Strong Efficient Edge Domination Number of Some Cycle Related Graphs

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Abstract

Let $G = (V, E)$ be a simple graph. A subset $S$ of $E(G)$ is a strong (weak) efficient edge dominating set of $G$ if $\|N[\{e\} \cap S]\| = 1$ for all $e \in E(G)$ if $\|N[\{e\} \cap S]\| = 1$ for all $e \in E(G)$, where $N[\{e\}] = \{f / f \in E(G) \& \deg f \geq \deg e\}$ and $N[\{e\}] = \{f / f \in E(G) \& \deg f \leq \deg e\}$. The minimum cardinality of a strong efficient edge dominating set of $G$ (weak efficient edge dominating set of $G$) is called a strong efficient edge domination number of $G$ and is denoted by $\gamma'_{se}(G)$ ($\gamma'_{we}(G)$). In this paper, the strong efficient edge domination number of some cycle related graphs are studied.

Keywords - Domination, edge domination, strong edge domination, efficient edge domination, strong efficient edge domination.

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I. INTRODUCTION

Throughout this paper, only finite, undirected and simple graphs are considered. The concept of domination in graphs was introduced by Ore. Two volumes on domination have been published by T.W. Haynes, S.T. Hedetniemi and P.J. Slater [5, 6]. Let $G = (V, E)$ be a graph with $p$ vertices and $q$ edges. The degree of an edge is defined to be $\deg u + \deg v - 2$ [4]. An edge $uv$ is called an isolated edge if $\deg uv = 0$. Also $\Delta(G)$ denotes the minimum degree and $\Delta'(G)$ denotes the maximum degree among the edges of $G$. A subset $D$ of $V(G)$ is called an efficient dominating set of $G$ if for every vertex $u \in V(G)$, $|N[u] \cap D| = 1$ [1, 2]. Edge dominating sets were studied by S.L. Mitchell and S.T. Hedetniemi [8]. A set $F$ of edges in a graph $G$ is called an edge dominating set of $G$ if every edge in $E - F$ is adjacent to at least one edge in $F$. Equivalently, a set $F$ of edges in $G$ is called an edge dominating set of $G$ if every edge $e \in E - F$ there exists an edge $e_i \in F$ such that $e$ and $e_i$ have a vertex in common. The edge domination number $\gamma'(G)$ of a graph $G$ is the minimum cardinality of an edge dominating set of $G$. The strong (weak) domination number $\gamma'_s(G)$ ($\gamma'_w(G)$) of $G$ is the minimum cardinality of a strong (weak) dominating set of $G$ and $\Gamma_s(G)$ is the maximum cardinality of a minimal strong dominating set of $G$ [9]. A subset $D$ of $E(G)$ is called an efficient edge dominating set if every edge in $E(G)$ is dominated by exactly one edge in $D$ [3, 7, 10]. The cardinality of the minimum efficient edge dominating set is called the efficient edge domination number of $G$. Motivated by these definitions, the authors define strong efficient edge domination in graphs. In this paper, the strong efficient edge domination numbers of some cycle related graphs are studied.

II. MAIN RESULTS

Definition 2.1: Let $G = (V, E)$ be a simple graph. A subset $S$ of $E(G)$ is a strong (weak) efficient edge dominating set of $G$ if $\|N[\{e\} \cap S]\| = 1$ for all $e \in E(G)$ if $\|N[\{e\} \cap S]\| = 1$ for all $e \in E(G)$, where $N[\{e\}] = \{f / f \in E(G) \& \deg f \geq \deg e\}$ and $\{N[\{e\}] = N[\{e\}] \cup \{e\} \}$ and $N[\{e\}] = \{N[\{e\}] \cup \{e\} \}$. The minimum cardinality of a strong efficient edge dominating set of $G$ (weak efficient edge dominating set of $G$) is called a strong(weak) efficient edge domination number of $G$ and is denoted by $\gamma'_{se}(G)$ ($\gamma'_{we}(G)$).
Example 2.2: Consider the following graph

\[ \text{Let } S = \{e_4, e_7\}. \text{ Now } |N_2[e_1] \cap S| = |\{e_1, e_2, e_3\} \cap S| = |\{e_3\}| = 1, \quad |N_2[e_2] \cap S| = |\{e_2, e_3, e_4\} \cap S| = |\{e_4\}| = 1, \quad |N_2[e_4] \cap S| = |\{e_4\} \cap S| = |\{e_6\} \cap S| = |\{e_6\} \cap S| = |\{e_7\} \cap S| = 1. \text{ In G, S is the unique strong efficient edge dominating set of G. Therefore } \gamma'_s(G) = 2. \]

Theorem 2.3: Every strong efficient edge dominating set is independent.

