Optimal EPQ Model with Weibullly Distributed Deterioration Rate and Time Varying IHC

Devani A. Chatterji*, U. B. Gothi*

#Assistant Professor, Department of Statistics, S. M. Patel Institute of Commerce, GLS University, Ahmedabad, Gujarat, India.

*Associate Professor, Department of Statistics, St. Xavier’s College(Autonomous), Ahmedabad, Gujarat, India.

Abstract - Many real life situations, the stock level of the inventoried items is continuously depleting due to the combined effects of its demand and deterioration. This paper develops an optimal lot-sizing EPQ model for Weibull deteriorated items with constant rate of demand and time-varying holding cost over a finite planning horizon. Specifically a 3-parameter Weibull distribution is used to represent the deterioration rate. Shortages are permitted and are completely backlogged. Numerical example along with sensitivity analysis is given to support the model.

Keywords - Inventory, Time-varying holding cost, 3-parameter Weibull deterioration, Fully backlogged shortages.

I. INTRODUCTION

One of the assumptions in the traditional inventory model is that the items preserved their physical characteristics while they are kept or stored in the inventory. This assumption is evidently true for most of the items, but not for all. In any company, the major problem a supplier confronts is the control and maintenance of inventories of deteriorating items. From the past one and a half decade, inventory problems for deteriorating items have been widely studied, as most physical goods deteriorate with time. Deterioration is defined as the decay, damage, vaporization, obsolescence, pilferage, loss of utility or loss of marginal value of the products that results in decrease in their original properties of the goods to satisfy the demand. Deterioration rate is almost nil in some of the items like hardware, glassware, toys, steel etc. but for the items like food grains, vegetables, fish, medicines, gasoline, alcohol, radioactive chemicals, fashion goods, electronic substances etc. have finite shelf life and deteriorate rapidly over time. Owing to this fact, controlling and maintaining the inventory of deteriorating items become a challenging problem for decision makers.

Goswami and Chaudhuri [8] developed an EOQ model for deteriorating items with shortages and linear trend in demand.


Recently, Gothi and Chatterji [9] have developed an inventory model for imperfect items under constant demand rate with time varying IHC. Parmar and Gothi [15] developed an EPQ model for deteriorating items using three parameter Weibull distribution with constant production rate and time varying holding cost. Parmar and Gothi [14] developed an order level inventory model for deteriorating items under quadratic demand with time dependent IHC. Gothi and Parmar [10] had also developed an order level inventory model for deteriorating items under quadratic demand with time dependent IHC and partial backlogging.

Kawale and Bansode [12] have developed an inventory model for time varying holding cost and weibull distribution for deterioration with fully backlogged shortages. In this paper, it is observed that the p.d.f. which is constant and so deterioration rate is also constant which seems to be an old work.

In this paper, an effort has been made to redevelop Sunil Kanwale and Pravin Bansode’s [11] EPQ model of deteriorating items with three parameter Weibull deterioration rate and with constant demand and production rates; time varying holding cost and fully backlogged shortages.

Mathematical model and solution procedures are derived followed by numerical examples, graphical and sensitivity analysis to demonstrate the effects of changing parameter values on the optimal solution of the system. The paper ends with concluding remarks.

II. NOTATIONS

The following notations are used for developing the model:

1. \( Q(t) \): Inventory level of the product at time \( t (\geq 0) \).
2. \( d \): Demand rate.
3. \( p \): Rate of production per unit time (\( > d \)).
4. \( k \): Production cost per unit.
5. \( A \): Operating cost.
6. \( C_h \): Inventory holding cost per unit per unit time.
7. \( C_d \): Deterioration cost per unit per unit time.
8. \( C_s \): Shortage cost per unit per unit time.
9. \( T \): Duration of a cycle.
10. \( TC \): Total cost per unit time.
III. ASSUMPTIONS

The model is developed under the following assumptions:
1. Demand rate is known and finite.
2. Production rate is known and finite which is always greater than the demand rate.
3. Shortages occur and they are completely backlogged.
4. An infinite planning horizon is assumed.
5. Once a unit of the product is produced, it is available to meet the demand.
6. As soon as the production stops, products start deterioration.
7. Time to deteriorate follows three parameter Weibull distribution.
8. No replacement or repairs for the deteriorated items will be done.

