Abstract — Let \((L, \wedge, \vee)\) be a lattice. A graph \(G(V, E)\) is said to be \(L\)-magic if there exists a labeling \(f\) of the edges of \(G\) with the elements of \(L\) induces the vertex labeling \(f^+\) defined as \(f^+(v) = \bigvee_{u \in V} f(uv)\) taken over all edges \(uv\) incident at \(v\) is a constant and the constant is nothing but the least upper bound of \(L\) and also induces another vertex labeling \(f^-\) defined as \(f^-(v) = \bigwedge_{u \in V} f(uv)\) is also a constant and the constant is the greatest lower bound of \(L\). A graph is said to be \(L\)-magic if it admits \(L\)-magic labeling.

Keywords — \(L\) - magic labeling, \(L\) - magic graph, least upper bound, greatest lower bound.

I. INTRODUCTION

By a graph \(G(V, E)\) we mean \(G\) is a finite, simple, and undirected graph. Magic labelings were introduced by Sedlacek in 1963. Kong, Lee and Sun [3] used the term magic labeling for the labeling of edges with non negative integers such that for each vertex \(v\) the sum of the labels of all edges incident at \(v\) is same for all \(v\).

For any non trivial abelian group \(A\), under addition, a graph \(G\) is said to be \(A\) magic if there exists a labeling \(f\) of the edges of \(G\) with non zero elements of \(A\) such that, the vertex labeling \(f^+\) defined as \(f^+(v) = \sum_{u \in V} f(uv)\) taken over all edges \(uv\) incident at \(v\) is a constant.

This idea motivate us to define \(L\)-magic labeling. Let \((L, \wedge, \vee)\) be a Lattice, A graph \(G(V, E)\) is said to be \(L\)-magic if there exists a labeling \(f\) of the edges of \(G\) with the elements of \(L\) such that the vertex labeling \(f^+\) defined as \(f^+(v) = \bigvee_{u \in V} f(uv)\) considered overall edges \(uv\) incident at \(v\) is a constant and the constant is the least upper bound of the set \(L\) and another vertex labeling \(f^-\) defined as \(f^-(v) = \bigwedge_{u \in V} f(uv)\) taken over all edges incident at \(v\) is a constant and the constant is the greatest lower bound of \(L\).

A graph is said to be \(L\)-magic if it admits \(L\)-magic labeling.

In this paper, we consider a lattice with \(L = \{1, 2, 3\}\) and \(\leq\) is the "less than or equal to" relationship among numbers. By lub and glb we mean the least upper bound and greatest lower bound.

II. BASIC DEFINITIONS

Definition 2.1

A non empty set \(A\) on which a partial ordering relationship, (generally denoted by \(\leq\)) is defined is called a partially ordered set or poset and it is written as \((A, \leq)\).
Definition 2.2 [5]
A lattice is a poset (partially ordered set) \((L, \leq)\) in which every 2-element subset \(\{a, b\}\) has a lub and glb. That is, poset \((L, \leq)\) is a lattice if for every \(a, b \in L\), lub\((a, b)\) and glb\((a, b)\) exist in \(L\).

Definition 2.3
A nonempty set \(L\) closed under two binary operations \(\land\) and \(\lor\) is called a lattice \((L, \land, \lor)\) provided the following axioms hold.

1. (i) \(a \land a = a\)
   (ii) \(a \lor a = a\) for all \(a \in L\)
2. (i) \(a \land b = b \land a\)
   (ii) \(a \lor b = b \lor a\) for all \(a, b \in L\)
3. (i) \((a \land b) \land c = a \land (b \land c)\)
   (ii) \((a \lor b) \lor c = a \lor (b \lor c)\) for all \(a, b, c \in L\)
4. (i) \(a \land (a \lor b) = a\)
   (ii) \(a \lor (a \land b) = a\) for all \(a, b, c \in L\).

Definition 2.4
If \(b \leq a \lor b\) and \(b \leq a \land b\) then \(a \lor b\) is the upper bound of elements \(a, b \in L\). Also if \(c \leq a\) and \(c \leq b\) then \(a \lor b \leq c\) then \(a \lor b = \text{lub}\{a, b\}\) for all \(a, b \in L\). It is also denoted as \(a \lor b = \text{sup}(a, b)\).