Proof: Let S be a strong efficient edge dominating set of G. Let \( e_1, e_2 \in S \). Suppose the edges \( e_1 \) and \( e_2 \) are adjacent. Without loss of generality, \( \deg e_1 \geq \deg e_2 \). Then \( |N_2[e_2] \cap S| \geq 2 \), a contradiction. Therefore S is independent.

Remark 2.4: Not all graphs have strong efficient edge dominating set.

Proof: Consider the following graph G

\[ S_1 = \{e_2, e_3\}, S_2 = \{e_1, e_4\}. |N_2[e_1] \cap S_1| = |\{e_1, e_2, e_3\} \cap S_1| = |\{e_2, e_3\}| = 2 > 1. \text{ Therefore } S_1 \text{ is not a strong efficient edge dominating set of G and } |N_2[e_3] \cap S_2| = |\{e_3, e_4\} \cap S_2| = |\{e_4\}| = 2 > 1. \text{ Therefore } S_2 \text{ is not strong efficient edge dominating set. Since the edges } e_2 \text{ and } e_4 \text{ have maximum degree and they are adjacent, any strong efficient edge dominating set must have either } e_2 \text{ or } e_4. \text{ Hence the graph G does not have a have strong efficient edge dominating set.} \]

Definition 2.5: Let \( G = (V, E) \) be a simple graph. An edge dominating set S is a strong edge dominating set if for every edge \( e \in E(G) - S \), there is an edge \( f \in S \) with \( \deg f \geq \deg e \), and \( e \) is adjacent to \( f \). The strong edge domination number is denoted by \( \gamma'_s(G) \). An edge dominating set S is a weak edge dominating set if for every edge \( e \in E(G) - S \), there is an edge \( f \in S \) with \( \deg f \leq \deg e \), and \( e \) is adjacent to \( f \). The weak edge domination number is denoted by \( \gamma'_w(G) \).

Remark 2.6: \( \gamma'_s(G) \leq \gamma'_s(G) \)

Proof: Let S be a strong efficient edge dominating set of G. Let \( e_1 \in E(G) - S \). Then \( |N_2[e] \cap S| = 1 \). Then there exists \( e_2 \in S \) such that the edges \( e_1 \) and \( e_2 \) are adjacent and \( \deg e_2 \geq \deg e_1 \). Hence S is a strong efficient edge dominating set of G. Therefore \( \gamma'_s(G) \leq \gamma'_s(G) \).

Remark 2.7: The strong edge dominating set of a graph G need not be strong efficient edge dominating set.
**Proof:** Consider the following graph $G$.

![Graph Image]

$S_1 = \{e_2, e_6\}$, $S_2 = \{e_5, e_7\}$ are the strong edge dominating sets of $G$. But $|N[e_2] \cap S_1| = |\{e_7, e_2, e_6\} \cap S_1| = 2 > 1$ and $|\{e_2, e_6\} | = |\{e_7, e_2, e_6\} \cap S_1| = 1$. Hence $S_1$ & $S_2$ are not strong efficient edge dominating sets of $G$. Since the edges $e_2$ and $e_3$ have maximum degree and they are adjacent, no strong efficient edge dominating set without $e_2$ or $e_3$ exists in $G$.

**Theorem 2.8:** A graph $G$ does not admit a strong efficient edge dominating set if any two non adjacent maximum degree edges are joined by a single edge.

**Proof:** Let $G = (V, E)$ be a simple graph and any two non adjacent maximum degree edges are joined by a single edge. Suppose $G$ admits a strong efficient edge dominating set $S$. Let $e_1$ and $e_2$ be the edges of degree $\Delta(G)$. Let $e_3 \in E(G)$ such that the edges $e_1$ and $e_2$ are joined by the edge $e_3$. Since $e_1$ and $e_2$ are not adjacent, they belong to $S$. Therefore $|N[e_3] \cap S| = 2 > 1$, a contradiction. Hence $G$ does not admit a strong efficient edge dominating set of $G$.

**Definition 2.9:** Let $G = (V, E)$ be a simple graph. Let $E(G) = \{e_1, e_2, e_3, e_4, \ldots, e_n\}$. An edge $e_i$ is said to be full degree edge if and only if $\deg e_i = n-1$.

**Observation 2.10:** $\gamma_{se}(G) = 1$ if and only if $G$ has a full degree edge.

**Theorem 2.11:** $\gamma_{se}(C_{3n}) = n, \forall n \in N$.

**Proof:** Let $G = C_{3n}, n \in N$ Let $E(G) = \{e_1, e_2, e_3, e_4, \ldots, e_{3n-1}, e_{3n}\}$. Then $S_1 = \{e_1, e_3, e_5, \ldots, e_{3n-2}\}$, $S_2 = \{e_2, e_4, e_6, \ldots, e_{3n-3}\}$, $S_3 = \{e_3, e_5, e_7, \ldots, e_{3n-4}\}$ are the three strong efficient edge dominating sets of $G$ and $|S_1| = |S_2| = |S_3| = n$. Therefore $\gamma_{se}(C_{3n}) \leq n$. Since $n = \gamma_{se}(C_{3n}) \leq \gamma_{se}(C_{3n})$. Therefore $\gamma_{se}(C_{3n}) = n, \forall n \in N$.

