The Non-Linear Oscillation of the Centre of Mass of the System in Elliptic Orbit under the Influence of the Shadow of the Earth due to Solar Radiation Pressure, Magnetic Force and Oblateness of the Earth.

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Abstract
We have studied the non-linear oscillation of the system of the satellites connected by light, flexible and extensible cable under the influence of Earth’s magnetic force, the shadow of the earth due to solar radiation pressure and earth oblateness in the case of elliptic orbit of the Centre of mass of the system. The non-linear terms present in the equations of motion of the system are taken into consideration. First of all we have derived equations of motion for non-linear oscillations and a system of equation representing almost periodic oscillations due to Malkin. An attempt has been made to analyse the motion and stability of the system analytically. As there is no periodic terms in the equation of motion, so only non-resonant solution have been obtained and shown to be stable.

Keywords: Stability, Non-linear oscillation, Solar radiation pressure, Earth Magnetic force, Satellites, Elliptic orbit.

1. Introduction
This paper deals with the study the effect of shadow of the earth due to solar radiation pressure, magnetic force and earth’s oblateness on non-linear oscillation and stability of two satellites connected by light, flexible and extensible cable in the central gravitational field of earth in an elliptic orbit of the centre of mass of the system in case of two dimensional motion. Beletsky, V.V. is the pioneer worker in this field. This paper is an attempt towards the generalization of works done by him.

2. Equations of Motion for Non-Linear oscillation of the Centre of Mass of the System.
The equations of motion in elliptic orbit of the centre of mass of a system of two satellites connected by a light, flexible and extensible cable under the influence of the shadow of the earth due to solar radiation pressure, magnetic force and oblateness of the earth in two dimension case in Nechvile’s coordinates system are given by

\[ x'' - 2y' + 3x - \frac{4Bx}{\rho} + A\rho^3 \psi \cos \varepsilon \cos (\nu - \alpha) = \frac{-\mu}{\rho^4} \left(1 - \frac{\nu}{\rho^2}\right) x - \frac{C \cos i}{\rho} \]

and

\[ y'' + 2x' + \frac{B y}{\rho} - A\rho^3 \psi \cos \varepsilon \sin (\nu - \alpha) = \frac{-\mu}{\rho^4} \left(1 - \frac{\nu}{\rho^2}\right) y - \frac{\rho^3 \cos i}{\rho^2} \]

Where, \( r = \sqrt{x^2 + y^2}, \rho = \frac{1}{1 + \epsilon \cos \varphi} \), \( A = \frac{p^3}{\mu} \left[ \frac{B_1}{m_1} - \frac{B_2}{m_2} \right] \) = Solar pressure parameter

\( \psi \) = The shadow function parameter

\( B = \frac{3k_2}{p^2} \) = Oblatness parameter

\( C = \left[ \frac{Q_1}{m_1} - \frac{Q_2}{m_2} \right] \frac{\mu_e}{\sqrt{\mu_0}} \) = Magnetic force parameter
\[ \bar{\lambda}_a = \frac{p^3}{\mu} = \frac{p}{\mu m_1 m_2} \lambda \]

\[ \lambda_{\text{mod}} = \frac{\rho^3}{\mu} = \frac{p_3}{\mu} \left( m_1 + m_2 \right) \lambda \]

\[ \bar{\lambda}_a = \frac{1}{1 + e \cos \nu} = 1 - e \cos \nu + e^2 \cos^2 \nu - \ldots \]

\[ \rho = \frac{1}{1 + e \cos \nu} = (1 + e \cos \nu)^{-1} = 1 - e \cos \nu + e^2 \cos^2 \nu - \ldots \]

\[ \rho^3 = (1 + e \cos \nu)^{-3} = 1 - 3e \cos \nu + \ldots \]

\[ \rho^4 = (1 + e \cos \nu)^{-4} = 1 - 4e \cos \nu + \ldots \]

\[ \rho' = \frac{e \sin \nu}{(1 + e \cos \nu)^2} = e \sin \nu (1 + e \cos \nu)^{-2} = e \sin \nu - 2e^2 \sin \nu \cos \nu + \ldots \]

\[ \rho'' = \frac{e \sin \nu}{(1 + e \cos \nu)^3} = e \sin \nu (1 + e \cos \nu)^{-3} = e \sin \nu - 3e^2 \sin \nu \cos \nu + \ldots \]

\[ \rho''' = \frac{e \sin \nu}{(1 + e \cos \nu)^4} = e \sin \nu (1 + e \cos \nu)^{-4} = e \sin \nu - 4e^2 \sin \nu \cos \nu + \ldots \]

\[ \rho'''' = \frac{e \sin \nu}{(1 + e \cos \nu)^5} = e \sin \nu (1 + e \cos \nu)^{-5} = e \sin \nu - 5e^2 \sin \nu \cos \nu + \ldots \]