Definition 2.5
If \(a \land b \leq a\) and \(a \land b \leq b\) then \(a \land b\) is the lower bound of elements \(a, b \in L\). Also if \(c \leq a\) and \(c \leq b\) then \(c \leq a \land b\) then \(a \land b = \text{glb}\{a, b\}\) for all \(a, b \in L\). It is also denoted as \(a \land b = \text{inf}(a, b)\).

Definition 2.6
Let \(G_1(V_1, E_1)\) and \(G_2(V_2, E_2)\) be two graphs. Then their union \(G = G_1 \cup G_2\) is a graph with the vertex set \(V = V_1 \cup V_2\) and edge set \(E = E_1 \cup E_2\).

Definition 2.7
The join \(G_1 + G_2\) of \(G_1\) and \(G_2\) consists of \(G_1 \cup G_2\) and all lines joining \(V_1\) with \(V_2\). The graph \(P_n + K_1\) is called a fan \(P_n + 2K_1\) is called the double fan. It is denoted as \(DF_n\). The graph \(C_n + K_1\) is called a cone or a wheel \(W_n\) with \(n\) spokes and the graph \(C_n + 2K_1\) is called the double cone. It is denoted as \(DC_n\).

Definition 2.8
The helm \(H_n\) is the graph obtained from a \(W_n\) by attaching a pendant edge at each vertex of the \(n\)-cycle of the wheel.

### III Main Results

Let us learn through the following theorem about \(L\)-magic labeling

**Theorem 3.1.**
\(C_n\) is \(L\)-magic for \(n \equiv 0\) (mod 2).

**Proof.** Let \(f : E(C_n) \rightarrow L\)

\[
\begin{align*}
f(u_{2i-1}u_{2i}) &= 1, \quad 1 \leq i \leq n/2 \\
f(u_{2i}u_{2i+1}) &= 3, \quad 1 \leq i \leq n/2 \quad (u_{n+1} \equiv u_1)
\end{align*}
\]
Let \(f^+ : V(C_n) \rightarrow L\)
By Definition
\[
f^+(u_i) = \bigvee_{u \in V} f(uu_i) = f(u_{i-1}u_i) \lor f(u_iu_{i+1})
\]
\[
= 1 \lor 3 = \text{lub}\{1,3\}
\]
\[
= 3, \ 1 \leq i \leq n \ (u_0 = u_n)
\]

Let \( f^- : V(c_n) \rightarrow L \)
\[
f^-(u_i) = \bigwedge_{u \in V} f(uu_i) = f(u_iu_{i+1}) \land f(u_{i-1}u_i)
\]
\[
= 1 \land 3
\]
\[
= \text{glb}\{1,3\} = 1, \ 1 \leq i \leq n \ (u_0 = u_n)
\]
\( f^+(v) \) is a constant and \( f^-(v) \) is also a constant for all \( v \in C_n \). Therefore \( C_n \) is \( L \)-magic for \( n \equiv 0 \ (\text{mod} \ 2) \).

**Example 3.2.** \( L \)-magic labeling is given for \( C_6 \).

![Fig. 1 L - magic labeling of C_6](image)

**Theorem 3.3.**

\( W_n \) is \( L \)-magic for \( n \geq 3 \).

**Proof.** Let \( V(W_n) = \{u\} \cup \{u_i/1 \leq i \leq n\} \).

\( E(W_n) = \{uu_i/1 \leq i \leq n\} \cup \{u_iu_{i+1}/1 \leq i \leq n\}\{u_{n+1} \equiv u_1\} \)

**case 1** \( n \equiv 1 \ (\text{mod} \ 2) \)

Let \( f : E(W_n) \rightarrow L \) be defined as
\[
f(uu_i) = 1 \text{ and } f(uu_n) = 3
\]
\[
f(uu_i) = 2, \ 2 \leq i \leq n-1
\]
\[
f(u_{2i-1}u_{2i}) = 3, \ 1 \leq i \leq \frac{n-1}{2}
\]
\[ f(u_{2i}, u_{2i+1}) = 1, \quad 1 \leq i \leq \frac{n-1}{2} \]
\[ f(u_{n}, u_{1}) = 2. \]

