Prime Labeling of Duplication of Some Star related Graphs

S. Meena¹, P. Kavitha²

¹Department of Mathematics, Government Arts College, C-Muttar, Chidambaram– 608 102, Tamil Nadu, India.
²Department of Mathematics, S.R.M University, Chennai– 603 203, Tamil Nadu, India.

Abstract:
A graph $G = (V, E)$ with $n$ vertices is said to admit prime labeling if its vertices can be labeled with distinct positive integers not exceeding $n$ such that the label of each pair of adjacent vertices are relatively prime. A graph $G$ which admits prime labeling is called a prime graph. In this paper we investigate prime labeling for some star related to graph. We also discuss prime labeling in the context of duplication of graph elements.

Keywords: Graph Labeling, Prime Labeling, Prime Graph.

1.Introduction:
We begin with finite, undirected and non trivial graph $G = (V(G), E(G))$ with vertex set $V(G)$ and edge set $E(G)$. The elements of $V(G)$ and $E(G)$ are commonly termed as graph elements. Throughout this work, the Subdivision of Star $S_{1,n}$ is obtained from $K_{1,n}$ by joining $n$ pendant edges with $K_{1,n}$ that is a graph with a vertex of $n$ degree called apex, $n$ vertices of degree 2 and $n$ vertices of degree one called pendant vertices. Throughout this paper $|V(G)|$ and $|E(G)|$ denote the cardinality of the vertex set and edge set respectively. For various graph theoretic notation and terminology we refer to Bondy and Murthy [1]. We give brief summary of definitions and other information which are useful for the present investigation.

Definition 1.1:
If the vertices of the graph are assigned values subject to certain condition(s) then it is known as graph labeling.

Definition 1.2:
A prime labeling of a graph $G$ is an injective function $f : V(G) \rightarrow \{1, 2, \ldots, |V(G)|\}$ such that for every pair of adjacent vertices $u$ and $v$, $\gcd(f(u), f(v)) = 1$. The graph which admits a prime labeling is called a prime graph.

The notion of a prime labeling was originated by Entringer and it was discussed by Tout et al [8]. Fu and Huang [3] proved that $P_n$ and $K_{1,n}$ are prime graphs. Lee et al [5] proved that $W_n$ is a prime graph if and only if $n$ is even. Deretsky et al [2] proved that $C_n$ is a prime graph. Vaidya and Prajapati [9] discussed prime labeling in the context of duplication of graph elements. Meena and Vaithilingam [7] have investigated existence of the prime labeling for some crown related graph. In [6] Meena and Kavitha proved the prime labeling for some butterfly related graphs. A variant of prime labeling known as vertex – edge prime labeling is also introduced by Venkatachalam and Antoni Raj in [10]. For latest Dynamic survey on graph labeling we refer to [4] (Gallian J.A., 2009). Vast amount of literature is available on different types of graph labeling. More than 1000 research papers have been published so far in last four decades.

Definition 1.3:
Duplication of a vertex $v$ of a graph $G$ produces a new graph $G'$ by adding a new vertex $v'$ such that $N(v') = N(v)$. In other words a vertex $v$ is said to be duplication of $v$ if all the vertices which are adjacent to $v$ in $G$ are also adjacent to $v'$ in $G'$.

Definition 1.4:
Duplication of a vertex $v_k$ by a new edge $e = v_kv_k'$ in a graph $G$ produces a new graph $G'$ such that $N(v_k') = \{v_k, v_k'\}$ and $N(v_k) = \{v_k, v_k'\}$.

Definition 1.5:
Duplication of an edge $e = uv$ by a new vertex $w$ in a graph $G$ produces a new graph $G'$ such that $N(w) = \{u, v\}$.
**Definition 1.6:**
Duplication of an edge $e=uv$ of a graph $G$ produces a new graph $G'$ by adding an edge $e'=u'v'$ such that $N(u') = N(u) \cup (v') - \{v\}$ and $N(v') = N(v) \cup (u') - \{u\}$.

In this paper, we investigate prime labeling for some graphs obtained by duplication of graph elements and also we derive some result for subdivision of star $S_{1,n}$ in this context.