Putting the value of \( \rho, \rho^3, \rho^4 \) from [2.3] in [2.1] and neglecting the 2nd higher order terms containing \( e \) in their expansions, we get

\[ x'' - 2y' - 3(1 - e \cos \nu) x - 4B(1 + e \cos \nu) x + A(1 - 3e \cos \nu) \mu \cos \nu \cos \nu = \cos (\nu - \alpha) \]

\[ = -\bar{\lambda}_a \left[ (1 - 4e \cos \nu) - \frac{\ell_0}{r} (1 - 3e \cos \nu) \right] x - c(1 + e \cos \nu) \cos \nu \cos \nu \]

\[ y'' + 2x' + B(1 + e \cos \nu) y - A(1 - 3e \cos \nu) \mu \cos \nu \cos \nu = \sin (\nu - \alpha) \]

\[ = -\bar{\lambda}_a \left[ (1 - 4e \cos \nu) - \frac{\ell_0}{r} (1 - 3e \cos \nu) \right] y - c \cos \nu \cos \nu \]

Now, we want to examine the effect of the shadow of the earth due to the solar radiation pressure, magnetic force and oblateness of the earth on the equilibrium position \( (a, o) \) for the non-linear oscillation of the system.

For this, Let \( \eta_1 \) and \( \eta_2 \) be the small variations in \( x \) and \( y \) coordinates at the given equilibrium point \( (a, o) \) of the system. Then we have

\[ x = a + \eta_1 \quad \text{and} \quad y = \eta_2 \]

\[ x' = \eta_1' \quad \text{and} \quad y' = \eta_2' \]

\[ x'' = \eta_1'' \quad \text{and} \quad y'' = \eta_2'' \]

\[ r^2 = x^2 + y^2 = (a + \eta_1)^2 + \eta_2^2 = a^2 + 2a \eta_1 + \eta_2^2 = a^2 \left[ 1 + \frac{2a \eta_1 + \eta_2^2}{a^2} \right] \]

\[ \therefore r = a \left[ 1 + \frac{2a \eta_1 + \eta_2^2}{a^2} \right]^{\frac{1}{2}} \]

\[ r_o = a \]

\[ r_{\text{eq}} = a \]

From (2.6) and (2.7), we have
\[
\frac{1}{r} = \frac{1}{r_0} \left[ 1 + \frac{\left(2r_0 \eta_1 + \eta_1^2 + \eta_2^2\right)}{r_0^2} \right]^{1/2}
\]  
...(2.8)

Now expanding the right hand side of [2.8] and retaining terms only up to third order in infinitesimals \(\eta_1\) and \(\eta_2\), we get after some simplifications.

\[
\frac{1}{r} = \frac{1}{r_0} - \frac{\eta_1}{r_0^2} + \frac{\eta_1^2}{2r_0^3} + \frac{3\eta_1^3}{2r_0^4} + \frac{3 \eta_1 \eta_2}{2r_0^4}
\]  
............. (2.9)

Substituting the values of \(x\) and \(y\) and their derivatives from [2.5] in [2.4], we get the variational equations of motion in the form:

\[
\eta_1' - 2\eta_2' - \left[\left(3 + 4B\right) + \left(4B - 3\right)e \cos \nu\right] \left(r_o + \eta_1\right) = -\lambda_o \left[(1 - 4e \cos v) - \frac{e}{r} \left(1 - 3e \cos v\right)\right] \left(r_o + \eta_1\right)
\]  
\[\text{and}\]
\[
A(1 - 3e \cos v) \nu_1 \cos \in \cos (\nu - \alpha) - c(1 + e \cos v) \cos i
\]
\[
\eta_2' - 2\eta_1' + B(1 + e \cos v) \eta_2 = -\lambda_o \left[(1 - 4e \cos v) - \frac{e}{r} \left(1 - 3e \cos v\right)\right] \eta_2
\]  
\[\text{and}\]
\[
+ A \psi_1 \cos \in \left(1 - 3e \cos v\right) \sin (\nu - \alpha) - C \sin v \cos i
\]  
.............. (2.10)