**Remark 2.12:** $C_{3n+1}$, $C_{3n+2}$ do not have efficient edge dominating sets, they do not have strong efficient edge dominating sets.

**Theorem 2.13:** For any path $P_m$, $\gamma_{se}(P_m) = \begin{cases} \frac{n}{m} & \text{if } m = 3n + 1, n \geq 1 \\ n + 1 & \text{if } m = 3n, n \geq 2 \\ n + 1 & \text{if } m = 3n + 2, n \geq 2 \end{cases}$

**Proof:**

**Case 1:** Let $G = P_{3n}, n \geq 2$. Let $E(G) = \{e_1, e_2, e_3, e_4, \ldots, e_{3n-1}\}$. Then $S = \{e_1, e_3, \ldots, e_{3n-3}, e_{3n-1}\}$ is the unique strong efficient edge dominating set of $G$ and $|S| = n + 1, n \geq 2$. Therefore $\gamma_{se}(P_{3n}) = n + 1, n \geq 2$.

**Case 2:** Let $G = P_{3n+1}, n \geq 1$. Let $E(G) = \{e_1, e_2, e_3, e_4, \ldots, e_{3n-1}, e_{3n}\}$. Then $S = \{e_2, e_5, e_8, \ldots, e_{3n-1}\}$ is the unique strong efficient edge dominating set of $G$ and $|S| = n, n \geq 1$. Therefore $\gamma_{se}(P_{3n+1}) = n, n \geq 1$.

**Case 3:** Let $G = P_{3n+2}, n \geq 1$. Let $E(G) = \{e_1, e_2, e_3, e_4, \ldots, e_{3n-1}, e_{3n}, e_{3n+1}\}$. Then $S = \{e_1, e_3, e_6, e_9, \ldots, e_{3n}\}$, $S_2 = \{e_2, e_5, e_{8}, \ldots, e_{3n+1}\}$ are the two strong efficient edge dominating sets of $G$ and $|S_1| = n + 1, n \geq 1$, $|S_2| = n + 1, n \geq 1$. Therefore $\gamma_{se}(P_{3n+2}) \leq n + 1, n \geq 1$. Since $n + 1 = \gamma_{se}(P_{3n+2}) \leq \gamma_{se}(P_{3n+2}), n \geq 1$. Therefore $\gamma_{se}(P_{3n+2}) = n + 1, n \geq 1$.

**Definition 2.14:** Wheel $W_n$ is defined as the join of $C_{n+1} + K_1$. The vertex corresponding to $K_1$ is said to be apex vertex, the vertices corresponding to the cycle are called the rim vertices. The edges corresponding to cycle are called the rim edges and edges joining apex and vertices of the cycle are called spoke edges.
Theorem 2.15: Let $W_m$, $m \geq 2$ be a wheel graph. Then $W_m$ has a strong efficient edge dominating set if and only if $m = 3n, m \geq 2$ and $\gamma'_{se}(W_{3n}) = n, n \geq 2$.

Proof: Let $m = 3n, n \geq 2$. Let $u$ be the central vertex. Let $v_1, v_2, \ldots, v_{3n}$ be the vertices of the cycle $C_{3n-1}$. Let $E(G) = \{ e_i = u_i, 1 \leq i \leq 3n-1 \} \cup \{ f_i = v_i, 1 \leq i \leq 3n-1 \} \cup \{ f_{3n-1} = v_{3n} \} \cup \{ e \}$ and $deg u = 3n-1, deg v_i = 3, 1 \leq i \leq 3n-1, deg e_i = deg u + deg v_i = 2n, 1 \leq i \leq 3n$.

Case (i): $e_i = v_i, 1 \leq i \leq 3n-1 \}$ and $\gamma'_{se}(W_{3n}) = n, n \geq 2$.

Case (ii): $v_i \in C_i \cup \{ e \}$, strongly dominates $e_i$. Each $e_i$ strongly dominates all the spokes edges and two rim edges adjacent with $e_i$. The sub graph induced by remaining rim edges is $P_{3n-2}$. By Theorem 2.13, $P_{3n-2}$ has unique strong efficient edge dominating set containing $n-1$ elements. These $n-1$ elements together with $e_i$ form a strong efficient edge dominating set $S_i$ for $W_{3n}$. Therefore $\gamma'_{se}(W_{3n}) = n, n \geq 2$.

