)$ and $\tilde{\psi}_a(x)$ are defined as in Theorems 3.2 and 4.1 respectively.

**Proof.** Since $\psi(4^j x, 4^j y, 4^j z) \leq 16^j L \psi(x, y, z)$ and $\psi(4^j x, 4^j y, 4^j z) \leq 256^j L \psi(x, y, z)$ for all $x, y, z \in X$, by Theorems 3.2 and 4.1, there exist a quadratic mapping $B_0 : X \to Y$ and a quartic mapping $D_0 : X \to Y$ such that

$$
\|f(2x) - 16f(x) - B_0(x)\|_Y \leq \frac{1}{16^j |1 - L^j|} \tilde{\psi}_b(x) \quad \text{and}
$$

$$
\|f(2x) - 4f(x) - D_0(x)\|_Y \leq \frac{1}{256^j |1 - L^j|} \tilde{\psi}_b(x)
$$

for all $x \in X$. Therefore the result follows from the last two inequalities.

**Corollary 5.3.** Let $\nu \geq 0$ and $r, s, t$ which are all $> 0$ be real numbers such that $r, s$ and $t$ are all $\neq \frac{2\beta}{\alpha}$ and $\frac{4\beta}{\alpha}$. Suppose that a mapping $f : X \to Y$ with $f(0) = 0$ satisfies the inequality (3.26) for all $x, y, z \in X$. Then there exist a quadratic mapping $B : X \to Y$ and a quartic mapping $D : X \to Y$ such that

$$
\|f(x) - B(x) - D(x)\|_Y \leq \frac{K}{12} \left\{ \begin{array}{l}
\alpha_b + \alpha_d, r > 0, s = 0, t = 0; \\
\beta_b(x) + \beta_d(x), r = 0, s > 0, t = 0; \\
\gamma_b(x) + \gamma_d(x), r = 0, s > 0, t > 0; \\
\delta_b(x) + \delta_d(x), r > 0, s > 0, t > 0; \\
\zeta_b(x) + \zeta_d(x), r > 0, s > 0, t > 0;
\end{array} \right. \quad (5.4)
$$

for all $x \in X$, where $\alpha_b, \alpha_d, \beta_b(x), \beta_d(x), \gamma_b(x), \gamma_d(x), \delta_b(x), \delta_d(x), \zeta_b(x)$ and $\zeta_d(x)$ are defined as in Corollaries 3.3 and 4.2.

**Corollary 5.4.** Let $\nu \geq 0$ and $r, s, t$ which are all $> 0$ be real numbers such that $\lambda = r + s + t \neq \frac{2\beta}{\alpha}$ and $\frac{4\beta}{\alpha}$. Suppose that a mapping $f : X \to Y$ with $f(0) = 0$ satisfies the inequality (3.28) for all $x, y, z \in X$. Then there exist a quadratic mapping $B : X \to Y$ and a quartic mapping $D : X \to Y$ such that

$$
\|f(x) - B(x) - D(x)\|_Y \leq \frac{K}{12} \left\{ \begin{array}{l}
\rho_b(x) + \rho_d(x), r = 0, s = 0, t > 0; \\
\tau_b(x) + \tau_d(x)
\end{array} \right. \quad (5.4)
$$

for all $x \in X$, where $\rho_b(x), \rho_d(x), \tau_b(x), \tau_d(x)$ are defined as in Corollaries 3.4 and 4.3.

