Some Results on Semi-Compactness

Navpreet Singh Noorie #1, Sandeep Kaur #2
#Department of Mathematics, Punjabi University, Patiala- 147002, India.

Abstract — We give necessary and sufficient conditions for a semi-regular space to be semi-compact and for a map to be semi-compact preserving (semi-compact) when domain (co-domain) of map is SCS.

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I. INTRODUCTION

Since the introduction of semi-open sets by Levine [4], various authors have investigated the corresponding concepts of semi-compactness, semi-regularity of SCS spaces (see [2], [3], [4] etc). It is known that a regular space is compact if and only if there exist a dense set D in X such that every net in D has a cluster point in X [7]. In this paper, we give the corresponding result for semi-regular spaces (Theorem 2.1 below). Further we give necessary and sufficient condition for a map $f : X \to Y$ to be semi-compact preserving (semi-compact) by using the concept of semi-closure when domain (range) of map is SCS (Theorem 2.3 below). A subset A in a topological space X is said to be semi-open [4] if and only if $A \subseteq \text{cl}(\text{Int}(A))$, or equivalently, if there exists an open subset U of X such that $U \subseteq A \subseteq \text{cl}(U)$. A is called semi-closed if $X - A$ is semi-open. The semi-closure $\text{scl}(A)$ of a subset A of a space X is the intersection of all semi-closed subsets of X that contain A, or equivalently, the smallest semi-closed subset of X that contains A. A space X is called semi-compact [2] if any semi-open cover of X has a finite subcover. A space X is said to be semi-regular [3] if for each $x \in X$ and every semi-open set $U$ containing x there exist semi-open set $V$ containing $x$ such that $V \subseteq \text{scl}(V) \subseteq U$. A net $\{x_n\}$ is semi-converges [2] (semi-clusters [1]) at $x$ if and only if $\{x_n\}$ is eventually (frequently) in every semi-open set containing x. A space is said to be SCS [8] if any subset of X which is semi-compact is semi-closed.

Notation: Throughout this paper, X and Y will denote arbitrary topological spaces. For a subset A of a space X, $\text{scl}(A)$ will denote the semi-closure of A.

We will also make use of following results:

Theorem 1.1: ([1]) A space X is semi-compact if and only if every net in X has a semi-cluster point in X.

Theorem 1.2: (Theorem 2.4; [5]) Let X be a topological space. Then D is dense in X if and only if $\text{scl}(D) = X$.

II. RESULTS

We begin with the following definitions.

Definition 2.1. A subset A of X will be called relatively semi-compact if $\text{scl}(A)$ is semi-compact.
Remark 2.1. Since semi-closed subsets of semi-compact spaces are semi-compact [6], therefore every subset of semi-compact space is relatively semi-compact.

Definition 2.2. A map $f : X \to Y$ is called semi-compact preserving (semi-compact) if the image (inverse image) of a semi-compact subset of $X$ ($Y$) is semi-compact in $Y$ ($X$).

The following characterization of compactness for regular spaces is known.

Theorem (Ex 201 and 202, Sec 7.2 of [7]). A regular space $X$ is compact if and only if there exists a dense subset $D$ of $X$ such that every net in $D$ has a cluster point in $X$.

Our first result gives a similar characterization of semi-compactness for semi-regular spaces.

Theorem 2.1. A semi-regular space $X$ is semi-compact if and only if there exists a dense subset $D$ of $X$ such that every net in $D$ has a semi-cluster point in $X$.

Proof: Equivalently, we prove if there exist a dense subset $D$ of $X$ such that every filterbase in $D$ has a semi-cluster point in $X$ then $X$ is semi-compact. Let $D$ be such a dense set. Assume $X$ is not semi-compact, then there exist a cover $\{U_\alpha\}$ of semi-open set in $X$ with no finite subcover. Since $X$ is semi-regular, there exists semi-open cover $\{V_\beta\}$ of $X$ such that for each $\beta$ there exist $\alpha$ such that $\text{scl}(V_\beta) \subseteq U_\alpha$. By Theorem 1.2 above, since $X = \text{scl}(D)$, $\{V_\beta\}$ is a semi-open cover of $\text{scl}(D)$ with no finite subcover. Therefore, the collection $B = \{D \cup V_\beta \mid \text{for any positive integer } n\}$ is a filterbase in $D$. By assumption, $B$ has a semi-cluster point $x$. Then $x \in \text{scl}(D)$ implies $x \in V_\beta$ for some $\beta$ and so $V_\beta$ is a semi-open set containing $x$. Then $(D - V_\beta) \cap V_\beta = \emptyset$ contradicts the fact that $x$ is semi-cluster point of $B$. Hence $\text{scl}(D) = X$ is semi-compact. The converse follows immediately from Theorem 1.1 above.