Conversely:

Case (iii): $m = 3n+2$. Suppose $S$ is a strong efficient edge dominating set of $W_m$. Clearly $S$ contains one of $e_i$'s and two rim edges adjacent with $e_i$. Without loss of generality $e_i$ belongs to $S$, $e_i$ strongly dominates $e_1, e_2, e_3, e_4\ldots e_{3n+1}, f_1$ and $f_{3n+1}$. Clearly $f_2$ and $f_{3n+1}$ do not belong to $S$. Then the edges $f_1, f_2, \ldots, f_{3n+1}$ belong to $S$. $f_{3n+1}$ strongly dominates $f_{3n+2}$. If the edge $f_{3n+1}$ belongs to $S$ then $f_{3n+1}$ is strongly dominated by two edges $f_{3n+1}$ and $f_{3n+2}$. Hence $f_{3n+1}$ does not belong to $S$. Therefore there is no edge in $S$ to strongly dominate $f_{3n+1}$, a contradiction. Hence $W_{3n+1}$ has no strong efficient edge dominating set.

Case (iv): $m = 3n+1$. Suppose $S$ is a strong efficient edge dominating set of $W_m$. Clearly $S$ contains one of $e_i$'s and two rim edges adjacent with $e_i$. Without loss of generality $e_i$ belongs to $S$, $e_i$ strongly dominates $e_1, e_2, e_3, e_4\ldots, e_{3n+1}, f_1$ and $f_{3n+1}$. Clearly $f_2$ and $f_{3n+1}$ do not belong to $S$. Then the edges $f_1, f_2, \ldots, f_{3n+1}$ belong to $S$. $f_{3n+1}$ strongly dominates $f_{3n+2}$. If the edge $f_{3n+1}$ belongs to $S$ then $f_{3n+2}$ is strongly dominated by two edges $f_{3n+1}$ and $f_{3n+2}$. If $f_{3n+1}$ belongs to $S$ then $f_{3n+1}$ is dominated by two edges $f_{3n+1}$ and $f_e$. Hence $f_{3n+1}$ does not belong to $S$. Therefore there is no edge in $S$ to strongly dominate $f_{3n+1}$, a contradiction. Hence $W_{3n+1}$ has no strong efficient edge dominating set.

Definition 2.16: Consider two cycles $C_1$ and $C_2$. Connect a vertex of $C_1$ to a vertex of $C_2$ with a new edge. The new graph obtained joining the sum of $C_1$ and $C_2$.

Theorem 2.17: Let $G$ be the joint sum of two cycles $C_1$ and $C_2$ where $i, j \in N$ then $\gamma'_{se} (G) = 2n+1, n \geq 1$.

Proof:

Case (i): $i = 3n, j = 3n, n \geq 1$. Let $V(G) = \{ v_i, u_i / 1 \leq i \leq 3n \}$. Let $e$ be an edge joining a vertex $v_i$ of $C_i$ and a vertex $u_j$ of $C_j$ and $E(G) = \{ v_i, u_i / 1 \leq i \leq 3n \} \cup \{ v_i, u_j / 1 \leq j \leq 3n \} \cup \{ u_j, v_i \} \cup \{ v_i, v_{i+1} \} \cup \{ u_j, u_{j+1} \} \cup \{ u_{j+1}, v_{i+1} \} \cup \{ e \}$. Without loss of generality, let $e = v_1, u_1$. $deg v_1 = 3, deg v_i = 3, deg u_j = 3, deg u_i = 2, deg v_{i+1} = 2, 2 \leq i \leq 3n$. The edge $e$ is the only maximum degree edge of $G$. $deg v_1 u_1 = 4, deg v_1 v_2 = 3, deg v_2 v_3 = 3$. Similarly $deg u_1 u_2 = 3, deg u_3 u_4 = 3$. The remaining edges have degree two. $S = \{ v_1 u_1, v_2 v_3, v_3 v_4, \ldots, v_{3n} v_{3n+1}, u_1 u_2, u_2 u_3, \ldots, u_{3n} u_{3n+1} \}$ is the unique strong efficient edge dominating set of $G$. Therefore $S = 2n+1$. Hence $\gamma'_{se} (G) = 2n+1, n \geq 1$.

Case (ii): $i = 3n, j = 3n+1, n \geq 1$. Let $e = v_1 u_i$. $V(G) = \{ v_i, u_i / 1 \leq i \leq 3n \} \cup \{ u_1, u_j / 1 \leq j \leq 3n+1 \}$, $E(G) = \{ v_i, v_{i+1} / 1 \leq i \leq 3n \} \cup \{ v_1, v_2 \} \cup \{ u_j, u_{j+1} \} \cup \{ e \}$. Then degree of all the vertices and all edges are same as in case (i). $S = \{ v_1 u_1, v_2 u_2, v_3 u_3, \ldots, v_{3n} u_{3n}, u_1 u_2, u_2 u_3, \ldots, u_{3n} u_{3n+1} \}$ is the unique strong efficient edge dominating set of $G$. Therefore $|S| = 2n+1$. Hence $\gamma'_{se} (G) = 2n+1, n \geq 1$.