IV. MATHEMATICAL MODEL AND ANALYSIS

The distribution of time to deteriorate is a random variable which follows a three parameter Weibull distribution. The probability density function for three parameter Weibull distribution is given by

\[ f(t) = \alpha \beta (t - \mu)^{\beta-1} e^{-\alpha(t-\mu)^\beta} \]

where \( t \geq \mu, 0 < \alpha < 1, \beta, \mu > 0 \).

The instantaneous rate of deterioration \( \theta(t) \) of the non-deteriorated inventory at time \( t \) can be obtained from \( \theta(t) = \frac{f(t)}{F(t)} \), where \( F(t) = 1 - e^{-\alpha(t-\mu)^\beta} \) is the cumulative distribution function for the three parameter Weibull distribution. Thus, the instantaneous rate of deterioration of the on-hand inventory is \( \theta(t) = \alpha \beta (t - \mu)^{\beta-1} \). The probability density function represents the distribution of the time to deteriorate which may be decreasing, constant or increasing rate of deterioration. The three parameter Weibull distribution is suitable for items with any initial value of the rate of deterioration (Begum et al. [3]).

This paper assumes that the production starts at time \( t = 0 \) and stops at time \( t = \mu \). During \([0, \mu]\) the inventory is built up at the rate of \( p - d \), there is no deterioration at this interval and stock level attains a level \( Q \). During \([\mu, t_1]\), the inventory level gradually decreases because of deterioration and supply is also there with discounted rate and the stock falls to zero level till time \( t = t_1 \). Then shortages occur and get accumulated to the level \( S \) in the interval \([t_1, t_2]\). The production starts again at time \( t = t_2 \) to fulfill the backlog by the time \( t = T \) to attain the stock level upto zero. The cycle then repeats itself after time \( t = T \).

The pictorial presentation is shown in the Figure – 1.
The differential equations which describe the instantaneous state of Q(t) over the period (0, T) are given by

\[
\frac{dQ(t)}{dt} = p - d \quad (0 \leq t \leq \mu) \quad \text{....(1)}
\]

\[
\frac{dQ(t)}{dt} + \alpha \beta (t - \mu)^{\beta - 1} Q(t) = -d \quad (\mu \leq t \leq t_1) \quad \text{....(2)}
\]

\[
\frac{dQ(t)}{dt} = -d \quad (t_1 \leq t \leq t_2) \quad \text{....(3)}
\]

\[
\frac{dQ(t)}{dt} = p - d \quad (t_2 \leq t \leq T) \quad \text{....(4)}
\]

Under the boundary conditions Q(0) = 0, Q(\mu) = Q_1, Q(t_1) = 0, Q(t_2) = -S and Q(T) = 0 solutions of equations (1) to (4) are given by

\[
Q(t) = (p - d)t \quad (0 \leq t \leq \mu) \quad \text{....(5)}
\]

\[
Q(t) = d \left[ (t_1 - \mu) + \frac{\alpha}{\beta + 1} (t_1 - \mu)^{\beta + 1} \right] \quad (\mu \leq t \leq t_1) \quad \text{....(6)}
\]

\[
Q(t) = -d (t - t_1) \quad (t_1 \leq t \leq t_2) \quad \text{....(7)}
\]
\[ Q(t) = -S + (p - d)(t - t_2) \quad (t_2 \leq t \leq T) \quad \ldots \ldots (8) \]

Putting \( Q(t_2) = -S \) in equation (7), we get

\[ \Rightarrow S = d(t_2 - t_1) \]

Putting \( Q(T) = 0 \) in equation (8), we get

\[ \Rightarrow S = (p - d)(T - t_2) \]