Putting the value of \(\frac{1}{r}\) from [2.9] in [2.10], we get after neglecting the higher order terms than the third in infinitesimals \(\eta_1\) and \(\eta_2\) and after some simplifications.

\[
\eta_1'' - 2\eta_2'' - m_1^2 \eta_1 = \frac{\lambda_o}{r_o^2} \left[\frac{3\eta_1^4}{\lambda_o} + \frac{4Br_0^4}{\lambda_o^2} - r_0^4 - \frac{e}{2r_o^2} \left(1 - 3e \cos v\right) \left(2r_o^3 - r_o \eta_2^2 + 5 \eta_1^2 + 2 \eta_2^2\right)\right]
\]
\[\text{+} \lambda_o \left\{ \frac{r_o^2}{6} \left[3A \psi_1 \cos \in \cos (\nu - \alpha) + \left(4\lambda_o + 4B - 3\right) \eta_1\right]
\]
\[\text{+} \cos i \left[\left(4\lambda_o + B - 3\right) r_0\right] \left[r_o^3 \cos v\right] - Q \cos i \left(\nu - \alpha\right)\right]
\]
\[\text{and}\]
\[
\eta_2'' + 2\eta_1'' - m_2^2 \eta_2 = \frac{\lambda_o}{r_o^2} \left[\frac{e}{2r_o} \left(3e \cos v - 1\right) \left(r_o \eta_1 \eta_2 - \eta_1^2 \eta_2 + \eta_2^3\right)\right]
\]
\[\text{+} \lambda_o \left\{ \left[4r_o^2 - B - 3e r_0\right] \eta_2 \cos v \cos \in \sin (\nu - \alpha) - \frac{c r_o^2}{\lambda_o \cos i} \sin \cos i\right]\]  
\[\text{+} Q \sin \left(\nu - \alpha\right)\]

\[
\text{Where}\ Q = A \psi_1 \cos \in
\]
\[\text{[2.11] can be re-written as}\]
\[
\eta_1'' - 2\eta_2'' - m_1^2 \eta_1 = \mu f_1 - Q \cos (\nu - \alpha)
\]
\[\text{and}\]
\[
\eta_2'' + 2\eta_1'' - m_2^2 \eta_2 = \mu f_2 + Q \sin \left(\nu - \alpha\right)
\]

\[\text{Where}\]
\[
m_1^2 = 3 + 4B - \lambda_o, \quad m_2^2 = \frac{\lambda_o}{r_o^3} \left(3 - B - \lambda_o\right). \quad Q = A \psi_1 \cos \in, \quad \mu = \frac{\lambda_o}{r_o^3}
\]
\[
f_1 = \left[\frac{3\eta_1^4}{\lambda_o} + \frac{4Br_0^4}{\lambda_o^2} - r_0^4\right] c. \cos i - \frac{e}{2r_o^2} \left(1 - 3e \cos v\right) \left(2r_o^3 - r_o \eta_2^2 + 5 \eta_1^2 + 2 \eta_2^2\right)
\]
The genera

\[ H_1 = \frac{\mu_0}{2} (3e\cos \nu - 1) \left( r_0 \eta_1 \eta_2 - \eta_2^2 + \eta_2^3 \right) \]

\[ + e \left( 4r_0^2 - B - 3e r_0^2 \right) \eta_2 \cos \nu - \frac{3r_0^3}{\lambda_\alpha} A \psi_1 \cos \psi \cos \nu + \frac{e r_0^3}{\lambda_\alpha} \sin \psi \cos \nu \]

Thus, the system of equations given by [2.12] represents the non-linear oscillation of the system at the equilibrium point \((a_0, 0)\). We see that it represents the almost periodic oscillation due to Malkin.