Case (iii): $i = 3n+2, j = 3n+2, n \geq 1$. Let $e = v_1 u_i$. $V(G) = \{ v_i, u_i / 1 \leq i \leq 3n \} \cup \{ u_1, u_j / 1 \leq j \leq 3n+2 \}$, $E(G) = \{ v_i, v_{i+1} / 1 \leq i \leq 3n \} \cup \{ v_1, v_2 \} \cup \{ u_j, u_{j+1} \} \cup \{ e \}$. Then degree of all the vertices and all the edges are same as in case (i). $S = \{ v_1 u_1, v_2 u_2, v_3 u_3, \ldots, v_{3n} u_{3n}, u_1 u_2, u_2 u_3, \ldots, u_{3n} u_{3n+1} \}$ is the unique strong efficient edge dominating set of $G$. Therefore $|S| = 2n+1$. Hence $\gamma'_{se} (G) = 2n+1, n \geq 1$.

Case (iv): $i = 3n+1, j = 3n+1, n \geq 1$. Let $e = v_1 u_i$. $V(G) = \{ v_i, u_i / 1 \leq i \leq 3n \} \cup \{ u_1, u_j / 1 \leq j \leq 3n+1 \}$, $E(G) = \{ v_i, v_{i+1} / 1 \leq i \leq 3n \} \cup \{ v_1, v_2 \} \cup \{ u_j, u_{j+1} \} \cup \{ e \}$. As in case (i), the degree of all the vertices and all the edges are same. $S = \{ v_1 u_1, v_2 u_2, v_3 u_3, \ldots, v_{3n} u_{3n}, u_1 u_2, u_2 u_3, \ldots, u_{3n} u_{3n+1} \}$ and $S_2 = \{ v_1 u_1, v_2 u_2, v_3 u_3, \ldots, v_{3n} u_{3n}, u_1 u_2, u_2 u_3, \ldots, u_{3n} u_{3n+1} \}$ are the four strong efficient edge dominating set of $G$. Therefore $|S_1| = |S_2| = 2n+1$. Hence $\gamma'_{se} (G) \leq 2n+1, n \geq 1$. Any strong efficient edge dominating set must contain the edge $e$. Also it contains the edge $u_{3n} u_{3n+1}$ or $u_3 u_{3n+1}$ or $v_{3n+1} v_{3n}$. Therefore no set of $2n$ edges is a strong efficient edge dominating set. Hence $\gamma'_{se} (G) = 2n+1, n \geq 1$. 

ISSN: 2231 – 5373  http://www.ijmttjournal.org  Page 205
**Theorem 2.18**: Let $G = C_{3n} \circ mK_1$, $n \geq 1$. Then $\gamma_{se}'(G) = 2n, n \geq 1$.

**Proof**: Let $G = C_{3n} \circ mK_1$, $m \geq 1$, $n \geq 1$. Let $V(C_{3n}) = \{v_i \mid 1 \leq i \leq 3n\}$. Let $v_1, v_2, \ldots, v_{3n}$ be the vertices joined with $v_1$, $1 \leq i \leq 3n$. Let $e_i = v_i v_{i+1}$, $1 \leq i \leq 3n - 1$, $e_{3n} = u_{3n} u_1$, and let $e_k = v_i v_k$, $1 \leq k \leq m$, $1 \leq i \leq 3n$. Clearly $S_1 = \{e_i \mid i = 1, 4, 7, \ldots, 3n-2\} \cup \{e_k \mid k \neq i, i+1, 1 \leq j \leq 3n, 1 \leq k \leq m\}$, $S_2 = \{e_i \mid i = 2, 5, 8, \ldots, 3n-2\} \cup \{e_k \mid k \neq i, i+1, 1 \leq j \leq 3n, 1 \leq k \leq m\}$, $S_3 = \{e_i \mid i = 3, 6, 9, \ldots, 3n\}$. Therefore $\gamma_{se}'(G) \leq 2n$, $n \geq 1$. It is verified that no set of $2n - 1$ edges are not strong efficient edge dominating set of $G$. Hence $\gamma_{se}'(G) = 2n$. Therefore $\gamma_{se}'(G) = 2n, n \geq 1$.

**Definition 2.20**: A gear graph $G_n$ is obtained from the wheel graph $W_n$ by adding a vertex between every pair of adjacent vertices in the cycle $C_n$.

**Theorem 2.20**: Let $G_m$ be a gear graph. Then $G_m$ has a strong efficient edge dominating set if and only if $m = 3n + 2$, $n \geq 1$ and $\gamma_{se}'(G_{3n+2}) = 2n + 1, n \geq 1$.