Now, \( f^+ : V(W_n) \rightarrow L \)
\[ f^+(u_i) = \bigvee_{u \in V} f(uu_i) \]
\[ = f(u_{i-1}u_i) \lor f(u_{i+1}u_i) \lor f(uu_i) \]
\[ = 2 \lor 3 \lor 1 \]
\[ = \text{lub}\{1,2,3\} = 3, \quad 1 \leq i \leq n. \]

\[ f^+(u) = \bigvee_{u \in V} f(uu_i) \]
\[ = f(uu_1) \lor f(uu_2) \lor ... \lor f(uu_n) \]
\[ = 1 \lor 2 \lor ... \lor 3 \]
\[ = \text{lub}\{1,2,3\} = 3. \]

\[ f^- : V(W_n) \rightarrow L \]
\[ f^-(u_i) = \bigwedge_{u \in V} f(uu_i) \]
\[ = f(u_{i-1}u_i) \land f(u_{i+1}u_i) \land f(uu_i) \]
\[ = 2 \land 3 \land 1 \]
\[ = \text{glb}\{1,2,3\} = 1, \quad 1 \leq i \leq n. \]

\[ f^-(u) = \bigwedge_{u \in V} f(uu_i) \]
\[ = f(uu_1) \land f(uu_2) \land ... \land f(uu_n) \]
\[ = 1 \land 2 \land ... \land 3 \]
\[ = \text{glb}\{1,2,3\} = 1. \]

Hence \( f^+(v) \) and \( f^-(v) \) are constant for all \( v \in V \).

\textbf{case 2} \quad n \equiv 0 \pmod{2}

Let \( f : E(W_n) \rightarrow L \) be defined as
\[ f(uu_i) = 1 \quad \text{and} \quad f(uu_n) = 3 \]
\[ f(uu_i) = 2, \quad 2 \leq i \leq n-1 \]
\[ f(u_{2i}, u_{2i+1}) = 3, \quad 1 \leq i \leq n/2 \]
\[ f(u_{2i}, u_{2i+1}) = 1, \quad 1 \leq i \leq n/2 \]

Now, \( f^+ : V(W_n) \rightarrow L \)
\[ f^+(u_i) = \bigvee_{u \in V} f(uu_i) \]
\[ = f(u_{i-1}u_i) \lor f(u_{i+1}u_i) \lor f(uu_i) \]
\[ = 3 \lor 1 \lor 2 \]
\[ = \text{lub}\{1,2,3\} \]
\[ f^+(u_i) = f(u_i, u_i) \lor (u_i, u_2) \lor f(uu_i) \]

\[ = 1 \lor 3 \lor 1 \]

\[ = lub\{1,3\} \]

\[ = 3 \]

\[ f^+(u_n) = f(u_{n-1}, u_n) \lor f(u_n, u_1) \lor f(uu_n) \]

\[ = 3 \lor 1 \lor 3 \]

\[ = lub\{1,3\} \]

\[ = 3. \]

\[ f^+(u) = \lor_{u\in V} f(uu_i) \]

\[ = f(uu_1) \lor f(uu_2) \lor \ldots f(uu_n) \]

\[ = 1 \lor 2 \lor \ldots \lor 3 \]

\[ = lub\{1,2,3\} \]

\[ = 3. \]

Now, \( f^- : V(W_n) \to L \)

\[ f^-(u_i) = \land_{u\in V} f(uu_i) \]

\[ = f(u_{i-1}, u_i) \land (u_i, u_{i+1}) \land f(uu_i) \]

\[ = 3 \land 1 \land 2 \]

\[ = glb\{1,2,3\} \]

\[ = 1, \; 2 \leq i \leq n-1. \]

\[ f^-(u_1) = f(u_1, u_2) \land f(u_2, u_3) \land f(uu_1) \]

\[ = 1 \land 3 \land 1 \]

\[ = glb\{1,3\} = 1 \]

\[ f^-(u_n) = f(u_{n-1}, u_n) \land f(u_n, u_1) \land f(uu_n) \]

\[ = 3 \land 1 \land 3 \]

\[ = glb\{1,3\} = 1 \]

\[ f^-(u) = \land_{u\in V} f(uu_i) \]

\[ = f(uu_1) \land f(uu_2) \land \ldots f(uu_n) \]

\[ = 1 \land 2 \land \ldots \land 3 \]

\[ = glb\{1,2,3\} = 1. \]

Hence \( f^+(v) \) and \( f^-(v) \) are constant for all \( v \in V(W_n) \)

\[ f^+(v) = 3 \] which is the least upper bound of \( L \) and \( f^-(v) = 1 \) which is the greatest lower bound of \( L \) for all \( v \in V(W_n) \).