Equation (7) and (8) coincide at \( t = t_2 \) hence

\[ \Rightarrow T = \frac{pt_2 - dt_1}{p - d} \]

The total cost per unit time comprises of the following costs

1) Production Cost

\[ PC = pk(\mu + T - t_2) \quad \ldots \ldots (9) \]

2) Setup Cost

\[ OC = A \quad \ldots \ldots (10) \]

3) Holding Cost

\[ IHC = \int_{0}^{\mu} (h + rt)Q(t)dt + \int_{\mu}^{t_1} (h + rt)Q(t)dt \]

\[ = \left\{ \begin{array}{c}
\int_{0}^{\mu} (h + rt)(p - d)t \ dt \\
+ d\int_{\mu}^{t_1} (h + r\mu) + r(t - \mu) \left[ \frac{(t_1 - \mu) - (t - \mu)}{\beta + 1} (t_1 - \mu)^{\beta+1} - (t - \mu)^{\beta+1} \\
- \alpha (t_1 - \mu) - (t - \mu)(t - \mu)^\beta \right] dt \end{array} \right\} \]
\[ \begin{align*}
\text{IHC} = & \left( p - d \right) \left[ \frac{1}{2} h \mu^2 + \frac{1}{3} r \mu^3 \right] \\
& + d \left[ (h + r \mu) \left\{ \frac{(t_1 - \mu)^2}{2} + \frac{\alpha}{\beta + 1} \frac{(t_1 - \mu)^{\beta+2}}{2} - \frac{\alpha}{(\beta + 1)(\beta + 2)}(t_1 - \mu)^{\beta+2} \right\} \\
& + r \left\{ \frac{(t_1 - \mu)^3}{2} + \frac{\alpha}{\beta + 1} (t_1 - \mu)^{\beta+2} - \frac{\alpha}{(\beta + 2)(\beta + 3)}(t_1 - \mu)^{\beta+3} \right\} \right] \\
\end{align*} \]

.....(11)

4) Deterioration Cost

\[ DC = C_d \int_{\mu}^{t_1} \alpha \beta (t - \mu)^{\beta-1} Q(t) \, dt \]

\[ \Rightarrow DC = C_d d \frac{\alpha}{\beta + 1} t_1 - \mu \beta+1 \]

.....(12)

5) Shortage Cost

\[ SC = -C_s \int_{t_1}^{t_2} Q(t) \, dt \]

\[ \Rightarrow SC = C_s d \frac{(t_2 - t_1)^2}{2} \]

.....(13)

Hence, the average total cost for the time period [0,T] is given by

\[ TC = \frac{1}{T} \left( PC + OC + \text{IHC} + DC + SC \right) \]
$$TC = \frac{1}{(pt_2 - dt_1)^2} \left[ p k \left( \frac{\mu + pt_2 - dt_1}{p - d} - t_2 \right) + A + (p - d) \left\{ \frac{1}{2} h \mu^2 + \frac{r \mu^3}{3} \right\} \right] + d \left\{ \frac{(t_1 - \mu)^2}{2 \beta + 1} + \frac{\alpha (t_1 - \mu) + \beta}{2} - \frac{\alpha}{(\beta + 2)(\beta + 3)} (t_1 - \mu)^{\beta + 2} \right\} 
+ r \left\{ \frac{(t_1 - \mu)^3}{2 \beta + 1} + \frac{\alpha}{(\beta + 2)(\beta + 3)} (t_1 - \mu)^{\beta + 2} \right\} 
+ C_d \frac{\alpha}{\beta + 1} t_1 - \mu^{\beta + 1} + \frac{1}{2} C_s d (t_2 - t_1)^2$$

.....(14)

