Fixed Point Theorem on Complete N - Metric Spaces

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ABSTRACT: *In this paper we have proved the generalized fixed point theorem on complete N-Metric Space with the help of N-mappings. This theorem generalizes and extends the result of R.K. Jain, H.K. Sahu and B. Fisher [3], Kikina [2] and chaube, Gupta and Shau [1] from three, four, five metric space to N-Metric Spaces.*

KEY WORDS: *Complete metric spaces, Cauchy sequences, fixed point. 2010 AMS subject classification 47 H10; 54H25*

INTRODUCTION: In this paper $X_1, X_2, \cdots\cdots X_N$ are taken as complete N-Metric Spaces and $f_N : X_1 \to X_2,\ f_{N-1} : X_2 \to X_3 \cdots\cdots f_1 : X_N \to X_1$ as N-mappings which are all the self mappings. The elements of the metric spaces are taken as $x_{1n} \in X_1,\ X_{2n} \in X_2 \cdots\cdots x_{nN} \in X_N$ where the first suffix indicating the respective metric space and the second one are the number of elements where $n \in N$ (Set of natural number).

MAIN RESULT:

Theorem [1.1] Let $(X_1, d_1),\ (X_2, d_2),\ (X_3, d_3) \cdots\cdots (X_N, d_N)$ are N-complete metric spaces and $f_1\ f_2\ f_3 ----f_N$ are self mappings such that

$$f_N : X_1 \to X_2,\ f_{N-1} : X_2 \to X_3,\ f_{N-2} : X_3 \to X_4 \cdots\cdots\cdots\cdots$$

$$f_3 : X_{N-2} \to X_{N-1},\ f_2 : X_{N-1} \to X_N,\ f_1 : X_N \to X_1$$

Satisfying following conditions

$$d_1\left(f_1\ f_2\ f_3 \cdots\cdots f_{N-1}\ x_{21},\ f_1\ f_2\ f_3 \cdots\cdots f_N\ x_{11}\right) \le \frac{\alpha\ \beta_1\ (x_{11}, x_{21})}{\gamma_1\ (x_{11}, x_{21})} \qquad (1.1.1)$$

$$d_2\left(f_N\ f_1\ f_2\ f_3 \cdots\cdots f_{N-1}\ x_{31},\ f_N\ f_1\ f_2\ f_3 \cdots\cdots f_{N-1}\ x_{21}\right) \le \frac{\alpha \beta_2\ (x_{21}, x_{3_1})}{\gamma_2\ (x_{21}, x_{3_1})} \qquad (1.1.2)$$

$$d_3\left(f_{N-1}\ f_N\ f_1\ f_2\ f_3 \cdots\cdots f_{N-3}\ x_{11},\ f_{N-1}\ f_N\ f_1\ f_2\ f_3 \cdots\cdots f_{N-2}\ x_{31}\right) \le \frac{\alpha \beta_3\ (x_{31}, x_{41})}{\gamma_3\ (x_{31}, x_{41})} \qquad (1.1.3)$$

$$d_4\left(f_{N-2}\ f_{N-1}\ f_N\ f_1\ f_2\ f_3 \cdots\cdots f_{N-4}\ x_{51}, f_{N-2}\ f_{N-1}\ f_N\ f_1\ f_2\ f_3 \cdots\cdots f_N\ x_{41}\right) \le \frac{\alpha \beta_4\ (x_{41}, x_{51})}{\gamma_4\ (x_{41}, x_{51})} \qquad (1.1.4)$$

continuing the same process we have the relative result for Nth complete metric spaces

$$d_N\left(f_2\ f_3\ f_4\cdots\cdots f_{N-1}\ f_N\ x_{11},\ f_N\ f_{N-1}\cdots\cdots f_3\ f_2\ f_1\ x_{N1}\right)\le\frac{\alpha\beta_N\left(x_{N1},x_{11}\right)}{\gamma_N\left(x_{N1},x_{11}\right)}\quad(1.1.5)$$

$$\forall\ x_{11}\in X_1,\ x_{21}\in X_2,\ x_{31}\in X_3\cdots\cdots x_{N1}\in X_N$$

and $\alpha\in(0,1)$, Also $\beta_1\ \beta_2\cdots\cdots\beta_N$ and $\gamma_1,\gamma_2,\gamma_3\cdots\cdots\gamma_N$ indicated in above inequality are defined as follows,

$$\gamma_1\ \left(x_{11},\ x_{21}\right)\neq 0,\ \gamma_2\left(x_{21},x_{31}\right)\neq 0\cdots\cdots\gamma_3\left(x_{31},\ x_{41}\right)\neq 0\cdots\cdots\gamma_N\left(x_{N1},x_{11}\right)\neq 0$$

$$\begin{aligned}\beta_1\ \left(x_{11},\ x_{21}\right)= max\ \{&d_1\left(x_{11},f_1\ f_2\,f_3\cdots\cdots f_N\,x_{N1}\right), d_N\left(f_2\ f_3\cdots\cdots f_{N-2}\ f_{N-1}\ x_{21},\right.\\ &\left.f_2\ f_3\cdots\cdots f_{N-1}\ f_N\ x_{11}\right), d_1\ \left(x_{11},\ f_1\ f_2\ f_3\cdots\cdots f_N\ x_{N1}\right),\\ &d_{N-1}\left(f_3\ f_4\ f_5\cdots\cdots f_{N-2}\ x_{31}, f_3\ f_4\cdots\cdots f_{N-1}\ f_N\ x_{11}\right),\\ &d_1\ \left(x_{11},\ f_1\ f_2\ f_3\cdots\cdots f_N\ x_{11}\right),\ d_{N-2}\left(f_4\ f_5\ f_6\cdots\cdots f_{N-1}\,x_{21}\right.\\ &\left.f_4\ f_5\cdots\cdots f_N\,x_{11}\right)\cdots\cdots d_1\ \left(x_{11},\ f_1\ f_2\ f_3\cdots\cdots f_{N-1}\ x_{21}\right),\\ &d_2\ \left(x_{21},\ f_N\ x_{11}\right)\}\end{aligned}\quad(1.1.6)$$