**2. Main Results:**

**Theorem 2.1:**
The graph obtained by duplication of all vertices by an edges in subdivision of star $S_{1,n}$ is not a prime graph.

**Proof:**
Let $v_0$ be the apex vertex $v_1, v_2, ..., v_n$ be consecutive vertices of degree 2 and $u_1, u_2, ..., u_k$ be the consecutive pendant vertices of subdivision of star. Let $G$ be the graph obtained by duplication of each and every vertex $v_0, v_i, u_j$ by each of the edges $v_0v_0', u_iu_i', v_i'v_i'$ for $1 \leq i \leq n$. Then $G$ is a graph with $6n+3$ vertices and having $2n+1$ vertex disjoint cycles each of length three. Any prime labeling of $G$ must contains at the most one even label in each of these $2n+1$ cycles as it is not possible to assign even labels for two adjacent vertices. Consequently at the most $2n+1$ vertices will receive even labels out of $6n+3$ vertices. Hence the number of even integers which are left to be used as vertex labels is at least 

$$\left\lceil \frac{6n+3}{2} \right\rceil - (2n+1) = \left\lceil \frac{6n+3}{2} - (2n+1) \right\rceil = \left\lceil \frac{2n+1}{2} \right\rceil \geq 1$$

That means at least one even integer from $\{1, 2, ..., 6n+3\}$ is left for label assignment. This labeling is not a prime labeling. Hence $G$ is not a prime graph.

**Theorem 2.2:**
The graph obtained by duplication of a vertex in subdivision of star $S_{1,n}$ is a prime graph.

**Proof:**
Let $v_0$ be the apex vertex $v_1, v_2, ..., v_n$ be consecutive vertices of degree 2 and $u_1, u_2, ..., u_k$ be the consecutive pendant vertices of subdivision of star. Let $G$ be the graph obtained by duplication of vertex $v$ in a subdivision of star by a vertex $v'$. We consider the following cases depending on degree of $v$:

**Case (i):** If $\deg(v) = n$ then $v = v_0$. Let $v_0'$ be the duplication of $v_0$ in $G$ then define $f : V(G) \rightarrow \{1, 2, ..., 2n+1, 2n+2\}$ as

$$f(v) = \begin{cases} 2i + 1 & \text{if } v = v_i \text{ for } i = 0, 1, 2, 3, ..., n; \\ 2i & \text{if } v = v_0. \end{cases}$$

then $f$ is an injection and it is a prime labeling for $G$.

**Case (ii):** If $\deg(v) = 2$ then $v = v_k$ and let $v_k'$ be the duplication of $v_k$ in $G$ then define $f : V(G) \rightarrow \{1, 2, ..., 2n+1, 2n+2\}$ as

$$f(v) = \begin{cases} 1 & \text{if } v = v_0; \\ 2i & \text{if } v = v_i \text{ for } 1 \leq i \leq k; \\ 2i + 1 & \text{if } v = u_j \text{ for } 1 \leq j \leq n; \\ 2i + 2 & \text{if } v = v_i \text{ for } k + 1 \leq i \leq n; \\ 2k + 2 & \text{if } v = v_k. \end{cases}$$

Then $f$ is an injection and it is a prime labeling for $G$.

**Case (iii):** If $\deg(v) = 1$ then $v = u_k$ and let $u_k'$ be the duplication of $u_k$ in $G$ where $k = 1, 2, ..., n$ then define $f : V(G) \rightarrow \{1, 2, ..., 2n+1, 2n+2\}$ as

$$f(v) = \begin{cases} 2i + 1 & \text{if } v = v_i \text{ for } i = 1, 2, ..., k, ..., n; \\ 2i + 2 & \text{if } v = u_j \text{ for } i = 1, 2, ..., k, ..., n; \\ 2 & \text{if } v = v_k. \end{cases}$$

Then $f$ is an injection and it is a prime labeling for $G$.

Thus from all cases described above $G$ is a prime graph.

![Fig. 1 A prime labeling of a graph obtained by duplication of the apex vertex in $S_{1,5}$](http://www.ijmttjournal.org)
Theorem 2.3:
The graph obtained by duplicating all the pendant vertices in subdivision of star $S_{1,n}$ is a prime graph.