3. Non-resonant solution of the equations and its stability

The general solution of linear part of [2.12] which can be obtained by putting \(\mu=0\), can be written in the form:

\[ \eta_1 = a_1 \sin \phi_1 + a_2 \sin \phi_2 + A_1 \cos \nu - \alpha \]

\[ \eta_2 = a_1 \omega_1 \sin \phi_1 + a_2 \omega_2 \sin \phi_2 - A_1 \sin \nu + \alpha \]

\[ \eta_1' = a_1 K_1 \cos \phi_1 + a_2 K_2 \cos \phi_2 + A_1 \sin \nu - \alpha \]

\[ \eta_2' = -a_1 K_1 \omega_1 \sin \phi_1 - a_2 K_2 \omega_2 \sin \phi_2 + A_1 \cos \nu - \alpha \]

Where

\[ \phi_1 = \omega_1 \nu + \alpha_1, \quad \phi_2 = \omega_2 \nu + \alpha_2 \]

Here \(\alpha_1, \alpha_2, a_1, a_2\) are constants to be determined from the initial conditions and \(\omega_1, \omega_2\) are the roots of the characteristic equation.

\[ \omega^4 + (m_1^2 + m_2^2 - \theta) \omega^2 + m_1^2 + m_2^2 = 0 \]  

(3.2)

From (3.1), we have:

\[ \eta_1'' = -a_1 \omega_1^2 \sin \phi_1 - a_2 \omega_2^2 \sin \phi_2 - A_1 \cos \nu - \alpha \]

\[ \eta_2'' = -a_1 \omega_1^2 K_1^2 \cos \phi_1 - a_2 \omega_2^2 K_2^2 \cos \phi_2 - A_1 \sin \nu - \alpha \]

(3.3)

Thus, putting the values of \(\eta_1, \eta_2, \eta_1', \eta_2', \eta_1''\) and \(\eta_2''\) from [3.1] and [3.3] respectively in [2.12] when \(\mu = 0\), we get

\[ -\left[ a_1 \left( w_1^2 - 2K_1 w_1 + m_1^2 \right) \sin \phi_1 - a_2 \left( w_2^2 - 2K_2 w_2 + m_2^2 \right) \sin \phi_2 \right] \sin \phi_2 \]

\[ - A_1 + 2A_2 + m_1^2 A_1 \]  

\[ \cos \nu - \alpha \]

and

\[ -a_1 \left( K_1 w_1^2 - 2w_1 + m_1^2 K_1 \right) \cos \phi_1 - a_2 \left( K_2 w_2^2 - 2w_2 + m_2^2 K_2 \right) \cos \phi_2 \]

\[ - A_1 + 2A_2 + m_2^2 A_2 \sin \nu - \alpha \]

(3.4)

Equations of (3.4) will be identically satisfied if the coefficients of \(\sin \phi_1, \sin \phi_2, \cos (\nu - \alpha), \cos \phi_1, \cos \phi_2\) and sin \(\nu - \alpha\) vanish separately, so we get

\[ w_1^2 - 2K_1 w_1 + m_1^2 = 0 \]

\[ w_2^2 - 2K_2 w_2 + m_2^2 = 0 \]

\[ A_1 + 2A_2 + m_1^2 A_1 = Q \]

(3.5)

\[ K_1 w_1^2 - 2w_1 + m_1^2 K_1 = 0 \]

\[ K_2 w_2^2 - 2w_2 + m_2^2 K_2 = 0 \]

\[ A_1 + 2A_2 + m_2^2 A_2 = -Q \]

(3.6)

From (3.5) and (3.6), we get
K_1 = \frac{w_1 + m_1^2}{2w_1}, \quad K_2 = \frac{w_2 + m_2^2}{2w_2}
A_1 = \frac{(m_1^2 + 3)Q}{m_1^2 + m_2^2 + m_1^2 m_2^2 - 3}, \quad A_2 = \frac{-(m_1^2 + 3)}{m_1^2 + m_2^2 + m_1^2 m_2^2 - 3}

(3.7)

Now, we shall study the general solution of the entire non-linear equations (2.12) with \( \mu \neq 0 \) (i.e. \( f_1 \neq 0, H_1 \neq 0 \)). For this, we exploit the method of variation of arbitrary constants in our further studies.

Here the amplitude and the phase will now be taken as functions of \( v \) but not constants as in linear case.