similarly for $\beta_2,\ \beta_3\cdots\cdots$ we get

$$\beta_N \ (x_{N1}, x_{11}) = max \{d_N \ (x_{N1}, f_2 \ f_3 \ f_4 \ f_5 \cdots\cdots f_N \ f_1 \ x_{N-1}), d_{N-1} \ (f_3 \ f_4 \ f_5 \cdots\cdots f_N \ x_{11},$$

$$f_3 \ f_4 \ f_5 \cdots\cdots f_N \ f_1 \ x_{N1}), d_N (x_{N1}, f_2 \ f_3 \ f_4 \cdots\cdots f_N \ f_1, x_{N1}),$$

$$d_{N-2} \ (f_4 \ f_5 \ f_6 \cdots\cdots f_N \ x_{11}, f_3 \ f_4 \ f_5 \cdots\cdots f_N \ f_1 \ x_{N1}),$$

$$d_N \ (x_{N1}, f_2 \ f_3 \ f_4 \cdots\cdots f_N \ f_1 \ x_{N1}), d_{N-3} \ (f_5 \ f_6 \ f_7 \cdots\cdots f_N \ x_{11},$$

$$f_1 \ f_N \ f_{N-1} \cdots\cdots f_3 \ f_2 \ x_{N-11}), d_N \ (x_{N-1}, f_2 \ f_3 \ f_4 \cdots\cdots f_{N-1} \ x_{N-11}),$$

$$d_1 \ (x_{11}, f_1 \ f_2 \ f_3 \cdots\cdots f_N \ x_{11}), d_N \ (x_{N1}, f_2 \ f_3 \cdots\cdots f_N \ x_{11}),$$

$$d_1 \ (x_{11}, f_1 \ x_{N-11}) \qquad (1.1.7)$$

Again $\gamma_1 \ \gamma_2 \ \gamma_3 \cdots\cdots \gamma_N$ are defined as -

$$\gamma_1 \ (x_{11}, x_{21}) = max \{d_1 \ (x_{11}, f_1 \ f_2 \ f_3 \cdots\cdots f_{N-1} \ x_{N-11}), d_1 \ (x_{11}, f_1 \ f_2 \ f_3 \cdots\cdots f_N \ x_{11}),$$

$$d_2 (f_N \ x_{11}, f_N \ f_1 \ f_2 \ f_3 \cdots\cdots f_{N-1} \ x_{N-11})\}$$

$$r_2 \ (x_{21}, x_{31}) = max \{d_2 (x_{21}, f_N \ f_1 \ f_2 \ f_3 \cdots\cdots f_{N-2} \ x_{N-21}), d_2 \ (x_{21}, f_1 \ f_2 \ f_3 \cdots\cdots f_N \ x_{11}),$$

$$d_3 (f_{N-1} \ x_{21}, f_{N-1} \ f_N \ f_1 \ f_2 \ f_3 \cdots\cdots f_{N-2} \ x_{N-21})\} \qquad (1.1.8)$$

$$r_N \ (x_{N1}, x_{11}) = \max \{d_N \ (x_{N1}, f_2 \ f_3 \ f_4 \cdots\cdots f_N \ x_{11}), d_N \ (x_{N1}, f_2 \ f_3 \ f_4 \ f_5 \cdots\cdots$$
$$f_N \ f_{N-1} \ x_{21}), d_1 \ (f_1 \ x_{N1}, f_1 \ f_2 \ f_3 \ f_4 \cdots\ldots f_N \ x_{11})\} \qquad (1.1.9)$$

i) Then $f_1 \ f_2 \ f_3 \cdots\cdots f_N$ has a unique fixed point $\alpha_1 \in X_1$

ii) $f_N \ f_1 \ f_2 \ f_3 \ f_4 \cdots\cdots f_{N-1}$ has a unique fixed point $\alpha_2 \in X_2$

iii) $f_{N-1}, \ f_N \ f_1 \ f_2 \ f_3 \cdots\cdots f_{N-2}$ has a unique fixed point $\alpha_3 \in X_3$

iv) $f_{N-2,}\ f_{N-1}\ f_N\ f_1\ f_2 \cdots\cdots f_{N-3}$ has a unique fixed point $\alpha_4 \in X_4$ and so on for $\alpha_5 \in X_5$, $\alpha_6 \in X_6$ and finally we obtain $f_2\ f_3\ f_4\ f_5 \cdots\cdots f_{N-1}\ f_N\ f_1$ has a unique fixed point $\alpha_N \in X_N$.

Further $f_N\ \alpha_1 = \alpha_2$, $f_{N-1}\ \alpha_2 = \alpha_3$, $f_{N-2}\ \alpha_3 = \alpha_4$, $f_{N-3}\ \alpha_4 = \alpha_5, \cdots\cdots f_1\ \alpha_N = \alpha_1$