\(\mu^*, t_1^*\) and \(t_2^*\) are the optimum values of \(\mu, t_1\) and \(t_2\) respectively, which minimize the cost function \(TC\) and they are the solutions of the equations 
\[
\frac{\partial TC}{\partial \mu} = 0, \quad \frac{\partial TC}{\partial \alpha} = 0, \quad \frac{\partial TC}{\partial \beta} = 0, \quad \text{satisfying the sufficient condition } H > 0, \text{ at } \mu^*, t_1^* \text{ and } t_2^* \text{ where}
\]

\[
H = \begin{vmatrix}
\frac{\partial^2 TC}{\partial \mu^2} & \frac{\partial^2 TC}{\partial \mu \partial t_1} & \frac{\partial^2 TC}{\partial \mu \partial t_2} \\
\frac{\partial^2 TC}{\partial t_1 \partial \mu} & \frac{\partial^2 TC}{\partial t_1^2} & \frac{\partial^2 TC}{\partial t_1 \partial t_2} \\
\frac{\partial^2 TC}{\partial t_2 \partial \mu} & \frac{\partial^2 TC}{\partial t_2 \partial t_1} & \frac{\partial^2 TC}{\partial t_2^2}
\end{vmatrix}
\]
is Hessian determinant. 

.....(15)

V. NUMERICAL EXAMPLE

Let us consider the following example to illustrate the above developed model, taking \(A = 500, \ p = 50, \ d = 25, \ k = 1, \ \alpha = 0.02, \ \beta = 2, \ h = 2, \ r = 1, \ C_d = 1, \ C_s = 2\) (with appropriate units).

The optimal values of \(\mu, t_1\) and \(t_2\) are \(\mu^* = 0.6983193493, \ t_1^* = 2.094466538, \ t_2^* = 5.959701114\) units and the optimal total cost per unit time \(TC = 121.6308644\) units.

VI. SENSITIVITY ANALYSIS

Sensitivity analysis depicts the extent to which the optimal solution of the model is affected by the changes in its input parameter values. Here, we study the sensitivity for the cycle length \(T\) and total cost per time unit \(TC\) with respect to the changes in the values of the parameters \(A, p, d, k, \alpha, \beta, h, r, C_d\) and \(C_s\).

The sensitivity analysis is performed by considering variation in each one of the above parameters keeping all other remaining parameters as fixed. The last column of the Table – 1 gives the percentage changes in \(TC\) as compared to the original solution for the relevant costs.
### Table – 1: Partial Sensitivity Analysis