**Example 5.5.** Let $\alpha : \mathbb{R} \to \mathbb{R}$ be a function defined by

$$
\alpha(x) = \begin{cases} \\ \mu(x^2 + x^4), & \text{if } |x| < 1 \\
\mu, & \text{otherwise} \\
\end{cases}
$$

where $\mu > 0$ is a constant, and define a function $f : \mathbb{R} \to \mathbb{R}$ by

$$
f(x) = \sum_{n=0}^{\infty} \frac{\alpha(4^n x)}{4^n} \quad \text{for all } x \in \mathbb{R}.
$$
Then $f$ satisfies the functional inequality

$$|Df(x, y, z)| \leq \frac{1352 \cdot 16^3}{15} \mu \left(|x|^2 + |y|^2 + |z|^2\right)$$

(5.5)

for all $x, y, z, w \in \mathbb{R}$. Then there do not exist a quadratic mapping $B : \mathbb{R} \to \mathbb{R}$ and a quartic mapping $D : \mathbb{R} \to \mathbb{R}$ and a constant $\eta > 0$ such that

$$|f(x) - B(x) - D(x)| \leq \eta |x| \quad \text{for all} \quad x \in \mathbb{R}.$$  

(5.6)

6 Stability of Eq. (1.4) using various substitutions

In this section, the generalized Hyers-Ulam stability of (1.4) using various substitutions is investigated. The proofs of the following theorems and corollaries are similar to that of the Theorems 3.2, 4.1, and 5.1 and the corollaries 3.3, 4.2, 4.3, 5.3 and 5.4. Hence the details of the proofs are omitted.

**Theorem 6.1.** Let $j \in \{-1, 1\}$ be fixed, and let $\psi_h : X^3 \to [0, \infty)$ be a mapping such that there exists an $L < 1$ with $\psi_h(jx, jy, jz) \leq 9^jL\psi_h(x, y, z)$ for all $x, y, z \in X$. Let a mapping $f : X \to Y$ with $f(0) = 0$ satisfies the inequality (3.12) for all $x, y, z \in X$. Then there exists a unique quadratic mapping $B : X \to Y$ such that

$$\|f(2x) - 16f(x) - B(x)\|_Y \leq \frac{1}{9^j - 1} \tilde{\psi}_h(x)$$

(6.1)

for all $x \in X$, where

$$\tilde{\psi}_h(x) = K[M_h(x, x, x) + K\left(\frac{1}{2}\right)M_h(x, 0, x) + K^24^\beta M_h(x, 0, 0) + K^2\left(\frac{1}{8}\right)^\beta M_h(0, 0, x)]$$

for all $x \in X$.

**Corollary 6.2.** Let $\nu \geq 0$ and $r, s, t$ which are all $> 0$ be real numbers such that $r, s$ and $t$ are all $\neq \frac{1}{\alpha}$. Suppose that a mapping $f : X \to Y$ with $f(0) = 0$ satisfies the inequality (3.26) for all $x, y, z \in X$. Then there exists a unique quadratic mapping $B : X \to Y$ such that

$$\|f(2x) - 16f(x) - B(x)\|_Y \leq \begin{cases} \alpha_{\nu}(x), & r > 0, s = 0, t = 0; \\ \beta_{\nu}(x), & r = 0, s > 0, t = 0; \\ \gamma_{\nu}(x), & r = 0, s > 0, t > 0; \\ \delta_{\nu}(x), & r = 0, s = 0, t > 0; \\ \zeta_{\nu}(x), & r > 0, s > 0, t > 0; \end{cases}$$

for all $x \in X$, where

$$\alpha_{\nu} = K\nu\left\{1 + \left(\frac{1}{2}\right)^\beta K + K^2\left[4^\beta + \left(\frac{1}{2}\right)^\beta\right]\right\} \|x\|_X$$

$$\beta_{\nu} = K\nu\left\{\left(\frac{3^\nu}{9^\beta}\right)^\beta + \left(\frac{1}{2}\right)^\beta K + 4^\beta K^2\left[4^\beta + \left(\frac{1}{2}\right)^\beta\right]\right\} \|x\|_X$$

$$\gamma_{\nu} = K\nu\left\{\left(\frac{3^\nu}{9^\beta}\right)^\beta + \left(\frac{1}{2}\right)^\beta K + 4^\beta K^2\left[4^\beta + \left(\frac{1}{2}\right)^\beta\right]\right\} \|x\|_X$$

$$\delta_{\nu} = K\nu\left\{\left(\frac{3^\nu}{9^\beta}\right)^\beta + \left(\frac{1}{2}\right)^\beta K + 4^\beta K^2\left[4^\beta + \left(\frac{1}{2}\right)^\beta\right]\right\} \|x\|_X$$

$$\zeta_{\nu} = \beta_{\nu}(x) + \gamma_{\nu}(x) + \delta_{\nu}(x) \quad \text{for all} \quad x \in X.$$