Hence, \( W_n \) is \( L \)-magic for \( n \geq 3 \).
Example 3.4 The \( L \) - magic labeling of \( W_5 \) and \( W_8 \) are shown below.

![Fig. 2 L-magic labeling of \( W_5 \)](image)

![Fig. 3 L-magic labeling of \( W_8 \)](image)

Observation 3.5 In a similar way of labeling \( f(vu_i) \) as that of \( f(uu_i) \) in both the cases \( 1 \leq i \leq n \) we can prove the graph double cone is also \( L \) - magic.

![Fig. 4 L-magic labeling of \( DC_5 \)](image)

![Fig. 5 L-magic labeling of \( DC_6 \)](image)
Theorem 3.6

$F_n$ is $L$-magic for $n \geq 3$.

Proof. Let $V(F_n) = \{v_i/1 \leq i \leq n\} \cup \{u\}$ and $E(F_n) = \{uv_i/1 \leq i \leq n\} \cup \{v_i v_{i+1}/1 \leq i \leq n-1\}$

Case 1: $n$ be even

Let $f: E(F(n)) \rightarrow L$ be defined as

$f(v_{2i-1}v_{2i}) = 1, \ 1 \leq i \leq n/2$
$f(v_{2i})v_{2i+1} = 3, \ 1 \leq i \leq n/2 - 1$
$f(uv_i) = 3, \ 2 \leq i \leq n-2$ and $f(uv_{n-1}) = 1$
$f(uv_1) = f(uv_n) = 3$

Let $f^+: V(F_n) \rightarrow L$

$f^+(u) = \bigvee f(uv_i)$

$= f(uv_1) \lor f(uv_2) \lor ... \lor f(uv_{n-1}) \lor f(uv_n)$

$= 3 \lor 2 \lor ... \lor 1 \lor 3$

$= \text{lub}\{1,2,3\} = 3$

$f^+(v_i) = \bigvee f(v_{i-1}v_i)$

$= f(v_{i-1}v_i) \lor f(v_i v_{i+1}) \lor f(uv_i), \ 2 \leq i \leq n-2$

$= 1 \lor 3 \lor 2$

$= \text{lub}\{1,2,3\}$

$= 3, \ 2 \leq i \leq n-2$.

$f^+(v_{n-1}) = f(v_{n-2}v_{n-1}) \lor f(v_{n-1}v_n) \lor f(uv_{n-1})$

$= 3 \lor 1 \lor 1$

$= \text{lub}\{1,3\} = 3$

$f^+(v_1) = f(uv_1) \lor f(v_1 v_2)$

$= 3 \lor 1 = \text{lub}\{1,3\} = 3$
\( f^+(v_n) = f((uv_n) \lor f((v_{n-1}v_n)) \)
\[ = 3 \lor 1 \\
= lub\{1,3\} = 3 \]

Let \( f^- : V(F_n) \rightarrow L \)
\[ f^-(u) = \bigwedge_{v \in V} f(uv_v) \]
\[ = f(uv_1) \land f(uv_2) \land f(uv_3) ... \land f(uv_{n-1}) \land f(uv_n) \]
\[ = 3 \land 2 \land ... \land 1 \land 3 \]
\[ = glb\{1,2,3\} = 1 \]

\( f^-(v_i) = f(v_{i-1}v_i) \land f(v_{i+1}v_i) \land f(uv_i) \quad 2 \leq i \leq n-2 \)
\[ = 1 \land 3 \land 2 \]
\[ = glb\{1,2,3\} = 1 \quad 2 \leq i \leq n-2 \]

\( f^-(v_{n-1}) = f(v_{n-2}v_{n-1}) \land f(v_{n-1}v_n) \land f(v_{n-1}v_n) \land f(uv_{n-1}) \)
\[ = 3 \land 1 \land 1 \]
\[ = glb\{1,3\} = 1 \]

\( f^-(v_1) = f(uv_1) \land f(v_1v_2) \)
\[ = 3 \land 1 \]
\[ = glb\{1,3\} = 1 \]

\( f^+(v_n) = f(uv_n) \land f(v_{n-1}v_n) \)
\[ = 3 \land 1 \]
\[ = glb\{1,3\} = 1. \]