Above Theorem motivates the following Definition:

Definition 2.1. A set $A$ in a space $X$ will be called semi-clustering if every net in $A$ has a semi-cluster point in $X$.

Corollary 2.1. In a semi-regular space which is not semi-compact, every dense subset is non semi-clustering.

The following result characterizes relatively semi-compactness for semi-regular spaces which is corollary to the above Theorem 2.1.

Theorem 2.2. In a semi-regular space $X$, a set $A$ is relatively semi-compact if and only if $A$ is semi-clustering in $X$.

For our next result, we make the following Definition.

Definition. A net $\{x_\alpha\}$ in a space $X$ will be called relatively semi-compact in $X$ if its range is relatively semi-compact in $X$. 

The following Theorem characterizes semi-compact preserving (semi-compact) maps \( f : X \rightarrow Y \) in terms of semi-cluster point of relatively semi-compact nets, where \( X (Y) \) is assumed to be SCS.

**Theorem 2.3.** Let \( f : X \rightarrow Y \) be a map, where \( X (Y) \) is SCS. Then \( f \) is semi-compact preserving (semi-compact) if and only if for every relatively semi-compact net \( \{x_\alpha\} \) in \( X (\{f(x_\alpha)\} \) in \( Y \) with range \( S = \cup_\alpha \{x_\alpha\}, (S = \cup_\alpha \{f(x_\alpha)\}) \), the net \( \{f(x_\alpha)\} \) has a semi-cluster point in \( f(\text{scl}(S)) \) (the net \( \{x_\alpha\} \) has a semi-cluster point in \( f^{-1}(\text{scl}(S)) \)).

**Proof:** For arbitrary spaces \( X \) and \( Y \), if \( f \) is semi-compact preserving (semi-compact) and \( \{x_\alpha\} \) (\( \{f(x_\alpha)\} \)) is a relatively semi-compact net, then \( \{f(x_\alpha)\} \) is a net in the semi-compact set \( f(\text{scl}(S)) \) (\( \{x_\alpha\} \) is a net in the semi-compact set \( f^{-1}(\text{scl}(S)) \)) and so has a semi-cluster point in \( f(\text{scl}(S)) \) (\( f^{-1}(\text{scl}(S)) \)) by Theorem 1.1 above, proving the necessity of the condition. Conversely, let \( X (Y) \) be SCS and assume that for every net \( \{x_\alpha\} \) in \( X (\{f(x_\alpha)\} \) in \( Y \) with relatively semi-compact range \( S \), the net \( \{f(x_\alpha)\} \) has a semi-cluster point in \( f(\text{scl}(S)) \) (the net \( \{x_\alpha\} \) has a semi-cluster point in \( f^{-1}(\text{scl}(S)) \)). Let \( K \) be a semi-compact subset of \( X (Y) \) and \( \{f(x_\alpha)\} \) be any net in \( f(K) \) (\( \{x_\alpha\} \) be any net in \( f^{-1}(K) \)). Then \( \{x_\alpha\} \) may be taken to be a net in \( K \) (\( \{f(x_\alpha)\} \) is a net in \( K \)) and since \( X (Y) \) is SCS, \( K \) is semi-closed and so \( \{x_\alpha\} \) (\( \{f(x_\alpha)\} \)) is a relatively semi-compact net. Therefore, by assumption, the net \( \{f(x_\alpha)\} \) has a semi-cluster point in \( f(K) \) (the net \( \{x_\alpha\} \) has a semi-cluster point in \( f^{-1}(K) \)) and so \( f(K) \) (\( f^{-1}(K) \)) is semi-compact. Hence \( f \) is semi-compact preserving (semi-compact).

**References**