Proof : Let us take x_{10} an element in X_1 Now let us define the sequences $\{x_{1\,n}\}, \{x_{2\,n}\}, \{x_{3\,n}\} \cdots\cdots \{x_{N\,n}\}$ on the complete metric spaces $X_1, X_2, X_3, X_4 \cdots\cdots X_N$ respectively and the sequences defined on $X_1, X_2, X_3 \cdots\cdots X_N$ as follows

$$x_{1\,n} = (f_1\ f_2\ f_3 \cdots\cdots f_N)^n\ x_{10} \quad \text{----(1.1.10)}$$

$$x_{2\,n} = f_N\ x_{1\,n-1},\ x_{3\,n} = f_{N-1},\ x_{3\,n} \quad \text{----(1.1.11)}$$

$$x_{4\,n} = f_{N-2}\ x_{4\,n} \quad \text{----(1.1.12)}$$

$x_{N\,n} = f_1\ x_{N\,n}$ for all positive integer n First we take $x_{1\,n} \neq x_{1\,n+1}$, $x_{2\,n} \neq x_{2\,n+1}$, $x_{3\,n} \neq x_{3\,n+1}$ and $x_{N\,n} \neq x_{N\,n+1}$ and therefore we take -

$$x_{1\,n} = \alpha_1,\ x_{2\,n} = \alpha_2,\ x_{3\,n} = \alpha_3,\ \cdots\cdots x_{N\,n} = \alpha_N$$

$$x_{3\,n+1} = x_{3\,n+2} \quad \cdots\cdots \quad x_{N\,n+1} = x_{N\,n+2}$$

Again we can put

$$x_{1\,n} = \alpha_1,\ x_{2\,n} = \alpha_2,\ x_{3\,n} = \alpha_3, \cdots\cdots x_{N\,n} = \alpha_N$$

if $x_{2\,n} = x_{2\,n+1}$, $x_{3\,n} = x_{3\,n+1}$, $x_{N\,n} = x_{N\,n+1}$

then the inequality (1.1.1) gives $x_{1\,n} = x_{1\,n+1}$ and by the inequality (1.1.2) we obtain

$$d_2\left(x_{2n}, x_{2n+1}\right) = d_2\left(f_N\, f_1\, f_2\, f_3 \cdots\cdots f_{N-2}\, x_{3n-1},\, f_N\, f_1\, f_2\, f_3 \cdots\cdots f_{N-1}\, x_{1n}\right)$$

$$\leq \frac{\alpha\, \beta_2\left(x_{2n}, x_{3n-1}\right)}{\gamma_2\left(x_{2n}, x_{3n-1}\right)}$$

$$\alpha\ max\ \{d_2\left(x_{2n}, x_{2n+1}\right), d_1\left(x_{1n-1}, x_{1n}\right), d_2\left(x_{2n}, x_{2n+1}\right),$$

$$d_3\left(x_{3n-1}, x_{3n}\right), d_2\left(x_{2n}, x_{2n+1}\right), d_4\left(x_{4n-1}, x_{4n}\right) \cdots\cdots d_2\left(x_{2n}, x_{2n+1}\right),$$

$$\leq \frac{d_N\left(x_{Nn}, x_{Nn+1}\right)\}}{max\ \{d_2\left(x_{2n}, x_{2n}\right), d_2\left(x_{2n}, x_{2n+1}\right), d_3\left(x_{3n}, x_{3n}\right)}$$

$$\alpha\ \ max\ \ d_2\left(x_{2n}, x_{2n+1}\right), \{d_1\left(x_{1n-1}, x_{1n}\right), d_3\left(x_{3n-1}, x_{3n}\right), d_4\left(x_{4n-1}, x_{4n}\right) \cdots\cdots$$

$$\leq \frac{d_N\left(x_{Nn}, x_{Nn+1}\right)\}}{max\ \ \{0,\ \ d_2\left(x_{2n}, x_{2n+1}\right),\ 0\}}$$

$$\leq\ \ \alpha\ \ max\ \{d_1\left(x_{1n-1}, x_{1n}\right), d_3\left(x_{3n-1}, x_{3n}\right), d_4\left(x_{4n+1}, x_{4n}\right) \cdots\cdots$$

$$d_N\left(x_{Nn}, x_{Nn+1}\right)\} \tag{1.1.13}$$

Again substituting $x_{31} = x_{3n}$ and $x_{41} = x_{4n-1}$

we obtain from the inequality (1.1.3)

$$d_3\left(x_{3n}, x_{3n+1}\right) = d_3\left(f_{N-1}\, f_N\, f_1\, f_2\, f_3 \cdots\cdots f_{N-3}\, x_{3n-1},\, f_{N-1}\, f_N\, f_1\, f_2\, f_3 \cdots\cdots f_{N-3}\, f_{N-2}\, x_{2n}\right)$$

$$\leq \frac{\alpha\, \beta_3\left(x_{3n}, x_{3n-1}\right)}{\gamma_3\left(x_{3n}, x_{3n-1}\right)}$$

$$\alpha\ \ max\ [d_3\left(x_{3n}, x_{3n+1}\right), d_2\left(x_{2n}, x_{2n+1}\right), d_3\left(x_{3n}, x_{3n+1}\right), d_1\left(x_{1n-1}, x_{1n}\right), d_3\left(x_{3n}, x_{3n+1}\right),$$

$$\leq \frac{d_4\left(x_{4n-1}, x_{4n}\right) \cdots\cdots d_3\left(x_{3n}, x_{3n+1}\right) d_N\left(x_{Nn}, x_{Nn+1}\right)]}{max\ \{d_3\left(x_{3n}, x_{3n}\right), d_3\left(x_{3n}, x_{3n+1}\right), d_4\left(x_{4n}, x_{4n}\right)\}}$$

$$\leq \frac{\alpha\ max\left[d_3(x_{3n}, x_{3n+1})\{d_2(x_{2n}, x_{2n+1}),\ d_1(x_{3n}, x_{3n+1}),\ d_4(x_{4n}, x_{4n+1})\cdots\cdots d_N(x_{Nn}, x_{Nn+1})\}\right]}{max\{0,\ d_3(x_{3n}, x_{3n+1}),\ 0\}}$$

$$\leq \frac{\alpha\ max\left[d_3(x_{3n}, x_{3n+1}),\ \{d_2(x_{2n}, x_{2n+1}),\ d_1(x_{3n}, x_{3n+1}),\ d_4(x_{4n}, x_{4n+1})\cdots\cdots d_N(x_{Nn}, x_{Nn+1})\}\right]}{d_3(x_{3n}, x_{3n+1})}$$

$$\leq \alpha\ max\{d_2(x_{2n}, x_{2n+1}), d_1(x_{1n}, x_{1n+1}), d_4(x_{4n}, x_{4n+1})\cdots\cdots d_N(x_{Nn}, x_{Nn+1})\}$$