**Corollary 6.3.** Let $\nu \geq 0$ and $r, s, t$ which are all $> 0$ be real numbers such that $\lambda = r + s + t \neq \frac{2\beta}{\alpha}$. Suppose that a mapping $f : X \to Y$ with $f(0) = 0$ satisfies the inequality (3.28) for all $x, y, z \in X$. Then there exists a unique quadratic mapping $B : X \to Y$ such that

$$\|f(2x) - 16f(x) - B(x)\|_Y \leq \begin{cases} \rho_{\nu}(x), & r = 0, s = 0, t = 0; \\ \tau_{\nu}(x), & r, s, t > 0; \end{cases}$$

for all $x \in X$, where

$$\rho_{\nu}(x) = K\nu\left\{\left(\frac{3^\nu}{9^\beta}\right)^\beta + \left(\frac{1}{2}\right)^\beta K + 4^\beta K^2\left[4^\beta + \left(\frac{1}{2}\right)^\beta\right]\right\} \|x\|_X$$

$$\tau_{\nu}(x) = K\nu\left\{\left(\frac{3^\nu}{9^\beta}\right)^\beta + \left(\frac{1}{2}\right)^\beta K + 4^\beta K^2\left[4^\beta + \left(\frac{1}{2}\right)^\beta\right]\right\} \|x\|_X \quad \text{for all} \quad x \in X.$$

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Theorem 6.4. Let \( j \in \{-1, 1\} \) be fixed, and let \( \psi_d : X^3 \to [0, \infty) \) be a mapping such that there exists an \( L < 1 \) with \( \psi_d(3^j x, 3^j y, 3^j z) \leq 81^j L \psi_d(x, y, z) \) for all \( x, y, z \in X \). Let a mapping \( f : X \to Y \) with \( f(0) = 0 \) satisfies the inequality (4.1) for all \( x, y, z \in X \). Then there exists a unique quartic mapping \( D : X \to Y \) such that
\[
||f(2x) - 4f(x) - D(x)||_Y \leq \frac{1}{81^j - 1 - L^j} \tilde{\psi}_d(x)
\] (6.2)
for all \( x \in X \), where
\[
\tilde{\psi}_d(x) = K [M_d(x, x, x) + \left( \frac{1}{2} \right)^{\beta} M_d(x, 0, x) K + M_d(x, 0, 0) K^2 + \left( \frac{1}{6} \right)^{\beta} M_d(0, 0, x) K^2]
\]
for all \( x \in X \).

Corollary 6.5. Let \( \nu \geq 0 \) and \( r, s, t \) which are all \( > 0 \) be real numbers such that \( r, s, t \) are all \( \neq \frac{4 \beta}{\alpha} \). Suppose that a mapping \( f : X \to Y \) with \( f(0) = 0 \) satisfies the inequality (3.26) for all \( x, y, z \in X \). Then there exists a unique quartic mapping \( D : X \to Y \) such that
\[
||f(2x) - 4f(x) - D(x)||_Y \leq \begin{cases} 
\alpha_d(x), & r > 0, s = 0, t = 0; \\
\beta_d(x), & r = 0, s > 0, t = 0; \\
\gamma_d(x), & r = 0, s > 0, t > 0; \\
\delta_d(x), & r > 0, s > 0, t > 0;
\end{cases}
\] (6.3)
for all \( x \in X \), where
\[
\alpha_d = K \nu \left\{ 1 + \left( \frac{1}{2} \right)^{\beta} K + \left[ 1 + \left( \frac{1}{2} \right)^{\beta} \right] K^2 \right\},
\]
\[
\beta_d = K \nu \left\{ \frac{3^\nu}{81^\beta} \left[ 1 + \left( \frac{1}{2} \right)^{\beta} K + K^2 \right] \right\} \|x\|^2_X,
\]
\[
\gamma_d = K \nu \left\{ \frac{3^\nu}{81^\beta} \left[ 1 + \left( \frac{1}{2} \right)^{\beta} \right] \|x\|^2_X \right\},
\]
\[
\delta_d = K \nu \left\{ \frac{3^\nu}{81^\beta} \left[ 1 + \left( \frac{1}{2} \right)^{\beta} K + \left( \frac{1}{2} \right)^{\beta} K^2 \right] \right\} \|x\|^2_X
\]
and
\[
\zeta_d = \beta_d(x) + \gamma_d(x) + \delta_d(x) \quad \text{for all} \ x \in X.
\]