**Proof**: Let $m = 3n + 2$, $n \geq 1$. Let $V(G) = \{u, v_1, v_2, v_3, \ldots, v_3n, v_{3n+1}, v_{3n+2}, \ldots, v_{3n+3}, u, u_1, u_2, u_3, \ldots, u_{3n+1}\}$. Let $E(G) = \{f_i = uv_i \mid 1 \leq i \leq 3n+1\} \cup \{e_i = u_i u_{i+1} \mid 1 \leq i \leq 3n\} \cup \{g_i = uv_i \mid 1 \leq i \leq 3n+1\} \cup \{e_{3n+1} = u_{3n+1} v_1 \mid 1 \leq i \leq 3n\}$ and deg $u = 3n + 1$, deg $v_i = 3$, deg $u_i = 2$, deg $f_i = 3$, deg $e_i = 3$, deg $g_i = 3n+2, 1 \leq i \leq 3n + 1$. Each $g_i$ strongly dominates all the spoke edges and two rim edges adjacent with $g_i$. The subgraph induced by the remaining rim edges is the path $P_{3n} = P_{3(2n)}$. By theorem 2.12, $P_{3(2n)}$ has a unique strong efficient edge dominating set containing $2n + 1$ elements. These $2n + 1$ elements together with the $g_i$ form a strong efficient edge dominating set $S_i$ for $G_{3n+2}$. Therefore $|S| = 2n+1, 1 \leq i \leq 3n$. Hence $\gamma_{se}'(G_{3n+2}) = 2n + 1, n \geq 1$.

**Conversely**: 

**Case 1**: Let $m = 3n$. Suppose $S$ is the strong efficient edge dominating set of $G_m$. Clearly $S$ contains one of $g_i$’s and two rim edges adjacent with $g_i$. Without loss of generality $g_i$ belongs to $S$. $g_i$ strongly dominates all $g_j, 2 \leq j \leq 3n$ and $f_i$ and $e_{3n+1}$. Clearly $e_i$ and $f_{3n+1}$ do not belong to $S$. Then the edges $e_1, e_3, e_5, e_6, e_8, e_9, f_1, \ldots, e_{3n+1}, f_{3n+1}$ belong to $S$. The edge $f_{3n+1}$ strongly dominates $e_{3n+2}$ and $e_{3n+1}$. If the edge $f_{3n+1}$ belongs to $S$ then the edge $e_{3n+1}$ is strongly dominated by two edges $f_{3n+1}$ and $e_i$. Hence $e_{3n+1}$ does not belong to $S$. Therefore there is no edge in $S$ to strongly dominate $f_{3n+1}$. A contradiction. Hence $G_{3n+1}, n \geq 1$ has no strong efficient edge dominating set.

**Case 2**: Let $m = 3n+1$. Suppose $S$ is the strong efficient edge dominating set of $G_m$. As in case 1, suppose $g_i$ belongs to $S$. $g_i$ strongly dominates all $g_j, 2 \leq j \leq 3n$, $f_i$ and $e_{3n+1}$ do not belong to $S$. Then the edges $e_1, e_3, e_5, e_6, e_8, e_9, f_1, \ldots, e_{3n+1}, f_{3n+1}$ belongs to $S$. The edge $e_{3n+1}$ strongly dominates $e_{3n+2}$ and $f_{3n+1}$. If the edge $e_{3n+1}$ belongs to $S$ then the edge $f_{3n+1}$ is strongly dominated by two edges $f_{3n+1}$ and $e_i$. Hence $f_{3n+1}$ does not belong to $S$. Therefore there is no edge in $S$ to strongly dominate $e_{3n+1}$. A contradiction. Hence $G_{3n+1}, n \geq 1$ has no strong efficient edge dominating set.

**Definition 2.21**: The Helm $H_n$ is the graph obtained from the wheel $W_n$ with $n$ spokes by adding a pendant edge at each vertex on the wheel’s rim.

**Theorem 2.22**: Let $H_n = W_n \circ K_1$. Then $\gamma_{se}'(H) = 2n, n \geq 2, m = 3n$.

**Proof**: Let $m = 3n$, $n \geq 2$. Let $V(G) = \{u, v_1, u_1, / 1 \leq i \leq 3n - 1\}$, $E(G) = \{e_i = u_i v_i / 1 \leq i \leq 3n - 1\} \cup \{f_i = v_i v_{i+1} / 1 \leq i \leq 3n - 1\}$ and deg $u = 3n$, deg $v_1 = 4$, deg $u_1 = 1, 1 \leq i \leq 3n - 1$, deg $e_i = 3n + 1$, deg $f_i = 6$, deg $g_i = 3$, deg $f_{3n+1} = 6$. All $e_i$’s are adjacent with each other. To dominate them, any one $e_i$ is considered. Without loss of generality, let it be $e_1$, which is adjacent to two rim edges. The subgraph induced by the remaining rim edges is the path $P_{3n+1}$. By theorem 2.12, $P_{3n+1}$ has the unique strong efficient edge dominating
set containing \( n - 1 \) elements. The edges \( g_2, g_5, g_8, \ldots, g_{3n-1} \) also belongs to strong efficient edge dominating set. Therefore the total number of elements in the strong efficient edge dominating set is \( 2n \). Hence \( y'_{se}(H_{3n}) \leq 2n, n \geq 2 \). Since \( 2n = \gamma'_s(H_{3n}) \leq y'_{se}(H_{3n}) \). Therefore \( y'_{se}(H_{3n}) = 2n, n \geq 2 \).