<table>
<thead>
<tr>
<th>Parameters</th>
<th>Change</th>
<th>$\mu$</th>
<th>$t_1$</th>
<th>$t_2$</th>
<th>TC</th>
<th>% TC</th>
</tr>
</thead>
<tbody>
<tr>
<td>$A$</td>
<td>450</td>
<td>0.6497004</td>
<td>2.01285888</td>
<td>5.66922575</td>
<td>116.409172</td>
<td>-4.29</td>
</tr>
<tr>
<td></td>
<td>475</td>
<td>0.6745138</td>
<td>2.05445726</td>
<td>5.81661924</td>
<td>119.054050</td>
<td>-2.12</td>
</tr>
<tr>
<td></td>
<td>525</td>
<td>0.7212076</td>
<td>2.13302376</td>
<td>6.09881332</td>
<td>124.144739</td>
<td>2.07</td>
</tr>
<tr>
<td></td>
<td>550</td>
<td>0.7432568</td>
<td>2.17024813</td>
<td>6.23425535</td>
<td>126.600181</td>
<td>4.09</td>
</tr>
<tr>
<td>$p$</td>
<td>46</td>
<td>0.8026992</td>
<td>2.09819018</td>
<td>6.01214554</td>
<td>114.340286</td>
<td>-5.99</td>
</tr>
<tr>
<td></td>
<td>48</td>
<td>0.8478104</td>
<td>2.09634486</td>
<td>5.98490287</td>
<td>118.163369</td>
<td>-2.85</td>
</tr>
<tr>
<td></td>
<td>53</td>
<td>0.6295683</td>
<td>2.09162145</td>
<td>5.92519389</td>
<td>126.264178</td>
<td>3.81</td>
</tr>
<tr>
<td></td>
<td>56</td>
<td>0.5675723</td>
<td>2.08877247</td>
<td>5.89404522</td>
<td>130.324154</td>
<td>7.15</td>
</tr>
<tr>
<td>$d$</td>
<td>17</td>
<td>0.4413641</td>
<td>2.39997355</td>
<td>6.92683390</td>
<td>118.582746</td>
<td>-2.51</td>
</tr>
<tr>
<td></td>
<td>20</td>
<td>0.5429811</td>
<td>2.26597779</td>
<td>6.49117252</td>
<td>121.404674</td>
<td>0.19</td>
</tr>
<tr>
<td></td>
<td>27</td>
<td>0.7567441</td>
<td>2.03842823</td>
<td>5.79364252</td>
<td>120.279523</td>
<td>-1.11</td>
</tr>
<tr>
<td></td>
<td>30</td>
<td>0.8411634</td>
<td>1.96400501</td>
<td>5.5802013</td>
<td>116.788731</td>
<td>-3.98</td>
</tr>
<tr>
<td>$k$</td>
<td>0.8</td>
<td>0.7876776</td>
<td>2.11193926</td>
<td>5.98969045</td>
<td>116.943780</td>
<td>-3.85</td>
</tr>
<tr>
<td></td>
<td>0.9</td>
<td>0.7440563</td>
<td>2.10382004</td>
<td>5.97572890</td>
<td>119.297721</td>
<td>-1.92</td>
</tr>
<tr>
<td></td>
<td>1.1</td>
<td>0.6501843</td>
<td>2.08374113</td>
<td>5.94143095</td>
<td>123.942246</td>
<td>1.90</td>
</tr>
<tr>
<td></td>
<td>1.2</td>
<td>0.5992976</td>
<td>2.07147184</td>
<td>5.92070170</td>
<td>126.230747</td>
<td>3.78</td>
</tr>
<tr>
<td>$\alpha$</td>
<td>0.010</td>
<td>0.6917269</td>
<td>2.10412869</td>
<td>5.96682419</td>
<td>121.567387</td>
<td>-0.05</td>
</tr>
<tr>
<td></td>
<td>0.015</td>
<td>0.6950776</td>
<td>2.09923587</td>
<td>5.96321342</td>
<td>121.599439</td>
<td>-0.03</td>
</tr>
<tr>
<td></td>
<td>0.030</td>
<td>0.7045005</td>
<td>2.08527448</td>
<td>5.95295204</td>
<td>121.691939</td>
<td>0.05</td>
</tr>
<tr>
<td></td>
<td>0.040</td>
<td>0.7103145</td>
<td>2.07650899</td>
<td>5.94654086</td>
<td>121.750979</td>
<td>0.10</td>
</tr>
<tr>
<td>$\beta$</td>
<td>1.8</td>
<td>0.6972112</td>
<td>2.09615466</td>
<td>5.96126058</td>
<td>121.627648</td>
<td>-0.0026</td>
</tr>
<tr>
<td></td>
<td>1.9</td>
<td>0.6977599</td>
<td>2.09531706</td>
<td>5.96048644</td>
<td>121.629234</td>
<td>-0.0013</td>
</tr>
<tr>
<td></td>
<td>2.1</td>
<td>0.6988897</td>
<td>2.09360246</td>
<td>5.95890399</td>
<td>121.632539</td>
<td>0.0014</td>
</tr>
<tr>
<td></td>
<td>2.2</td>
<td>0.6994711</td>
<td>2.09272429</td>