Corollary 6.6. Let \( \nu \geq 0 \) and \( r, s, t \) which are all \( > 0 \) be real numbers such that \( \lambda = r + s + t \neq \frac{4 \beta}{\alpha} \). Suppose that a mapping \( f : X \to Y \) with \( f(0) = 0 \) satisfies the inequality (3.28) for all \( x, y, z \in X \). Then there exists a unique quartic mapping \( D : X \to Y \) such that
\[
||f(2x) - 4f(x) - B(x)||_Y \leq \begin{cases} 
\rho_d(x), & r > 0, s = 0, t = 0; \\
\tau_d(x), & r > 0, s > 0, t > 0;
\end{cases}
\] (6.4)
for all \( x \in X \), where
\[
\rho_d = K \nu \left\{ \frac{3^{2 \nu}}{81^\beta} \left[ 1 \right] \|x\|^2_X \right\} \quad \text{and}
\]
\[
\tau_d = K \nu \left\{ \frac{4 + 2 \left( \frac{1}{2} \right)^{\beta} K + K^2 \left[ 1 + \left( \frac{1}{2} \right)^{\beta} \right]}{81^\beta - 3^{\nu} \|x\|^2_X} \right\} \quad \text{for all} \ x \in X.
\]

Theorem 6.7. Let \( j \in \{-1, 1\} \) be fixed, and let \( \psi : X^3 \to [0, \infty) \) be a mapping such that there exists an \( L < 1 \) with \( \psi(3^j x, 3^j y, 3^j z) \leq 81^j L \psi(x, y, z) \) and \( \psi(3^j x, 3^j y, 3^j z) \leq 81^j L \psi(x, y, z) \) for all \( x, y, z \in X \). Let a mapping \( f : X \to Y \) with \( f(0) = 0 \) satisfies the inequality (5.1) for all \( x, y, z \in X \). Then there exists a quadratic mapping \( B : X \to Y \) and a quartic mapping \( D : X \to Y \) such that
\[
||f(x) - B(x) - D(x)||_Y \leq \frac{K}{12 - 81^j \|x\|^2_X} \left[ 9^\beta \tilde{\psi}_d(x) + \tilde{\psi}_d(x) \right]
\] (6.5)
for all \( x \in X \), where \( \tilde{\psi}_d(x) \) and \( \tilde{\psi}_d(x) \) are defined as in Theorems 6.1 and 6.4 respectively.
Theorem 6.8. Let $j \in \{-1, 1\}$ be fixed, and let $\psi : X^3 \to [0, \infty)$ be a mapping such that there exists an $L < 1$ with $\psi(3x, 3y, 3z) \leq 9^j L \psi(x, y, z)$ and $\psi(3x, 3y, 3z) \leq 81^{-j} L \psi(x, y, z)$ for all $x, y, z \in X$. Let a mapping $f : X \to Y$ with $f(0) = 0$ satisfies the inequality (5.1) for all $x, y, z \in X$. Then there exists a quadratic mapping $B : X \to Y$ and a quartic mapping $D : X \to Y$ such that

$$
\|f(x) - B(x) - D(x)\|_Y \leq \frac{K}{12 \cdot 81^j |1 - L|^j} [9^j \bar{\psi}_b(x) + \bar{\psi}_d(x)]
$$

for all $x \in X$, where $\bar{\psi}_b(x)$ and $\bar{\psi}_d(x)$ are defined as in Theorems 6.1 and 6.4 respectively.