**Conversely:**

**Case 1:** Let \( m = 3n+1 \). Suppose \( S \) is a strong efficient edge dominating set of \( H_m \). Clearly \( S \) contains atleast one of \( e_1 \)'s and two rim edges adjacent with \( e_i \). Without loss of generality \( e_1 \) belongs to \( S \). \( e_1 \) strongly dominates all \( g_2, g_5, \ldots, g_{3n-1} \). Clearly \( e_i \) and \( e_{3n-1} \) do not belong to \( S \). Then the edges \( e_1, e_2, \ldots, e_{3n-1}, f_1, g_1 \) and \( f_{3n-1} \). Clearly \( f_2 \) and \( f_{3n-1} \) do not belong to \( S \). Also the edges \( f_3, f_6, \ldots, f_{3n-3} \) belong to \( S \). The edge \( f_{3n-3} \) strongly dominates \( f_{3n-2} \) and \( e_{3n-1} \). If the edge \( f_{3n-1} \) belongs to \( S \) then the edge \( f_{3n-1} \) is strongly dominated by two edges \( f_{3n-1} \) and \( f_2 \). Hence \( f_{3n-1} \) does not belong to \( S \). Therefore there is no edge in \( S \) to strongly dominate \( f_{3n-1} \), a contradiction. Hence \( H_{3n+1}, n \geq 1 \) has no strong efficient edge dominating set.

**Case 2:** Let \( m = 3n+2 \). Suppose \( S \) is a strong efficient edge dominating set of \( H_m \). As in case 1, suppose \( e_1 \) belongs to \( S \). \( e_1 \) strongly dominates all \( e_1, 2 \leq j \leq 3n+2, f_1, g_1 \) and \( f_{3n+1} \). Clearly \( f_2 \) and \( f_{3n+1} \) do not belong to \( S \). Then the edges \( f_3, f_6, \ldots, f_{3n-2} \) belong to \( S \). The edge \( f_{3n-2} \) strongly dominates \( f_{3n-3} \) and \( f_{3n-1} \). If the edge \( f_{3n-1} \) belongs to \( S \) then the edge \( f_{3n-1} \) is strongly dominated by two edges \( f_{3n-1} \) and \( f_1 \). Hence \( f_{3n+1} \) does not belong to \( S \). Therefore there is no edge in \( S \) to strongly dominate \( f_{3n+1} \), a contradiction. Hence \( H_{3n+2} \) has no strong efficient edge dominating set.

**Theorem 2.23:** Let \( G \) be a graph obtained by joining the central vertices of two copies of a wheel \( W_m, m \geq 4 \). Then \( G \) has a strong efficient edge dominating set if and only if \( m = 3n + 1, n \geq 1 \).

**Proof:** Let \( m = 3n + 1 \). Let \( u \) and \( v \) be the central vertex of \( W_1 \) and \( W_2 \) respectively. Let \( u_i, 1 \leq i \leq 3n \) and \( v_i, 1 \leq i \leq 3n \) be the vertices of the wheel \( W_1 \) and \( W_2 \) respectively. Let \( e = uv \) and \( E(G) = \{ e_i = u_i v_i / 1 \leq i \leq 3n \} \cup \{ f_i = u_i v_{i+1} / 1 \leq i \leq 3n \} \cup \{ h_i = u_{3n+1} v_i / 1 \leq i \leq 3n \} \cup \{ g_i = v_i / 1 \leq i \leq 3n \} \cup \{ h_i = v_{i+1} / 1 \leq i \leq 3n \} \cup \{ h_1 = v_1 \} \cup \{ e = uv \} \). Clearly \( S \) contains one of \( e_1 \) and \( g_2, g_5, \ldots, g_{3n-1} \) belongs to \( S \). The edge \( f_{3n+1} \) strongly dominates \( f_{3n} \). Hence \( y'_{se}(H_{3n+1}) \leq 2n \). Therefore \( y'_{se}(H_{3n+1}) = 2n, n \geq 2 \).