**case 2:** Let \( n \) be odd

Let \( f : E(F_n) \rightarrow L \) be defined as

\( f(v_{2i-1}v_{2i}) = 1, \quad 1 \leq i \leq \frac{n-1}{2} \)

\( f(v_{2i}v_{2i+1}) = 3, \quad 1 \leq i \leq \frac{n-1}{2} \)

\( f(uv_i) = 2, \quad 2 \leq i \leq n-1 \)

\( f(uv_1) = 3 \)

\( f(uv_n) = 1 \)

Now \( f^+ : V(F_n) \rightarrow L \)
\[ f^+(u) = \bigvee_{v \in V} f(uv_v) \]
\[ = f(uv_1) \lor f(uv_2) \lor ... \lor f(uv_n) \]
\[ = 3 \lor 2 \lor ... \lor 1 \]
\[ = lub\{1,2,3\} = 3 \]

\( f^+(v_i) = f(v_{i-1}v_i) \lor f(v_{i+1}v_i) \lor f(uv_i) \quad 2 \leq i \leq n-1 \)
\[ = 1 \lor 3 \lor 2 \]
In both the cases $f^+(v) = 3$ and $f^-(v) = 1$ for all $v \in V(F_n)$. Hence, $F_n$ is $L$-magic for $n \geq 3$.

**Example 3.7.**

![Fig.7 L - magic labeling of $F_5$](image1)

![Fig. 8 L - magic labeling of $F_4$](image2)
Theorem 3.8
The graph \( (\text{double fan}) \, DF_n \) is \( L \)-magic \( n \geq 3 \)

Proof. The graph has the following vertex set and Edge set.

\[
V(DF_n) = \{u\} \cup \{v\} \cup \{v_i/1 \leq i \leq n\} \quad \text{and}
E(DF_n) = \{uv_i/1 \leq i \leq m\} \cup \{v_i v_{i+1}/1 \leq i \leq n-1\} \cup \{vv_i/1 \leq i \leq n\}.
\]

The labeling of edges are same like the previous theorem (3.6).

In addition to the labeling of case 1 of theorem (3.6) we add the labeling.

\[
f(vv_i) = 2, \quad 2 \leq i \leq n - 2
\]

\[
f(vv_{n-1}) = 1,
\]

\[
f(vv_i) = f(vv_n) = 3.
\]

We get \( f^+(v), f^-(v) \) are constant for all \( v \in V(DF_n) \) and in case 2, we have to add, additionally the following label as

\[
f(vv_i) = 2, \quad 2 \leq i \leq n - 1
\]

\[
f(vv_n) = 3 \quad \text{and} \quad f(vv_n) = 1.
\]

We can verify \( f^+(v), f^-(v) \) are constant for all \( v \in V(DF_n) \).

Hence, \( DF_n \) is \( L \)-magic.

Example 3.9.

Fig.9 \( L \)-magic labeling of \( DF_5 \)

Observation 3.10
1. The above labeling can be extended to any lattice \( L \) of integers, for the same relationship "less than or equal to" we can prove the above graphs are \( L \)-magic provided the labeling of the edges having "3 and 1" should be replaced by the greatest and lowest element of \( L \), and the edges having the label "2" may be labeled by any number in between the glb and lub of \( L \).
2. Similarly, a suitable edge labeling is possible for any lattice, according to the relationship defined in the poset, the glb and the lub are found.

For example, if \( L = (1,2,3,4,6,12) \) and the relation is "divisibility" then \( a \lor b \) and \( a \land b \) are defined as \( a \lor b = \text{least} \).
common multiple of \{a, b\} and \(a \wedge b\) = greatest common divisor of \{a, b\}, then it can be easily checked that the above graphs are \(L\)-magic.

**Theorem 3.11**

The graph having pendant edge(s) cannot be \(L\)-magic.

**Proof.** While we are labeling the edges of the graph, the pendant vertex (vertices) will get only one labeling as one edge incident to it (them). So for any pendant vertex \(v\), \(f^+ (v)\) and \(f^- (v)\) are the same constant. It can be either lub of the lattice or glb of the lattice but for other vertices of the graph the vertex labelings \(f^+, f^-\) give two different constants namely lub of the lattice and glb of the lattice. Hence, the graph having pendant edge(s) cannot be \(L\)-magic.

**Example 3.12** We show \(P_4\) and \(H_5\) are not \(L\)-magic.

![Fig. 10](image)

**REFERENCES**