Corollary 6.9. Let $\nu \geq 0$ and $r, s, t$ which are all $> 0$ be real numbers such that $r, s$ and $t$ are all $\neq \frac{2\beta}{\alpha}$ and $\frac{2\beta}{\delta}$. Suppose that a mapping $f : X \to Y$ with $f(0) = 0$ satisfies the inequality (3.26) for all $x, y, z \in X$. Then there exists a quadratic mapping $B : X \to Y$ and a quartic mapping $D : X \to Y$ such that

$$
\|f(x) - B(x) - D(x)\|_Y \leq \frac{K}{12} \begin{cases} \alpha_b + \alpha_d, & r > 0, s = 0, t = 0; \\
\beta_b(x) + \beta_d(x), & r = 0, s > 0, t = 0; \\
\gamma_b(x) + \gamma_d(x), & r = 0, s > 0, t > 0; \\
\delta_b(x) + \delta_d(x), & r = 0, s = 0, t > 0; \\
\zeta_b(x) + \zeta_d(x), & r > 0, s > 0, t > 0; \end{cases}
$$

for all $x \in X$, where $\alpha_b, \alpha_d, \beta_b(x), \beta_d(x), \gamma_b(x), \gamma_d(x), \delta_b(x), \delta_d(x), \zeta_b(x)$ and $\zeta_d(x)$ are defined as in Corollaries 6.2 and 6.5

Corollary 6.10. Let $\nu \geq 0$ and $r, s, t$ which are all $> 0$ be real numbers such that $\lambda = r + s + t \neq \frac{2\beta}{\alpha}$ and $\frac{2\beta}{\delta}$. Suppose that a mapping $f : X \to Y$ with $f(0) = 0$ satisfies the inequality (3.28) for all $x, y, z \in X$. Then there exists a quadratic mapping $B : X \to Y$ and a quartic mapping $D : X \to Y$ such that

$$
\|f(x) - B(x) - D(x)\|_Y \leq \frac{K}{12} \begin{cases} \rho_b(x) + \rho_d(x), & r > 0, s = 0, t = 0; \\
\tau_b(x) + \tau_d(x), & r = 0, s > 0, t = 0; \\
\nu \psi_b(x) = K \left[M_b(x, x, x) + \left(\frac{1}{4}\right)^\beta KM_b(2x, 0, 0) + 3^{\beta/2} K M_b(x, 0, x) + K^2 M_b(0, 0, x)\right] \text{ for all } x \in X.
$$

Theorem 6.11. Let $j \in \{-1, 1\}$ be fixed, and let $\psi : X^3 \to [0, \infty)$ be a mapping such that there exists an $L < 1$ with $\psi(2^j x, 2^j y, 2^j z) \leq 4^j L \psi(x, y, z)$ for all $x, y, z \in X$. Let a mapping $f : X \to Y$ with $f(0) = 0$ satisfies the inequality (3.12) for all $x, y, z \in X$. Then there exists a unique quadratic mapping $B : X \to Y$ such that

$$
\|f(2x) - 16f(x) - B(x)\|_Y \leq \frac{1}{4^j |1 - L|} \nu \psi_b(x)
$$

for all $x \in X$, where

$$
\nu \psi_b(x) = K \left[M_b(x, x, x) + \left(\frac{1}{4}\right)^\beta KM_b(2x, 0, 0) + 3^{\beta/2} K M_b(x, 0, x) + K^2 M_b(0, 0, x)\right] \text{ for all } x \in X.
$$