Again for $d_4(x_{4n}, x_{4n+1})$ we have,

$$d_4(x_{4n}, x_{4n+1}) \leq \alpha\ max\{d_3(x_{3n}, x_{3n+1}),\ d_2(x_{2n}, x_{2n+1}),\ d_1(x_{1n}, x_{1n+1})\cdots\cdots d_{N-1}(x_{N-1\,n}, x_{N-1\,n+1}),\ d_N(x_{Nn}, x_{Nn+1}) \qquad (1.1.14)$$

$$d_5(x_{5n}, x_{5n+1}) \leq \alpha\ max\{d_4(x_{4n}, x_{4n+1}),\ d_3(x_{3n}, x_{3n+1}),\ d_2(x_{2n}, x_{2n+1})\cdots\cdots d_1(x_{1n}, x_{4n+1}), d_{N-1}(x_{N-1\,n}, x_{N-1\,n+1}),\ d_N(x_{Nn}, x_{Nn+1}) \qquad (1.1.15)$$

continuing this process we obtain,

$$d_{N-1}(x_{N-1\,n}, x_{N-1\,n+1}) \leq \alpha\ max\{d_{N-2}(x_{N-2\,n}, x_{N-2\,n+1}),\ d_{N-3}(x_{N-3\,n}, x_{N-3\,n+1})\cdots\cdots d_2(x_{2n}, x_{2n+1})\ d_1(x_{1n}, x_{1n+1})\} \qquad (1.1.16)$$

$$d_N(x_{Nn}, x_{Nn+1}) \leq \alpha\ max\{d_{N-1}(x_{N-1\,n}, x_{N-1\,n+1}),\ d_{N-2}(x_{N-2\,n}, x_{N-2\,n+1}),\ d_{N-3}(x_{N-3\,n}, x_{N-3\,n+1})\cdots\cdots d_2(x_{2n}, x_{2n+1}),\ d_1(x_{1n}, x_{1n+1}) \qquad (1.1.17)$$

and thus we obtain for $d_1(x_{1n}, x_{1n+1})$

$$d_1\left(x_{1\,n}, x_{1\,n+1}\right) = d_1\left(f_1\ f_2\ f_3 \cdots\cdots f_{N-1}\ x_{21},\ f_1\ f_2\ f_3 \cdots\cdots f_n\ x_{11}\right)$$

$$\leq \frac{\alpha\ \beta_1\left(x_{11}, x_{21}\right)}{\gamma_1\left(x_{11}, x_{21}\right)}$$

$$\alpha\ max\left\{d_1\left(x_{1\,n}\ ,\ x_{1\,n+1}\right), d_2\left(x_{2\,n},\ x_{2\,n+1}\right), d_1\left(x_{1\,n},\ x_{1\,n+1}\right), d_3\left(x_{3\,n},\ x_{3\,n+1}\right),\right.$$

$$\leq \frac{d_1\left(x_{1\,n},\ x_{1\,n+1}\right), d_4\left(x_{4\,n},\ x_{4\,n+1}\right)\cdots\cdots d_1\left(x_{1\,n},\ x_{1\,n+1}\right),\ d_N\left(x_{N\,n},\ x_{N\,n+1}\right)\}}{max\ \left\{d_1\left(x_{1\,n}, x_{1\,n}\right), d_1\left(x_{1\,n},\ x_{1\,n+1}\right), d_2\left(x_{2\,n},\ x_{2\,n}\right)\right\}}$$

$$\alpha\ max\ \left\lfloor d_1\left(x_{1\,n}, x_{1\,n+1}\right)\left\{d_2\left(x_{2\,n}, x_{2\,n+1}\right), d_3\left(x_{3\,n}, x_{3\,n+1}\right), d_4\left(x_{4\,n}, x_{4\,n+1}\right),\cdots\cdots\right.\right.$$

$$\leq \frac{d_N\left(x_{N\,n}, x_{N\,n+1}\right)\}}{max\ \left\{0,\ d_1\ \left(x_{1\,n},\ x_{1\,n+1}\right),\ 0\right\}}$$

$$\leq\ \alpha\ max\ \left\{d_2\left(x_{2n}\ ,\ x_{2\,n+1}\right), d_3\left(x_{3\,n},\ x_{3\,n+1}\right), d_4\left(x_{4\,n},\ x_{4\,n+1}\right)\cdots\cdots\right.$$

$$\left. d_N\left(x_{N\,n},\ x_{N\,n+1}\right)\right\} \tag{1.1.18}$$

Now using all inequality (1.1.1) we obtain

$$d_1\left(x_{1\,n}, x_{1\,n+1}\right) \leq \alpha^2\ max\left\{d_2\left(x_{2\,n}\ ,\ x_{2\,n+1}\right), d_3\left(x_{3\,n},\ x_{3\,n+1}\right), d_4\left(x_{3\,n},\ x_{3\,n+1}\right),\cdots\cdots\right.$$

$$\left. d_{N-1}\left(x_{N-1\ \ n},\ x_{N-1\,n+1}\right), d_N\left(x_{N\,n},\ x_{N\ n+1}\right)\right\} \tag{1.1.19}$$