Thus, we get
\[
\eta_1' = a_1' \sin \phi_1 + a_2' \sin \phi_2 + A_4 \cos(v - \alpha)
\]
\[
\eta_2' = a_1' \cos \phi_1 + a_2' \cos \phi_2 + A_4 \sin(v - \alpha)
\]

Comparing the values of \( \eta_1 \) and \( \eta_2 \) in the system of equations \([4.3.8]\) and \([4.3.1]\), we get by subtraction:
\[
a_1' \sin \phi_1 + a_2' \sin \phi_2 + (a_1' \cos \phi_1 + a_2' \cos \phi_2 - a_1 k_1 \cos \phi_1 + a_2 k_2 \cos \phi_2) = 0
\]

In two cases when \( f_1 \neq 0, H_1 \neq 0 \) and \( f_1 = 0, H_1 = 0 \), substituting the values of \( \eta_1 \) and \( \eta_2 \) and their derivatives from \([3.1], [3.3]\) and \([3.8]\), we get on using \([3.7]\).
\[
a_1' w_1 \cos \phi_1 - a_1' w_1 \sin \phi_1 + a_1' w_2 \cos \phi_2 - a_2' w_2 \cos \phi_2 = \mu f_1
\]

Multiplying the first equation of \([3.9]\) by \( k_1 w_1 \) and then adding it to the 2nd equation of \([3.10]\), we get
\[
a_1' \sin \phi_1 + a_2' (w_1 K_1 - w_2 K_2) \cos \phi_2 = \mu H_1
\]

Again, multiply the 2nd equation of \([3.9]\) by \( k_2 w_2 \) and adding it the 2nd equation of \([3.10]\), we get
\[
a_1' (w_2 K_2 - w_1 K_1) \sin \phi_1 + a_1' (w_2 K_2 - w_1 K_1) \cos \phi_1 = \mu H_1
\]

Again, multiplying the 2nd equation of \([3.9]\) by \( w_1 \) and subtracting it from \( k_1 \) times the first equation of \([3.10]\), we get
\[
a_1' (w_2 K_2 - w_1 K_1) \cos \phi_2 + a_2' (w_2 K_2 - w_1 K_1) \sin \phi_2 = \mu K_1 f_1
\]

Lastly, multiply the 2nd equation of \([3.9]\) by \( w_2 \) and then subtracting it from \( k_2 \) times the first equation of \([3.10]\), we get
\[
a_1' (w_1 K_1 - w_2 K_2) \cos \phi_1 + a_1' (w_1 K_1 - w_2 K_2) \sin \phi_1 = \mu K_2 f_1
\]

Now, putting the value of \( w_1 \) and \( w_2 \) from \([3.7]\) in \([3.11]\), \([3.12]\), \([3.13]\) and \([3.14]\), we get,
After solving these four equations of \(3.15\) for and, we get
\[
\begin{align*}
\dot{a}_1' &= -\mu H_1' \sin \phi_1 + K_2 f_1^* \cos \phi_1 \\
\dot{a}_2' &= \mu [H_1' \sin \phi_2 + K_1 f_1^* \cos \phi_2] \\
\phi_1' &= w_1 + \frac{\mu}{a_1} \left[ -H_1^* \cos \phi_1 + K_2 f_1^* \sin \phi_1 \right] \\
\phi_2' &= w_2 + \frac{\mu}{a_2} \left[ H_1^* \cos \phi_2 - K_1 f_1^* \sin \phi_2 \right]
\end{align*}
\]  

Where,
\[
H_1' = \frac{-2H_1}{w_2 - w_1}, \quad f_1^* = \frac{-2m_2 f_1}{m_1(w_2^2 - w_1^2)} \quad \text{and} \quad \frac{w_1 w_2}{m_1^2} = \frac{m_2}{m_1}
\]  

Thus, on considering \(a_1, a_2, \phi_1\) and \(\phi_2\) as variables, we get a new system of four variations of motion given in \(3.16\).

It we put on the right hand side of the system of equations \(3.16\), the values of \(f_1^*\) and \(H_1^*\) in terms of \(f_1\) and \(H_1\) respectively from \(2.13\) and then the values of \(\eta_1\) and \(\eta_2\) from \(3.1\), then the right hand side forms of the expression are expanded into trigonometrical sums and averaged values of the variables are taken after dripping all the terms in the system of equations \(3.16\) except the free terms, we get the system of equations for first approximation as:
\[
\begin{align*}
\dot{a}_1' &= 0, \quad \dot{a}_2' = 0, \quad \phi_1' = w_1^*, \quad \phi_2' = w_2^*
\end{align*}
\]  