Proof:
Let $v_0$ be the apex vertex $v_1,v_2,...,v_n$ be consecutive vertices of degree 2 and $u_1,u_2,...,u_n$ be the consecutive pendant vertices of subdivision of star. Let $G$ be the graph obtained by duplicating all the vertices of degree 2 in subdivision of star and let the new vertices $v'_1,v'_2,...,v'_n$. Then the vertex set $V(G) = \{v_0,v_1,v_2,...,v_n,u_1,u_2,...,u_n,v'_1,v'_2,...,v'_n\}$ and edge set $E(G) = \{v_0v_i,v_iv_{i+1},v_iu_i,uv_i,1 \leq i \leq n\}$. Here $|V(G)|=3n+1$, $|E(G)|=3n$. Then define a labeling $f : V(G) \rightarrow \{1,2,3,...3n+1\}$ as

$$f(v) = \begin{cases} 1 & \text{if } v = v_0; \\ 3i & \text{if } v = v_i \text{ for } i = 1,2,...,n; \\ 3i - 1 & \text{if } v = u_i \text{ for } i = 1,2,...,n; \\ 3i + 1 & \text{if } v = u'_i \text{ for } i = 1,2,...,n, \end{cases}$$

then $f$ is an injection and it is a prime labeling for $G$. Hence $G$ is a prime graph.

Theorem 2.4:
The graph obtained by duplicating all the vertices of degree 2 in subdivision of star $S_{1,n}$ is a prime graph.

Proof:
Let $v_0$ be the apex vertex $v_1,v_2,...,v_n$ be consecutive vertices of degree 2 and $u_1,u_2,...,u_n$ be the consecutive pendant vertices of subdivision of star. Let $G$ be the graph obtained by duplicating all the vertices of degree 2 in subdivision of star and let the new vertices $v'_1,v'_2,...,v'_n$. Then the vertex set $V(G) = \{v_0,v_1,v_2,...,v_n,u_1,u_2,...,u_n,v'_1,v'_2,...,v'_n\}$ and edge set $E(G) = \{v_0v'_i,v'_iv_{i+1},v_iu_i,v_iu'_i,1 \leq i \leq n\}$. Here $|V(G)|=3n+1$, $|E(G)|=4n$. Then define a labeling $f : V(G) \rightarrow \{1,2,3,...4n+1\}$ as

$$f(v) = \begin{cases} 1 & \text{if } v = v_0; \\ 3i + 1 & \text{if } v = v'_i \text{ for } i = 1,2,...,n; \\ 3i & \text{if } v = u_i \text{ for } i = 1,2,...,n; \\ 3i - 1 & \text{if } v = u'_i \text{ for } i = 1,2,...,n, \end{cases}$$

then $f$ is an injection and it is a prime labeling for $G$. Hence $G$ is a prime graph.

Theorem 2.5:
The graph obtained by duplicating all the vertices of the subdivision of star $S_{1,n}$, except the apex vertex is a prime graph.

Proof:
Let $v_0$ be the apex vertex $v_1,v_2,...,v_n$ be the consecutive vertices of degree 2 and $u_1,u_2,...,u_n$ be the consecutive pendant vertices of subdivision of star. Let $G$ be the graph obtained by duplicating all the vertices in subdivision of star, except the apex vertex $v_0$. Now let $u_1,u_2,...,u_n$ and $v_1,v_2,...,v_n$ be the new vertices of $G$ by duplicating $u_1,u_2,...,u_n$ and $v_1,v_2,...,v_n$ then the vertex set $V(G) = \{v_0,u_1,u_2,...,u_n,v_1,v_2,...,v_n\}$ and edge set $E(G) = \{v_0v'_i,v'_iv_{i+1},v_iu_i,v_iu'_i,1 \leq i \leq n\}$. Here $|V(G)|=4n+1$, $|E(G)|=5n$. Then define a labeling $f : V(G) \rightarrow \{1,2,3,...4n+1\}$ as

$$f(v) = \begin{cases} 1 & \text{if } v = v_0; \\ 4i - 2 & \text{if } v = v'_i \text{ for } 1 \leq i \leq n; \\ 4i - 1 & \text{if } v = u_i \text{ for } 1 \leq i \leq n; \\ 4i & \text{if } v = u'_i \text{ for } 1 \leq i \leq n; \end{cases}$$

then $f$ is an injection and it is a prime labeling for $G$. Hence $G$ is a prime graph.
**Theorem 2.6:**
The graph obtained by duplicating all the vertices of the subdivision of star $S_{1,n}$ is a prime graph.