**Conversely:**

Let \( i \neq 3n, j \neq 3n, n \geq 1 \). The edge \( e \) strongly dominates all the edges of \( W_1 \) and \( W_2 \). The subgraph induced by the remaining rim edges are the cycles \( C_i \) and \( C_{i+1}, i \neq 3n, j \neq 3n, n \geq 1 \). Let \( \gamma_{se} (F_n) \geq 2n \). Clearly \( S \) contains one of \( e_1 \) and \( g_2, g_5, \ldots, g_{3n-1} \) belongs to \( S \). The edge \( f_{3n+1} \) does not belong to \( S \). Then the edges \( f_1, f_2, \ldots, f_{3n-2} \) belong to \( S \). The edge \( f_{3n-1} \) strongly dominates \( f_{3n-2} \). If the edge \( f_{3n-1} \) belongs to \( S \) then the edge \( f_{3n-1} \) is strongly dominated by two edges \( f_{3n-1} \) and \( f_1 \). Hence \( f_{3n-1} \) does not belong to \( S \). Therefore there is no edge in \( S \) to strongly dominate \( f_{3n-1} \), a contradiction. Hence \( y'_{se}(H_{3n+1}) = 2n, n \geq 2 \).

**Definition 2.24:** A flower \( F_n \) is constructed from a helm \( H_n \) by joining each vertex of degree one to the centre.

**Theorem 2.25:** A flower graph \( F_m \) has a strong efficient edge dominating set if and only if \( m = 3n, n \geq 1 \). Then \( y'_{se}(F_m) = 2n, n \geq 2 \).

**Proof:** Let \( m = 3n, n \geq 2 \). Let \( V(G) = \{ u, v, u_i / 1 \leq i \leq 3n - 1 \} \). Let \( E(G) = \{ e_i = u_i v_i / 1 \leq i \leq 3n - 1 \}. \cup \{ f_i = v_{i+1} u_i / 1 \leq i \leq 3n - 1 \}. \cup \{ g_i = v_i / 1 \leq i \leq 3n - 1 \}. \cup \{ h_i = u_i / 1 \leq i \leq 3n - 1 \}. \cup \{ e = uv \} \). Clearly \( S \) contains one of \( e_1 \) and \( g_2, g_5, \ldots, g_{3n-1} \) belongs to \( S \). The edge \( f_{3n+1} \) strongly dominates \( f_{3n} \). Therefore \( \gamma_{se}(F_m) = 2n, n \geq 2 \). Since \( P_{2n} \) has a unique strong efficient edge dominating set, no other set of \( 2n - 1 \) edges cannot be strong efficient edge dominating set of \( F_{3n} \). Therefore \( \gamma_{se}(F_{3n}) = 2n, n \geq 2 \).

**Conversely:**

**Case 1:** Let \( m = 3n+1 \). Suppose \( S \) is a strong efficient edge dominating set of \( F_m \). Clearly \( S \) contains one of \( e_1 \)'s and two rim edges adjacent with \( e_i \). Without loss of generality \( e_1 \) belongs to \( S \). \( e_1 \) strongly dominates all \( e_2, 2 \leq j \leq 3n, f_1, f_{3n-1}, g_1 \) and all the edges \( h_i, 1 \leq i \leq 3n \). Also the edges \( g_2, g_5, g_8, \ldots, g_{3n-1} \) belong to \( S \). Clearly \( f_2 \) and \( f_{3n-1} \) do not belong to \( S \). Then the edges \( f_1, f_2, \ldots, f_{3n-2} \) belong to \( S \). The edge \( f_{3n-1} \) strongly dominates \( f_{3n-2} \). If the edge \( f_{3n-1} \) belongs to \( S \) then the edge \( f_{3n-1} \) is strongly dominated by two edges \( f_{3n-1} \) and \( f_1 \). Hence \( f_{3n+1} \) does not belong to \( S \). Therefore there is no edge in \( S \) to strongly dominate \( f_{3n+1} \), a contradiction. Hence \( F_{3n+1} \) has no strong efficient edge dominating set.
Case 2: Let \( m = 3n+2 \). Suppose \( S \) is a strong efficient edge dominating set of \( F_m \). As in case 1, suppose \( e_1 \) belongs to \( S \). \( e_1 \) strongly dominates \( e_2, e_3, e_4, \ldots, e_{3n}, f_1, f_{3n}, g_1 \) and all the edges \( h_i, 1 \leq i \leq 3n+1 \). Also the edges \( g_2, g_3, g_4, \ldots, g_{3n-1} \) belong to \( S \). Clearly \( f_2 \) and \( f_{3n} \) do not belong to \( S \). Then the edges \( f_3, f_{3n}, \ldots, f_{3n-3} \) belong to \( S \). The edge \( f_{3n-3} \) strongly dominates \( f_{3n-4} \) \& \( f_{3n-2} \). If the edge \( f_{3n-1} \) belongs to \( S \) then \( f_{3n} \) is strongly dominated by two edges \( f_{3n-1} \) and \( f_{3n+1} \). Therefore there is no edge in \( S \) to strongly dominate \( f_{3n+1} \), a contradiction. Hence \( F_m, m = 3n+2 \) has no strong efficient edge dominating set.

III. CONCLUSION

In this paper, strong efficient edge domination number of some standard graphs and cycle related graphs are determined.

REFERENCES