Similarly for the other inequality we have,

$$d_2\left(x_{2n}, x_{2\,n+1}\right) \leq\ \alpha^2\, max\ \left\{d_1\left(x_{1\,n}\ ,\ x_{1\,n+1}\right), d_3\left(x_{3\,n},\ x_{3\,n+1}\right), d_4\left(x_{4\,n},\ x_{3\,n+1}\right),\right.$$

$$\left. d_5\left(x_{5\,n},\ x_{5\,n+1}\right)\cdots\cdots\ d_N\left(x_{N\ \ n},\ x_{N\ \ n+1}\right)\right\} \tag{1.1.20}$$

$$d_3\left(x_{3n},x_{3\,n+1}\right)\le\alpha^2\ max\ \{d_1\left(x_{1\,n}\ ,\ x_{1\,n+1}\right),\ d_2\left(x_{2\,n},\ x_{2\,n+1}\right),\ d_4\left(x_{4\,n},\ x_{4\,n+1}\right)$$

$$d_5\left(x_{5\,n},_{5\,n+1}\right)\cdots\cdots d_N\left(x_{N\,n},\ x_{N\,n+1}\right)\}\qquad(1.1.21)$$

$$d_N\left(x_{N\,n},x_{N\,n+1}\right)\le\alpha^2\ max\{d_1\left(x_{1\,n}\ ,\ x_{1\,n+1}\right),\ d_2\left(x_{2\,n},\ x_{2\,n+1}\right),\ d_3\left(x_{3\,n},\ x_{3\,n+1}\right),$$

$$d_4\left(x_{4\,n},\ x_{4\,n+1}\right)\cdots\cdots d_{N-1}\left(x_{N-1\,n},\ x_{N-1\,n+1}\right)\}\qquad(1.1.22)$$

Now by induction, we obtain from the above inequality

$$d_1\left(x_{1\,n},x_{1\,n+1}\right)\le\alpha^{n-1}\ max\{d_1\left(x_{11}\ ,\ x_{12}\right)\ ,\ d_2\left(x_{21},\ x_{22}\right),\ d_3\left(x_{31},\ x_{32}\right)\cdots\cdots$$

$$d_N\left(x_{N\,1},\ x_{N\,2}\right)\}\qquad(1.1.23)$$

$$d_2\left(x_{2\,n},x_{2\,n+1}\right)\le\alpha^{n-1}\ max\{d_1\left(x_{11}\ ,\ x_{12}\right),\ d_2\left(x_{21},\ x_{22}\right),\ d_3\left(x_{31},\ x_{32}\right)\cdots\cdots$$

$$d_N\left(x_{N1},\ x_{N\,2}\right)\}\qquad(1.1.24)$$

$$d_3\left(x_{3\,n},x_{3\,n+1}\right)\le\alpha^{n-1}\ max\{d_1\left(x_{11}\ ,\ x_{12}\right),\ d_2\left(x_{21},\ x_{22}\right)\cdots\cdots d_N\left(x_{N1},\ x_{N\,2}\right)\}\qquad(1.1.25)$$

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$$d_{N-1}\left(x_{N-11}\ x_{N-12}\right)\le\alpha^{n-1}\ max\{d_1\left(x_{11}\ ,\ x_{12}\right),\ d_2\left(x_{21},\ x_{22}\right),\ d_3\left(x_{31},\ x_{32}\right),$$

$$\cdots\cdots d_N\left(x_{N1}\ ,\ x_{N\,2}\right)\}\qquad(1.1.26)$$

$$d_N\left(x_{N1},x_{N\,2}\right)\le\alpha^{n-1}\ max\{d_1\left(x_{11}\ ,\ x_{12}\right),\ d_2\left(x_{21},\ x_{22}\right),\ d_3\left(x_{31},\ x_{32}\right),$$

$$\cdots\cdots d_{N-1}\left(x_{N+1}\ ,\ x_{N-12}\right)\ d_N\left(x_{N1}\ ,\ x_{N\,2}\right)\}\qquad(1.1.27)$$

as $\alpha\in(0,1)$ then $\alpha^{n-1}\to 0$ as $n\to\infty$ therefore,

$d_1() \to 0$

$$d_1\left(x_{1\,n}\;,\;x_{1\,n+1}\right)\quad\to\quad 0$$

$$d_2\left(x_{2\,n}\;,\;x_{2\,n+1}\right)\quad\to\quad 0$$

$$d_3\left(x_{3\,n}\;,\;x_{3\,n+1}\right)\quad\to\quad 0$$

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$$d_{N-1}\left(x_{N-1\;n}\;,\;x_{N-1\;n+1}\right)\quad\to\quad 0$$

$$d_N\left(x_{N\,n}\;,\;x_{N\;n+1}\right)\quad\to\quad 0$$

i.e. The Sequences $\{x_{1\,n}\}\ \{x_{2\,n}\}\ \{x_{3\,n}\}\cdots\cdots\{x_{N\,n}\}$ are cauchy sequence. Since $(X_1\ d_1), (X_2\ d_2), (X_3\ d_3)\cdots\cdots(X_N\ d_N)$ are complete metric space. So we have

$$\lim_{n\to\infty}\ \{x_{1\,n}\}\to\alpha_1\ \ ;\ \ \{x_{2\,n}\}\to\ \alpha_2$$

$$\lim_{n\to\infty}\ \{x_{3\,n}\}\to\alpha_3\ \ ;\ \ \{x_{4\,n}\}\to\ \alpha_4$$

$$\lim_{n\to\infty}\ \{x_{N-1\;n}\}\to\alpha_{N-1}\,;\ \ \{x_{N\;n}\}\to\ \alpha_n$$

where $\alpha_1\ \ \alpha_2\cdots\cdots\alpha_N\ \in\ X_1, X_2, X_3, X_4\cdots\cdots X_N$ respectively. Now substituting $x_{11}\ =\ x_{1\,n}$ and $x_{2\,1}\ =\ \alpha_2$ we obtain,