**Proof:**
Let $v_0$ be the apex vertex $v_1,v_2,...,v_n$ be consecutive vertices of degree 2 and $u_1,u_2,...,u_n$ be the consecutive pendant vertices of subdivision of star. Let $G$ be the graph obtained by duplicating all the vertices of the subdivision of star. Now let the new vertices $v_0',v_1',v_2',...,v_n'$ and $u_1',u_2',...,u_n'$ be the new vertices of $G$ by duplicating $v_1,v_2,...,v_n$ and $u_1,u_2,...,u_n$ respectively, then define a labeling $f : V(G) \to \{1,2,3,...,4n+2\}$ as

$$f(v) = \begin{cases} 
1 & \text{if } v = v_0'; \\
2 & \text{if } v = v_i'; \\
4i+1 & \text{if } v = v_i \text{ for } 1 \leq i \leq n; \\
4i-1 & \text{if } v = v_i \text{ for } 1 \leq i \leq n; \\
4i & \text{if } v = u_i \text{ for } 1 \leq i \leq n; \\
4i+2 & \text{if } v = u_i \text{ for } 1 \leq i \leq n,
\end{cases}$$

then $f$ is an injection and it is a prime labeling for $G$. Hence $G$ is a prime graph.

**Theorem 2.7:**
The graph obtained by duplication of the vertex by an edge in subdivision of star $S_{1,n}$ is a prime graph.

**Proof:**
Let $v_0$ be the apex vertex $v_1,v_2,...,v_n$ be consecutive vertices of degree 2 and $u_1,u_2,...,u_n$ be the consecutive pendant vertices of subdivision of star. Let $G$ be the graph obtained by duplication of a vertex $v_i$ by an edge $v_iv_i'$. Here $|V(G)| = 2n+3$, $|E(G)| = 2n+3$. We consider the following cases depending on degree of $v$:

**Case (i):** If $\deg(v_i) = n$ in subdivision of star then $v_i = v_0$. Now let $v_0,v_0'$ be the new vertices of $G$, then define a labeling $f : V(G) \to \{1,2,3,...,2n+3\}$ as

$$f(v) = \begin{cases} 
1 & \text{if } v = v_0; \\
2i+1 & \text{if } v = v_i \text{ for } i = 1,2,...,n; \\
2i+2 & \text{if } v = v_i \text{ for } i = 1,2,...,n; \\
2i+3 & \text{if } v = u_i \text{ for } i = 1,2,...,n; \\
2 & \text{if } v = v_0'; \\
3 & \text{if } v = v_i',
\end{cases}$$

then $f$ is an injection and it is a prime labeling for $G$.

**Case (ii):** If $\deg(v_i) = 2$ in subdivision of star then $v_i = v_j$. Let $v_j,v_j'$ be the new vertices of $G$, then define a labeling $f : V(G) \to \{1,2,3,...,2n+3\}$ as

$$f(v) = \begin{cases} 
1 & \text{if } v = v_0; \\
2i+4 & \text{if } v = v_i \text{ for } i = 1,2,...,n; \\
2i+2 & \text{if } v = v_i \text{ for } i = 1,2,...,n; \\
3 & \text{if } v = v_j; \\
5 & \text{if } v = v_j'; \\
4 & \text{if } v = u_k; \\
2i+5 & \text{if } v = u_j \text{ for } i = 1,2,...,n; \\
2i+3 & \text{if } v = u_i \text{ for } i = 1,2,...,n.
\end{cases}$$

then $f$ is an injection and it is a prime labeling for $G$.