Corollary 6.12. Let $\nu \geq 0$ and $r, s, t$ which are all $> 0$ be real numbers such that $r, s$ and $t$ are all $\neq \frac{2\beta}{\alpha}$. Suppose that a mapping $f : X \to Y$ with $f(0) = 0$ satisfies the inequality (3.26) for all $x, y, z \in X$. Then there exists a unique quadratic mapping $B : X \to Y$ such that

$$
\|f(2x) - 16f(x) - B(x)\|_Y \leq \begin{cases} \alpha_b, & r > 0, s = 0, t = 0; \\
\beta_b(x), & r = 0, s > 0, t = 0; \\
\gamma_b(x), & r = 0, s > 0, t > 0; \\
\delta_b(x), & r > 0, s = 0, t > 0; \\
\zeta_b(x), & r > 0, s > 0, t > 0; \end{cases}
$$

for all $x \in X$, where

$$
\alpha_b = K \nu \left\{ \frac{1 + \left(\frac{1}{4}\right)^\beta K + 3^{\beta/2} K^2 + K^2}{4^j - 1} \right\},
$$

$$
\beta_b(x) = K \nu \left\{ \left(\frac{1}{4}\right)^\beta K + 3^{\beta/2} K^2 \right\} \|x\|_X,
$$

$$
\gamma_b(x) = K \nu \left\{ \frac{1}{4^j - 2^{2\nu}} \right\} \|x\|_X,
$$

$$
\delta_b(x) = K \nu \left\{ \frac{1 + 3^{\beta/2} K^2 + K^2}{4^j - 2^{2\nu}} \right\} \|x\|_X \text{ and}
$$

$$
\zeta_b(x) = \beta_b(x) + \gamma_b(x) + \delta_b(x) \text{ for all } x \in X.
$$
Corollary 6.13. Let \( \nu \geq 0 \) and \( r,s,t \) which are all \( > 0 \) be real numbers such that \( \lambda = r + s + t \neq \frac{4\beta}{\alpha} \).

Suppose that a mapping \( f : X \to Y \) with \( f(0) = 0 \) satisfies the inequality \((3.28)\) for all \( x,y,z \in X \). Then there exists a unique quadratic mapping \( B : X \to Y \) such that

\[
\| f(2x) - 16f(x) - B(x) \|_Y \leq \begin{cases} 
\rho_b(x), \\
\tau_b(x)
\end{cases}
\]

for all \( x \in X \), where

\[
\rho_b(x) = K\nu \left( \frac{2\nu\lambda}{4^\beta} \right) \left( \frac{1}{|4^\beta - 2\nu\lambda|} \right) \|x\|_X^\lambda \quad \text{and}
\]

\[
\tau_b(x) = K\nu \left( \frac{2\nu\lambda}{4^\beta} \right) \left( \frac{4 + 2\nu\lambda \left( \frac{1}{4^\beta} \right) K + 2 \cdot 3^\beta K^2 + K^2}{|4^\beta - 2\nu\lambda|} \right) \|x\|_X^\lambda
\]

for all \( x \in X \).

Theorem 6.14. Let \( j \in \{-1,1\} \) be fixed, and let \( \psi_d : X^3 \to [0,\infty) \) be a mapping such that there exists an \( L < 1 \) with \( \psi_d(2x,2y,2z) \leq 16^{j\beta} L \psi_d(x,y,z) \) for all \( x,y,z \in X \). Let a mapping \( f : X \to Y \) with \( f(0) = 0 \) satisfies the inequality \((4.1)\) for all \( x,y,z \in X \). Then there exists a unique quartic mapping \( D : X \to Y \) such that

\[
\| f(2x) - 4f(x) - D(x) \|_Y \leq \frac{1}{16^j} \| x \|_Y \psi_d(x)
\]

for all \( x \in X \), where

\[
\psi_d(x) = K[M_d(x,x,x) + \left( \frac{1}{4} \right)^j K M_d(2x,0,0) + 3^\beta K^2 M_d(x,0,x) + K^2 M_d(0,0,0)]
\]

for all \( x \in X \).