<td>5.95809456</td>
<td>121.634257</td>
<td>0.0028</td>
</tr>
<tr>
<td>$h$</td>
<td>1.8</td>
<td>0.7073381</td>
<td>2.16226550</td>
<td>6.00167622</td>
<td>120.985268</td>
<td>-0.53</td>
</tr>
<tr>
<td></td>
<td>1.9</td>
<td>0.7031763</td>
<td>2.12880808</td>
<td>5.98063739</td>
<td>121.313733</td>
<td>-0.26</td>
</tr>
<tr>
<td></td>
<td>2.1</td>
<td>0.6929036</td>
<td>2.06142780</td>
<td>5.93891591</td>
<td>121.937203</td>
<td>0.25</td>
</tr>
<tr>
<td></td>
<td>2.2</td>
<td>0.6870389</td>
<td>2.02899036</td>
<td>5.91832029</td>
<td>122.233248</td>
<td>0.50</td>
</tr>
<tr>
<td>$r$</td>
<td>0.8</td>
<td>0.7536853</td>
<td>2.22767348</td>
<td>6.06482753</td>
<td>120.928851</td>
<td>-0.58</td>
</tr>
<tr>
<td></td>
<td>0.9</td>
<td>0.7442714</td>
<td>2.15754208</td>
<td>6.00934557</td>
<td>121.295087</td>
<td>-0.28</td>
</tr>
<tr>
<td></td>
<td>1.1</td>
<td>0.6747085</td>
<td>2.03728020</td>
<td>5.91490226</td>
<td>121.940552</td>
<td>0.25</td>
</tr>
<tr>
<td></td>
<td>1.2</td>
<td>0.6532366</td>
<td>1.98507637</td>
<td>5.87418228</td>
<td>122.227648</td>
<td>0.49</td>
</tr>
<tr>
<td>$C_d$</td>
<td>0.8</td>
<td>0.6973604</td>
<td>2.09554677</td>
<td>5.96041123</td>
<td>121.621611</td>
<td>-0.0076</td>
</tr>
<tr>
<td></td>
<td>0.9</td>
<td>0.6978408</td>
<td>2.09500604</td>
<td>5.96005575</td>
<td>121.626243</td>
<td>-0.0038</td>
</tr>
<tr>
<td></td>
<td>1.1</td>
<td>0.6987960</td>
<td>2.09392827</td>
<td>5.95934730</td>
<td>121.635476</td>
<td>0.0038</td>
</tr>
<tr>
<td></td>
<td>1.2</td>
<td>0.6992709</td>
<td>2.09339122</td>
<td>5.95899432</td>
<td>121.640077</td>
<td>0.0076</td>
</tr>
<tr>
<td>$C_s$</td>
<td>1.8</td>
<td>0.6615765</td>
<td>2.03275491</td>
<td>6.15141379</td>
<td>117.669825</td>
<td>-3.26</td>
</tr>
<tr>
<td></td>
<td>1.9</td>
<td>0.6804340</td>
<td>2.06439815</td>
<td>6.05139935</td>
<td>119.691279</td>
<td>-1.59</td>
</tr>
<tr>
<td></td>
<td>2.1</td>
<td>0.7153208</td>
<td>2.12309879</td>
<td>5.87528079</td>
<td>123.494778</td>
<td>1.53</td>
</tr>
<tr>
<td></td>
<td>2.2</td>
<td>0.7315152</td>
<td>2.15041584</td>
<td>5.87528079</td>
<td>125.305930</td>
<td>3.02</td>
</tr>
</tbody>
</table>
VII. GRAPHICAL PRESENTATION

Figure – 2

Figure – 3

Figure – 4

Figure – 5

Figure – 6

Figure – 7
VIII. CONCLUSION

- From the Partial Sensitivity Analysis table we can conclude that as the Operating Cost and Rate of Production increases, Total Cost also increases.
- From figure 4 it is analysed that as demand rate increases Total Cost of Production increases upto a certain level and then it decreases and then as the production stops and deterioration starts Total Cost decreases continuously.
- From figure 5 and figure 7 it is observed that as the scale parameter and shape parameter increases, the Total Cost again increases.
- From figure 8 and figure 10 it is observed that as h and r of Inventory Holding Cost increase, Total Cost increases.
- From figure 9 and figure 11 it is observed that as the Shortage cost and Deterioration cost increase, Total Cost increases.
- From figure 2, 3 and 6 it is observed that as operating cost, production rate and production cost per unit time increase, the Total Cost increases.
IX. REFERENCES