**Fig. 4** A prime labeling of a graph obtained by duplication of a vertex by an edge in $S_{1,6}$

**Fig. 5** A prime labeling of a graph obtained by duplication of a vertex of degree 2 by an edge in $S_{1,3}$
**Theorem 2.8:**
The graph obtained by duplication of an edge by a vertex in subdivision of star $S_{1,n}$ is a prime graph.

**Proof:**
Let $v_0$ be the apex vertex $v_1, v_2, ..., v_n$ be consecutive vertices of degree 2 and $u_1, u_2, ..., u_n$ be the consecutive pendent vertices of subdivision of star. Let $G$ be the graph obtained by duplication of an edge by a vertex. Here $|V(G)| = 2n + 2, |E(G)| = 2n + 2$ and we consider two cases.

**Case (i):** The duplication of an edge $v_0v_k$ by a vertex $v_i$ for $1 \leq k \leq n$ in subdivision of star, then define

$$f : V(G) \rightarrow \{1, 2, 3, ..., 2n + 2\}$$

as

$$f(v) =
\begin{cases}
1 & \text{if } v = v_0; \\
2 & \text{if } v = v_k; \\
2i + 1 & \text{if } v = v_i \text{ for } i = 1, 2, ..., n; \\
2i + 2 & \text{if } v = u_i \text{ for } i = 1, 2, ..., n,
\end{cases}$$

then $f$ is an injection and it is a prime labeling for $G$.

**Case (ii):** Here the duplication of an edge $v_0u_k$ in subdivision of star by a vertex $u_i$ for $1 \leq k \leq n$ then define a labeling $f_i$ using the labeling $f$ defined in case (i) as follows: $f_i(v_0) = f(v_0)$ and $f_i(v_k) = f(v_k)$ for $1 \leq k \leq n$ and $f_i(v) = f(v)$ for all the remaining vertices. Thus $f_i$ is a prime labeling. Hence from the above cases described above $G$ is a prime graph.

**Theorem 2.9:**
The graph obtained by duplication of an edge in subdivision of star $S_{1,n}$ is a prime graph.

**Proof:**
Let $v_0$ be the apex vertex $v_1, v_2, ..., v_n$ be consecutive vertices of degree 2 and $u_1, u_2, ..., u_n$ be the consecutive pendent vertices of subdivision of star. Let $G$ be the graph obtained by duplication of an edge $e$ by a new vertex $v_i$. Here $|V(G)| = 2n + 3$ then define a labeling $f : V(G) \rightarrow \{1, 2, 3, ..., 2n + 3\}$ by considering the following cases:

**Case (i):** If the edge $e$ is one of the pendent edges of subdivision of star say, $e = v_0u_i$ then $G$ can be thought as a graph with new $e' = v_ju_i'$ incident with $v_0v_i'$ and $v_ju_i'$, which is again a subdivision of star. Hence it is a prime graph as discussed [3].

**Case (ii):** Let $G$ be the graph obtained by duplication of an edge $e = v_iu_i$ by a new edge $e' = v_iu_i'$ which vertex $v_0$ is incident with...
Thus from all the cases described above $G$ is a prime graph.

\[ f(v) = \begin{cases} 
1 & \text{if } v = v_0; \\
2 & \text{if } v = v_i; \\
3 & \text{if } v = v_i \text{ for } i = 1, 2, \ldots, n; \\
2i + 3 & \text{if } v = v_i \text{ for } i = 1, 2, \ldots, n; \\
2i + 2 & \text{if } u = u_i \text{ for } i = 1, 2, \ldots, n.
\]

Then $f$ is an injection and it is a prime labeling for $G$. Thus from all the cases described above $G$ is a prime graph.

**Theorem 2.10:**
The graph obtained by duplication of every edge by a vertex in subdivision of star $S_{1,n}$ is a prime graph.