$$d_1\left(f_1\ f_2\ f_3\cdots\cdots f_{N-1}\ \alpha_2, x_{1\;n+1}\right) = d_1\left(f_1\ f_2\ f_3\cdots\cdots f_{N-1}\alpha_2,\ f_1\ f_2\ f_3\cdots\cdots f_N\ x_{1\,N}\right)$$

$$\leq \frac{\alpha\beta_1\left(x_{1\,n}, \alpha_2\right)}{\lambda_1\left(x_{1\,n}, \alpha_2\right)}$$

$$\alpha\, max\{d_1\left(x_{1\,n}\ ,\ x_{1\;n+1}\right),\ d_2\left(\alpha_2, f_N\ \ f_1\ f_2\ f_3\cdots f_{N-1}\ \alpha_2\right),\ d_1\left(x_{1\,n}, x_{1\,n+1}\right),$$

$$\leq\ \frac{d_3\left(f_{N-1}\ \alpha_2, x_{3\,n}\right)\cdots\cdots d_1\left(x_{1n}, x_{1\,n+1}\right), d_N\left(x_{N\,n}, f_1\ \ f_2\cdots\cdots f_{N-1}\ \alpha_2\right)\}}{max\ \{d_1\left(x_{1\,n}, f_1\ f_2\ f_3\cdots\cdots f_N\ \alpha_2\right), d_1\left(x_{1\,n}\ \ x_{1\,n+1}\right),\ d_2\left(\alpha_2, f_N\ \ f_1\ f_2\ f_3\cdots\cdots f_{N-1}\alpha_2\right)\}}$$

For $n \to \infty$ we get $d_1 \left(f_1 \; f_2 \; f_3 \; f_4 \cdots\cdots f_{N-1} \; \alpha_2 , \alpha_1 \right) \leq 0$ which is a contradiction and as $d_1 \left(f_1 \; f_2 \; f_3 \cdots\cdots f_{N-1} \; \alpha_2 , \alpha_1 \right) \geq 0$ Therefore $d_1 \left(f_1 \; f_2 \; f_3 \cdots\cdots f_{N-1} \; \alpha_2 , \alpha_1 \right) = 0$

$$\Rightarrow \quad f_1 \; f_2 \; f_3 \cdots\cdots f_{N-1} \; \alpha_2 \; = \; \alpha_1$$

Similarly we obtain that

$$f_N \;\; f_1 \; f_2 \; f_3 \cdots\cdots f_{N-2} \; \alpha_3 \; = \; \alpha_2$$

$$f_{N-1} \;\; f_N \;\; f_1 \;\; f_2 \cdots\cdots f_{N-3} \; \alpha_4 \; = \; \alpha_3$$

$$f_{N-2} \;\; f_{N-1} \;\; f_N \;\; f_1 \;\; f_2 \cdots\cdots f_{N-4} \; \alpha_5 \; = \; \alpha_4$$

....

....

.... $f_2 \;\; f_3 \;\; f_4 \cdots\cdots f_{N-1} \;\; f_N \;\; \alpha_1 \; = \; \alpha_N$

Now we claim that, α_1 is the fixed point of $f_1 \;\; f_2 \;\; f_3 \cdots\cdots f_N$ in X_1

α_2 is the fixed point of $f_2 \;\; f_3 \;\; f_4 \cdots\cdots f_N \;\; f_1$ in X_2

α_3 is the fixed point of $f_3 \;\; f_4 \;\; f_5 \cdots\cdots f_N \;\; f_1 \;\; f_2$ in X_3

....

....

....

α_N is the fixed point of $f_N \;\; f_{N+1} \;\; f_{N-2} \cdots\cdots f_2 \;\; f_1$ in X_N

Now let us take $x_{2\,1} \; = \; x_{2\,n}$ and $x_{3\,1} \; = \; f_{N-1} \; \alpha_2$ and substituting in (1.1.2)

we obtain

$$d_2 \;\left(f_N \;\; f_1 \;\; f_2 \;\; f_3 \cdots\cdots f_{N-1} \; \alpha_2 , \; x_{2\,n} \right)$$

$$= d_2 \;\left(f_N \;\; f_1 \;\; f_2 \;\; f_3 \;\; f_{N-1} \; \alpha_2 , \; f_N \;\; f_1 \;\; f_2 \cdots\cdots f_{N-1} \;\; x_{2\,n} \right)$$

$$\leq \; \frac{\alpha \; \beta_1 \left(x_{2\,n} , \; f_{N-1} \; \alpha_2 \right)}{\gamma_1 \left(x_{2\,n} , \; f_{N-1} \; \alpha_2 \right)}$$

$$max\{d_2\,(x_{2n}\,,\,x_{2\,n+1}),\ d_1(f_1\ f_2\ f_3\cdots\cdots f_{N-1}\ \alpha_2,x_{2n}),\ d_2(x_{2n}\,,x_{2\,n+1}),$$

$$d_3(f_N\ f_1\ f_2\cdots\cdots f_{N-1}\ \alpha_2,x_{2n}),\cdots\cdots d_2(x_{2n}\,,x_{2\,n+1}),$$

$$\leq \frac{d_N(f_2\ f_3\ f_4\cdots\cdots f_{N-1}\ \alpha_2\ x_{2n})\}}{max\{d_2\,(x_{2n}\,,\ f_N\ f_1\ f_2\ f_3\cdots\cdots f_{N-1}\alpha_2),\ d_2(x_{2n}\,,x_{2\,n+1})}$$

$$d_3(x_{3n}\,,f_{N-2}\ f_N\ f_1\ f_2\ f_3\cdots\cdots f_{N-1}\ \alpha_2)\}$$

Now let $n\rightarrow\infty$ and put $f_1\ f_2\ f_3\cdots\cdots f_{N-1}\ \alpha_2\ =\ \alpha_1$ we obtain