Where \(W_1^*\) and \(W_2^*\) are the new frequencies depending on \(w_1, w_2\) and constant quantities \(a_1, a_2, A_1, A_2, r_0, m_1, m_2, k_1\) and \(k_2\) and hence on integration, we get from \(3.18\).
\[
\begin{align*}
\phi_1 &= w_1^* v + \epsilon_1, \quad \phi_2 = w_2^* v + \epsilon_2
\end{align*}
\]  

where \(\epsilon_1\) and \(\epsilon_2\) are constants, Thus we see that in the relation \(3.19\) : \(a_1\) and \(a_2\) remain constant where as the values of \(\phi_1\) and \(\phi_2\) are slightly changed in the first approximation which indicates the change in the frequencies. But it has no effect on stability.

Thus, in the first approximation, the solutions of the equations of non-linear oscillation \(2.12\) can be written as -
\[
\begin{align*}
\eta_1 &= a_1^* \sin(w_1^* v + \epsilon_1) + a_2^* \sin(w_2^* v + \epsilon_2) + A_1 \cos(v - \alpha) \\
\eta_2 &= a_1^* K_1 \cos(w_1^* v + \epsilon_1) + a_2^* K_2 \cos(w_2^* v + \epsilon_2) + A_2 \sin(v - \alpha)
\end{align*}
\]  

Where \(a_1^*, a_2^*, \epsilon_1\) and \(\epsilon_2\) are arbitrary constant and \(w_1^*\) and \(w_2^*\) will be new frequencies, the values of \(A_1\) and \(A_2\) are given in \(3.6\).

Finally, we conclude that the solutions given in \(3.20\) will be stable.

4. **RESONANT SOLUTION OF THE EQUATION AND ITS STABILITY**

In this section, we shall examine the system of equations \(2.12\) with the supposition that the oscillation is of resonance type.
In case of resonance oscillation, we suppose that
\[ w_2 = 1 \quad \text{and} \quad Q = \mu Q' \] ........................ (4.1)

Which are customary in the resonance case of oscillation. In its absence, the generating system will have no almost periodic solution.

Thus, the system of equations [2.12] can be put in the form:
\[
\begin{align*}
\eta_1' - 2\eta_2' - m_1^2\eta_1 &= \mu f_1 - \mu Q' \cos(v - \alpha) \\
\eta_2' - 2\eta_2' - m_2^2\eta_2 &= \mu f_1 + \mu Q' \sin(v - \alpha)
\end{align*}
\] ........................ (4.2)

Where, \( f_1 \) and \( H_1 \) have their usual meanings given in [2.13].

In this case, the particular solutions of [4.2] for \( f_1 = 0 \) and \( H_1 = 0 \) can be assumed to be in the form -
\[
\begin{align*}
\eta_1 &= a \sin \phi + M_1 \cos(v - \alpha) + M_2 \sin(v - \alpha) \\
\eta_2 &= aK_1 \cos \phi + M_2 K_2 \cos(v - \alpha) - M_1 K_2 \sin(v - \alpha)
\end{align*}
\] ........................(4.3)

where \( \phi = w_1 v + \alpha_3 \); \( a, \alpha_3, M_1, M_2 \) are constants.

From (4.4.4), we get
\[
\begin{align*}
\eta_1' &= aw_1 \sin \phi - M_1 \sin(v - \alpha) + M_2 \cos(v - \alpha) \\
\eta_2' &= -aK_1w_1 \sin \phi - M_2 \sin(v - \alpha) - M_1K_2 \cos(v - \alpha)
\end{align*}
\] ........................ (4.4)

In a similar way just as in the preceding sections of this chapter and keeping in mind that \( w_2 = 1 \), we get from [4.3.7]
\[
\begin{align*}
K_1 &= \frac{w_1 + M_1^2}{2w_1}, \quad K_2 = 1 + \frac{M_2^2}{2} = \frac{2}{1 + M_2^2} 
\end{align*}
\] ........................ (4.5)

From [4.4.4], we get
\[
\begin{align*}
\eta_1' &= -aw_1^2 \sin \phi - M_1 \cos(v - \alpha) - M_2 \sin(v - \alpha) \\
\eta_2' &= -aK_1w_1^2 \sin \phi + M_2 \cos(v - \alpha) - M_1 K_2 \sin(v - \alpha)
\end{align*}
\] ........................ (4.6)

Now, similar to the non-resonance case, we shall investigate the general solution to the system of equations [4.2] representing the non-linear oscillation, when \( f_1 \neq 0 \) and \( H_1 \neq 0 \).