**Proof:**
Let $v_0$ be the apex vertex $v_1, v_2, \ldots, v_n$ be consecutive vertices of degree 2 and $u_1, u_2, \ldots, u_n$ be the consecutive pendant vertices of subdivision of star. Let $G$ be the graph obtained by duplication of every edges $v_0v_i$ by a vertex $v_i$ and $v_iu_i$ by a vertex $u_i$ then the vertex set $V(G) = \{v_0, v_1, v_2, \ldots, v_n, u_1, u_2, \ldots, u_n, v_1u_1, v_2u_2, \ldots, v_nu_n/1 \leq i \leq n\}$, and edge set $E(G) = \{v_0v_1, v_0v_2, \ldots, v_0v_n, v_1v_2, v_1v_3, \ldots, v_1v_n, v_2v_3, v_2v_4, \ldots, v_2v_n, \ldots, v_nv_{n-1}, v_nv_{n+1}, u_1u_2, u_1u_3, \ldots, u_1u_n, u_2u_3, u_2u_4, \ldots, u_2u_n, \ldots, u_{n-1}u_n, v_1u_1v_2, v_1u_1v_3, \ldots, v_1u_1v_n, v_2u_2v_3, v_2u_2v_4, \ldots, v_2u_2v_n, \ldots, v_nu_nv_{n+1}, u_1u_2u_3, u_1u_2u_4, \ldots, u_1u_2u_n, u_2u_3u_4, u_2u_3u_5, \ldots, u_2u_3u_n, \ldots, u_{n-1}u_nu_{n+1}/1 \leq i \leq n\}$.

Here $|V(G)| = 4n+1$ and $|E(G)| = 6n$, then define a labeling $f : V(G) \rightarrow \{1, 2, 3, \ldots, 6n+1\}$ as

\[ f(v) = \begin{cases} 
1 & \text{if } v = v_0; \\
3i & \text{if } v = v_i \text{ for } i = 1, 2, \ldots, n; \\
3i-1 & \text{if } v = v_i \text{ for } i = 1, 2, \ldots, n; \\
3i+1 & \text{if } v = u_i \text{ for } i = 1, 2, \ldots, n.
\]

then $f$ is an injection and it is a prime labeling for $G$. Thus from all the cases described above $G$ is a prime graph.

**Theorem 2.11:**
The graph obtained by duplication of the edges $v_0v_i$ by a vertex $v_i$ in subdivision of star $S_{1,n}$ is a prime graph.

**Proof:**
Let $v_0$ be the apex vertex $v_1, v_2, \ldots, v_n$ be consecutive vertices of degree 2 and $u_1, u_2, \ldots, u_n$ be the consecutive pendant vertices of subdivision of star. Let $G$ be the graph obtained by duplication of each of the edges $v_0v_i$ by a vertex $v_i$. Now the vertex set $V(G) = \{v_0, v_1, v_2, \ldots, v_n, v_0v_1, v_0v_2, \ldots, v_0v_n, v_1v_2, v_1v_3, \ldots, v_1v_n, v_2v_3, v_2v_4, \ldots, v_2v_n, \ldots, v_nv_{n-1}, v_nv_{n+1}, u_1u_2, u_1u_3, \ldots, u_1u_n, u_2u_3, u_2u_4, \ldots, u_2u_n, \ldots, u_{n-1}u_n, v_1u_1v_2, v_1u_1v_3, \ldots, v_1u_1v_n, v_2u_2v_3, v_2u_2v_4, \ldots, v_2u_2v_n, \ldots, v_nv_{n+1}/1 \leq i \leq n\}$.

Here $|V(G)| = 3n+1$ and $|E(G)| = 4n$ then define a labeling $f : V(G) \rightarrow \{1, 2, 3, \ldots, 3n+1\}$ as

\[ f(v) = \begin{cases} 
1 & \text{if } v = v_0; \\
3i & \text{if } v = v_i \text{ for } i = 1, 2, \ldots, n; \\
3i-1 & \text{if } v = v_i \text{ for } i = 1, 2, \ldots, n; \\
3i+1 & \text{if } v = u_i \text{ for } i = 1, 2, \ldots, n.
\]

then $f$ is an injection and it is a prime labeling for $G$. Thus from all the cases described above $G$ is a prime graph.

**Theorem 2.12:**
The graph obtained by duplication of the edges by a vertex which are incident with pendant vertices in subdivision of star $S_{1,n}$ is not a prime graph.