$$d_2(f_N\ f_1\ f_2\cdots\cdots f_{N-1}\ \alpha_2,\ \alpha_2)\leq\frac{\alpha\,d_2\,(\alpha_2,\,f_N\ f_1\ f_2\cdots\cdots f_{N-1}\ \alpha_2),d_3\,(f_{N-1}\ \alpha_2,\alpha_3)}{max\{d_2(f_N\ f_1\ f_2\cdots\cdots f_{N-1}\alpha_2),\,d_3(\alpha_3\,,f_{N-1}\ f_N\ \alpha_1)}$$ Now

if, $$max\{d_2\,(\alpha_2,f_N\ f_1\ f_2\cdots\cdots f_{N-1}\ \alpha_2),\ d_3(\alpha_3\,,f_{N-1}\ f_N\ \alpha_1)\}$$

$$=\ d_2\,(\alpha_2,\,f_N\ \ f_1\ \ f_2\cdots\cdots f_{N-1}\ \ \alpha_2)$$

Then we have $=\ d_2\,(f_N\ \ f_1\ \ f_2\cdots\cdots f_{N-1}\ \ \alpha_2,\alpha_2)=d_2\,(f_N\ \alpha_1,\alpha_2)$

$$d_2\,(f_N\ f_1\ f_2\cdots\cdots f_{N-1}\ \alpha_2,\alpha_2)\leq\frac{\alpha\ max\,\{d_2\,(\alpha_2,\,f_N\ f_1\ f_2\cdots\cdots f_{N-1}\ \alpha_2),d_3(f_{N-1}\ \alpha_2,\alpha_3)\}}{max\,\{d_2\,(\alpha_2,\,f_N\ f_1\ f_2\cdots\cdots f_{N-1}\ \alpha_2),d_3(\alpha_3\,,\,f_{N-1}\ f_N\ \alpha_1)\}}$$
$$\leq\ \alpha\ d_3\,(f_{N-1}\ \alpha_2,\,\alpha_3)$$

Again if $max\{d_2\,(\alpha_2\ f_N\ f_1\ f_2\cdots\cdots f_{N-1}\ \alpha_2),\ d_3(\alpha_3\,,f_{N-1}\ f_N\ \alpha_1)$

$$=\ d_3\,(\alpha_3\,,f_{N-1}\ f_N\ \alpha)$$

Then similarly we have,

$$d_2\,(f_N\ \ f_1\ f_2\cdots\cdots f_{N-1}\ \alpha_2,\ \alpha_2)=\ d_2(f_N\ \alpha_1\,,\alpha_2)\ \leq\ \alpha\,d_3\,(f_{N-1}\ \alpha_2\,,\alpha_3)$$

Again put $x_{31}\ =x_{3n}$ and $x_{41}\ =f_{N-3}\ \alpha_3$ we obtain,

$$d_3\,(f_{N-1}\ \ f_N\ f_1\ f_2\ \ f_3\cdots\cdots f_{N-2}\ \alpha_3,\ \alpha_3)\leq\ \alpha\,d_4(f_{N-2}\ \alpha_3\,,\alpha_3)$$

and $$d_3\,(f_{N-1}\ \ f_N\ f_1\ f_2\cdots\cdots f_{N-2}\ \alpha_3,\ \alpha_3)\leq\ \alpha\,d_4(f_{N-3}\ \alpha_3\,,\alpha_3)$$

similarly we have

$$d_4\left(f_{N-2}\ f_{N-1}\ f_N\ f_1\cdots\cdots f_{N-3}\ \alpha_4,\ \alpha_4\right)\le \alpha d_5\left(f_{N-4}\ \alpha_4,\alpha_4\right)$$

and $d_4\left(f_{N-2}\ f_{N-1}\cdots\cdots f_{N-3}\ \alpha_4,\ \alpha_4\right)\le \alpha d_5\left(f_{N-5}\ \alpha_5,\alpha_5\right)$ and Thus finally we get $d_1\left(f_1\ f_2\ f_3\cdots\cdots f_N\ \alpha_1,\ \alpha_1\right)\le \alpha d_2\left(f_N\ \alpha_1,\alpha_2\right)$ using the above inequality

$$d_2\left(f_N\ f_1\ f_2\cdots\cdots f_{N-1}\ \alpha_2,\ \alpha_2\right)= d_2\left(f_N\ \alpha_1,\alpha_2\right)$$

$$d_2\left(f_N\ \alpha_1,\alpha_2\right)\le \alpha d_3\left(f_{N-1}\ \alpha_2,\alpha_3\right)\le \alpha^2 d_4\left(f_{N-2}\ \alpha_3,\alpha_3\right)\le \alpha^3 d_5\left(f_{N-3}\alpha_4,\alpha_5\right)$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$\le \alpha^{n-1} d_2\left(f_N\ f_1\ f_2\cdots\cdots f_{N-1}\ \alpha_2,\alpha_2\right) \qquad (1.1.28)$$

as $\alpha^{n-1}\to 0$ as $\alpha\in(0,1)$ we have $f_N\ f_1\ f_2\cdots\cdots f_{N-1}\ \alpha_2=\alpha_2$ hence

$f_N\ f_1\ f_2\cdots\cdots f_{N-1}\ \alpha_2=\alpha_2$ has a fixed point α_2 in X_2.

Lastly we show that $\alpha_1\ \alpha_2\ \alpha_3\cdots\cdots\alpha_N$ are the unique fixed point on respective metric spaces.