Corollary 6.15. Let \( \nu \geq 0 \) and \( r,s,t \) which are all \( > 0 \) be real numbers such that \( r,s \) and \( t \) are all \( \neq \frac{4\beta}{\alpha} \).

Suppose that a mapping \( f : X \to Y \) with \( f(0) = 0 \) satisfies the inequality \((3.26)\) for all \( x,y,z \in X \). Then there exists a unique quartic mapping \( D : X \to Y \) such that

\[
\| f(2x) - 4f(x) - D(x) \|_Y \leq \begin{cases} 
\alpha_d(x), \\
\beta_d(x), \\
\gamma_d(x), \\
\delta_d(x), \\
\zeta_d(x)
\end{cases}
\]

for all \( x \in X \), where

\[
\alpha_d = K\nu \left( \frac{1 + \left( \frac{1}{4} \right)^j K + 3^\beta K^2 + K^2}{|16^{\beta} - 1|} \right),
\]

\[
\beta_d(x) = K\nu \left( \frac{2^{2\nu}}{16^\beta} \right) \left( \frac{1 + 2^{2\nu} \left( \frac{1}{4^\beta} \right) K + 3^\beta K^2 + K^2}{|16^\beta - 2^{2\nu}|} \right) \|x\|_Y^\lambda,
\]

\[
\gamma_d(x) = K\nu \left( \frac{2^{2\nu}}{16^\beta} \right) \left( \frac{1}{|16^\beta - 2^{2\nu}|} \right) \|x\|_X^s,
\]

\[
\delta_d(x) = K\nu \left( \frac{2^{2\nu}}{16^\beta} \right) \left( \frac{1 + 3^\beta K^2 + K^2}{|16^\beta - 2^{2\nu}|} \right) \|x\|_X^s
\]

and

\[
\zeta_d(x) = \beta_d(x) + \gamma_d(x) + \delta_d(x)
\]

for all \( x \in X \).

Corollary 6.16. Let \( \nu \geq 0 \) and \( r,s,t \) which are all \( > 0 \) be real numbers such that \( \lambda = r + s + t \neq \frac{4\beta}{\alpha} \).

Suppose that a mapping \( f : X \to Y \) with \( f(0) = 0 \) satisfies the inequality \((3.28)\) for all \( x,y,z \in X \). Then there exists a unique quartic mapping \( D : X \to Y \) such that

\[
\| f(2x) - 4f(x) - D(x) \|_Y \leq \begin{cases} 
\rho_d(x), \\
\tau_d(x)
\end{cases}
\]

for all \( x \in X \), where

\[
\rho_d(x) = K\nu \left( \frac{2\nu\lambda}{16^\beta} \right) \left( \frac{1}{|16^\beta - 2\nu\lambda|} \right) \|x\|_X^\lambda
\]

and

\[
\tau_d(x) = K\nu \left( \frac{2\nu\lambda}{16^\beta} \right) \left( \frac{4 + 2\nu\lambda \left( \frac{1}{4^\beta} \right) K + 2 \cdot 3^\beta K^2 + K^2}{|16^\beta - 2\nu\lambda|} \right) \|x\|_X^\lambda
\]

for all \( x \in X \).
Theorem 6.17. Let \( j \in \{-1, 1\} \) be fixed, and let \( \psi : X^3 \to [0, \infty) \) be a mapping such that there exists an \( L < 1 \) with \( \psi(2\lambda x, 2\lambda y, 2\lambda z) \leq 4^{j\lambda} L\psi(x, y, z) \) and \( \psi(2^{j\lambda} x, 2\lambda y, 2\lambda z) \leq 16^{j\lambda} L\psi(x, y, z) \) for all \( x, y, z \in X \).