**Proof:**

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Fig. 9 A prime labeling of a graph obtained by duplication of an edge by a vertex in $S_{1,6}$

Fig. 10 A prime labeling of a graph obtained by duplication of each edge by a vertex in $S_{1,6}$

Fig. 11 A prime labeling of a graph obtained by duplication of the edges $v_0v_i$ by vertices $v_i$ in $S_{1,6}$

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Let \( v_0 \) be the apex vertex \( v_1, v_2, ..., v_n \) be consecutive vertices of degree 2 and \( u_1, u_2, ..., u_n \) be the consecutive pendant vertices of subdivision of star. Let \( G \) be the graph obtained by duplication of the edges \( v_i u_i \) by a vertex \( u_i \) in subdivision of star. Then \( G \) is a graph with \( 3n+1 \) vertices and having \( n \) vertex disjoint cycles each of length three. Any prime labeling of \( G \) most contains at the most one even label in each of these \( n \) cycles as it is not possible to assign even labels for two adjacent vertices. Consequently at the most \( n \) vertices will receive even label out of \( 3n+1 \) vertices. Hence the number of even integer which are left to be used as vertex labels are at least 
\[
\left\lfloor \frac{3n+1}{2} \right\rfloor - n = \left\lfloor \frac{3n+1}{2} - n \right\rfloor = \frac{n+1}{2} \geq 1.
\]
That means at least one even integer from \( \{1, 2, 3, ..., 3n+1\} \) is left for label assignment. This not possible as prime labeling is bijective. Hence \( G \) is not a prime graph.

**Theorem 2.13:**
The graph obtained by duplication of every pendant vertex by an edge in subdivision of star \( S_{1,n} \) is a prime graph.

**Proof:**
Let \( v_0 \) be the apex vertex \( v_1, v_2, ..., v_n \) be consecutive vertices of degree 2 and \( u_1, u_2, ..., u_n \) be the consecutive pendant vertices of subdivision of star. Let \( G \) be the graph obtained by duplication of every pendant vertex \( u_i \) by an edges \( u_i u_i' \) for \( 1 \leq i \leq n \), then the vertex set \( V(G) = \{v_0, v_i, u_i, u_i' \mid 1 \leq i \leq n \} \) and the edge set \( E(G) = \{v_0 v_i / 1 \leq i \leq n \} \cup \{v_i u_i, u_i u_i' \mid 1 \leq i \leq n \} \). Here \( |V(G)| = 4n+1 \) and \( |E(G)| = 5n \), then define a labeling \( f_i \) using the labeling \( f \) defined in above theorem as follows: \( f_i(u_i) = f(v_i) \), \( f_i(v_i') = f(u_i) \), \( f_i(v_i') = f(u_i') \), \( f_i(v_i') = f(u_i) \) for \( i = 1, 2, ..., n \), and \( f_i(v) = f(v) \) for the remaining vertices. Then the resulting labeling \( f_i \) is a prime labeling. Hence \( G \) is a prime graph.

**Theorem 2.14:**
The graph obtained by duplication of every vertex of degree 2 by an edge in subdivision of star \( S_{1,n} \) is a prime graph.

**Proof:**
Let \( v_0 \) be the apex vertex \( v_1, v_2, ..., v_n \) be consecutive vertices of degree 2 and \( u_1, u_2, ..., u_n \) be the consecutive pendant vertices of subdivision of star. Let \( G \) be the graph obtained by duplication of every vertex of degree 2 by an edge \( v_i' v_j' \), then the vertex set \( V(G) = \{v_0, v_i, u_i, v_j'/1 \leq i \leq n \} \) and edge set 
\[
E(G) = \{v_0 v_i / 1 \leq i \leq n \} \cup \{v_i u_i, v_i v_j'/1 \leq i \leq n \}
\]
Here \( |V(G)| = 4n+1 \) and \( |E(G)| = 5n \), then define a labeling \( f_i \) as follows: \( f_i(u_i) = f(v_i) \), \( f_i(v_i') = f(v_i) \), \( f_i(v_i') = f(u_i) \) for \( i = 1, 2, ..., n \), and \( f_i(v) = f(v) \) for the remaining vertices. Then the resulting labeling \( f_i \) is a prime labeling. Hence \( G \) is a prime graph.

**References**