If possible let α_1 is not unique then let α_1' is another fixed point then putting $x_{21}=f_N\ \alpha_1$ and $x_{11}=\alpha_1'$ in (1.1.1) we get

$$d_1\left(\alpha_1,\alpha_1'\right) = d_1\left(f_1\ f_2\ f_3\cdots\cdots f_N\ \alpha_1,\ f_1\ f_2\ f_3\cdots\cdots f_N\ \alpha_1'\right)$$

$$\alpha\ max\ \{d_1\left(\alpha_1',\ \alpha_1'\right)\ d_N\left(f_2\ f_3\ f_4\cdots\cdots f_N\ \alpha_1,\ f_2\ f_3\ f_4\cdots\cdots f_N\ \alpha_1'\right)$$

$$\le \frac{d_1\ \left(\alpha_1',\ \alpha_1'\right)\ d_{N-1}\left(f_3\ f_4\cdots\cdots f_N\ \alpha_1,\ f_3\ f_4\cdots\cdots f_N\ \alpha_1'\right)\cdots\cdots\}}{max\ \{d_1\ \left(\alpha_1',\ \alpha_1\right),\ d_1\ \left(\alpha_1',\alpha_1'\right), d_2\ \left(f_N\ \alpha_1',\ f_N\ \alpha\right)\}}$$

$$\le \frac{\alpha\ max\ \{d_1\left(\alpha_1',\ \alpha\right),\ d_2\ \left(f_N\ \alpha_1',\ f_N\ \alpha\right)\}}{max\ \{d_1\ \left(\alpha,\alpha_1'\right),\ d_2\ \left(f_N\alpha_1',\ f_N\alpha\right)\}}$$

$$If,\qquad max\ \{d_1\ \left(\alpha_1,\ \alpha_1'\right),\ d_2\ \left(f_N\ ,\alpha_1',\ f_N\ \alpha_1\right)\} = d_2\ \left(f_N\ \alpha_1',\ f_N\ \alpha\right)$$

then we get $d_1\ (\alpha_1\ \ \alpha_1') \le \alpha\ d_1\ (\alpha_1'\ ,\ \alpha_1)$ which gives $\alpha_1 = \alpha_1'$

Again substituting $x_{31} = f_{N-1}\ f_N\ \alpha_1$ and $x_{21} = f_N\ \alpha_1'$ we obtain

$$d_2\ (f_N \alpha_1,\ f_N\ \alpha_1') \le d_2\ (f_N\ f_1\ f_2\ f_3 \cdots\cdots f_{N-1}\ \alpha_1,\ f_N\ f_1\ f_2\ f_3 \cdots\cdots f_{N-1}\ \alpha_1')$$

$$\alpha\ max\ \{d_2\ (f_N\ \alpha_1',\ f_N\ f_1\ f_2 \cdots\cdots f_{N-1}\ \alpha_1'),\ d_1\ (\alpha_1\ \alpha_1') \cdots\cdots$$

$$\le \frac{d_2\ (f_N \alpha,\ f_N \alpha_1'),\ d_N\ (f_2\ f_3 \cdots\cdots f_N\ \alpha_1,\ f_2\ f_3 \cdots\cdots f_N\ \alpha_1')}{\{max\ d_3\ ,(f_{N-1}\ f_N \alpha_1'\ ,\ f_{N-1}\ \ f_N\ \ \alpha_1)\ ,\ d_2\ (f_N\ \alpha_1\ ,\ f_N\ \alpha_1'))}$$

After simplifying we get,

$$d_2\ (f_N \alpha_1,\ f_N\ \alpha_1') \le \alpha d_3\ (f_{N-1}\ f_N\ \alpha_1'\ , f_{N-1}\ f_N\ \alpha_1')$$

similarly we get-----

$$d_3\ (f_{N-1}\ f_N\ \alpha_1\ ,\ f_{N-1}\ f_N \alpha_1') \le d_4\ (f_3\ f_2 \cdots\cdots f_{N-1}\ f_N\ \alpha_1'\ ,\ f_3 f_2 \cdots\cdots f_{N-1}\ f_N\ \alpha_1)$$

....

$$d_N\ (f_2\ f_3 \cdots\cdots\ f_N \alpha_1\ ,\ f_2\ f_3 \cdots\cdots f_N\ \alpha_1') \le \alpha\ d_1\ (\alpha_1'\ ,\ \alpha_1)$$

combining the above inequalities

$$\begin{aligned} d_1\ (\alpha_1\ ,\ \alpha_1') &\le \alpha d_2\ (f_N\ \alpha_1\ ,\ f_N\ \alpha_1') \\ &\le \alpha^2 d_3\ (f_{N-1}\ f_N\ \alpha_1'\ ,\ f_{N-1}\ f_N\ \alpha_1) \\ &\le \alpha^3\ d_4 (f_{N-2}\ f_{N-1}\ f_N\ \alpha_1'\ ,\ f_{N-2}\ f_{N-1}\ f_N\ \alpha_1) \\ &\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ &\le \alpha^n\ d_1\ (\alpha_1\ ,\ \alpha_1') \end{aligned}$$

as $\alpha \in (0,1)$ $\alpha^n \to 0$

$d_1\ (\alpha_1\ \ \alpha_1') \le 0$ which is a contradiction $\Rightarrow$ $\alpha_1 = \alpha_1'$

which proves that α_1 is a unique fixed point of $f_1 \; f_2 \; f_3 \cdots\cdots f_N$ in X_1 similarly we can show that $f_N \; f_1 \; f_2 \cdots\cdots f_{N-1}$ has a unique fixed point α_2 in X_2.

$f_{N-1} \; f_N \; f_1 \; f_2 \cdots\cdots f_{N-2}$ has an unique fixed point α_3 in X_3.

....
....
And $f_N \; f_{N-1} \; f_{N-3} ----f_1$ has an unique fixed point α_N in X_N

CONCLUSION

We observe that the self mappings $f_1 \; f_2 \; f_3 \cdots\cdots f_N$ play an essential role in contraction mapping. These mappings are effectively used in finding out the existence of fixed point on the complete metric spaces.

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