Let a mapping \( f : X \to Y \) with \( f(0) = 0 \) satisfies the inequality (5.1) for all \( x, y, z \in X \). Then there exists a quadratic mapping \( B : X \to Y \) and a quartic mapping \( D : X \to Y \) such that

\[
\|f(x) - B(x) - D(x)\|_Y \leq \frac{K}{12 \cdot 16^0 |1 - L^j|} [4^{j\lambda} \tilde{b}_B(x) + \tilde{b}_D(x)]
\]

(6.12)

for all \( x \in X \), where \( \tilde{b}_B(x) \) and \( \tilde{b}_D(x) \) are defined as in Theorems 6.11 and 6.14 respectively.

Theorem 6.18. Let \( j \in \{-1, 1\} \) be fixed, and let \( \psi : X^3 \to [0, \infty) \) be a mapping such that there exists an \( L < 1 \) with \( \psi(2\lambda x, 2\lambda y, 2\lambda z) \leq 4^{j\lambda} L\psi(x, y, z) \) and \( \psi(2^{j\lambda} x, 2\lambda y, 2\lambda z) \leq 8^{j\lambda} L\psi(x, y, z) \) for all \( x, y, z \in X \).

Let a mapping \( f : X \to Y \) with \( f(0) = 0 \) satisfies the inequality (5.1) for all \( x, y, z \in X \). Then there exists a quadratic mapping \( B : X \to Y \) and a quartic mapping \( D : X \to Y \) such that

\[
\|f(x) - B(x) - D(x)\|_Y \leq \frac{K}{12 \cdot 16^0 |1 - L^j|} [4^{j\lambda} \tilde{b}_B(x) + \tilde{b}_D(x)]
\]

(6.13)

for all \( x \in X \), where \( \tilde{b}_B(x) \) and \( \tilde{b}_D(x) \) are defined as in Theorems 6.11 and 6.14 respectively.

Corollary 6.19. Let \( \nu \geq 0 \) and \( r, s, t \) which are all \( > 0 \) be real numbers such that \( r, s, t \) are all \( \neq \frac{2\beta}{\alpha} \) and \( \frac{4\beta}{\alpha} \). Suppose that a mapping \( f : X \to Y \) with \( f(0) = 0 \) satisfies the inequality (3.26) for all \( x, y, z \in X \). Then there exists a quadratic mapping \( B : X \to Y \) and a quartic mapping \( D : X \to Y \) such that

\[
\|f(x) - B(x) - D(x)\|_Y \leq \frac{K}{12} \left\{ \begin{array}{l}
\alpha_d + \alpha_d, \\
\beta_d(x) + \beta_d(x), \\
\gamma_d(x) + \gamma_d(x), \\
\delta_d(x) + \delta_d(x), \\
\zeta_d(x) + \zeta_d(x) \\
\end{array} \right\}
\]

(6.14)

for all \( x \in X \), where \( \alpha_d, \beta_d(x), \gamma_d(x), \delta_d(x), \zeta_d(x) \) and \( \zeta_d(x) \) are defined as in Corollaries 6.12 and 6.15

Corollary 6.20. Let \( \nu \geq 0 \) and \( r, s, t \) which are all \( > 0 \) be real numbers such that \( \lambda = r + s + t \neq \frac{2\beta}{\alpha} \) and \( \frac{4\beta}{\alpha} \). Suppose that a mapping \( f : X \to Y \) with \( f(0) = 0 \) satisfies the inequality (3.28) for all \( x, y, z \in X \). Then there exists a unique quartic mapping \( B : X \to Y \) satisfying

\[
\|f(x) - B(x) - D(x)\|_Y \leq \frac{K}{12} \left\{ \begin{array}{l}
\rho_d(x), \\
\tau_d(x) + \tau_d(x) \\
\end{array} \right\}
\]

(6.14)

for all \( x \in X \), where \( \rho_d(x), \tau_d(x) \) are defined as in Corollaries 6.13 and 6.16

References