We shall assume that \( a, M_1, M_2 \) and \( \phi \) are new variables like the previous section of this chapter, we get on solving the system of equations for \( M_1', M_2' \), \( a' \) and \( \phi' \) obtained in the form:
\[
\begin{align*}
M_1' &= \mu [H_1'^{**} \cos(v - \alpha) - K_1 f_1'^{**} \sin(v - \alpha)] \\
M_2' &= \mu [H_1'^{**} \sin(v - \alpha) + K_2 f_1'^{**} \cos(v - \alpha)] \\
a' &= \mu [-H_1'^{**} \sin \phi - K_2 f_1'^{**} \cos \phi], \quad \phi' = w_1 + \frac{\mu}{a} [H_1'^{**} \cos \theta + K_2 f_1'^{**} \sin \theta]
\end{align*}
\] ........................ (4.7)

Where, \( H_1'^{**} = \frac{2Q' \sin(v - \alpha)}{w_1^2 - 1} + \frac{2M_2Q' \cos(v - \alpha)}{w_1^2 - 1} + f_1'^{**} \) ........................ (4.8)

The values of \( H_1'^{**} \) and \( f_1'^{**} \) can be given from [3.17] in case of resonance oscillation where \( \omega_2 = 1 \) as
\[
\begin{align*}
H_1'^{**} &= \frac{-2H_1^*}{1 - \omega_2^2}, \quad f_1'^{**} = \frac{-2M_2 f_1}{M_1 (1 - \omega_2^2)}
\end{align*}
\] ........................ (4.9)

In order to get the first approximate solution of the system of equation [4.7], we shall put in the right hand sides of [4.7], the different values from [4.8], [4.9] and [2.13] and then the values of \( \eta_1 \) and \( \eta_2 \) from [3.1]. Now after dropping the other terms except the free terms, we take the averaged terms into trigonometrical sums as mentioned in the previous section of this chapter, the set of equations [4.7] can be written as:
\[
M_1' = \frac{\mu}{4(w_1^2 - 1)} \left[ \left( M_1^2 K_2 + 3K_2^3 \right) - \frac{6K_1 m_2}{m_1} + \frac{K_1 K_2^2 m_2}{m_1} \right] + M_2^2 \left( -2K_2 + \frac{3}{2}K_2^2 - \frac{3K_1 m_2}{m_1} + \frac{K_1 K_2^2 m_2}{m_1} \right)
\]
\[ +a^2 \left( -2K_2 + 3K_1^2 K_2 - \frac{6K_1 m_2}{m_1} + \frac{K_1^3 m_2}{m_1} \right) M_2 + 6\epsilon_0 M_2^3 \left( K_2 - \frac{K_1 m_2}{m_1} \right) \sin \alpha - M_2^2 \left( K_2 + \frac{K_1 m_2}{m_1} \right) \sin \alpha \]

\[-\frac{2K_1 m_2}{m_1} M_1 M_2 \cos \alpha \]

\[ M'_2 = \frac{\mu}{4(w_i^2 - 1)} \left[ M_1^2 \left( 2K_2 - \frac{3}{2} K_2^2 + \frac{3K_1 m_2}{m_1} - \frac{K_1 K_2 m_2}{m_1} \right) + M_2^2 \left( -K_2 + 3K_2^2 + \frac{6K_1 m_2}{m_1} + \frac{K_1 K_2 m_2}{m_1} \right) \right] + \frac{Q^*}{(w_i^2 - 1)} \left( 1 + \frac{K_1 m_2}{m_1} \right) \]

\[ a' = -\frac{\mu^3 \epsilon_0 K_2}{2(w_i^2 - 1)} \left[ M_1 \sin \alpha + M_2 \cos \alpha \right] \]

\[ \phi' = w_i + \frac{\mu}{2(w_i^2 - 1)} \left[ M_1^2 \left( 2K_1 - 3K_1 K_2 + \frac{6K_2 m_2}{m_1} - \frac{K_2^2 m_2}{m_1} \right) + M_2^2 \left( 2K_1 - 3K_1 K_2 + \frac{6K_2 m_2}{m_1} + \frac{K_2^2 m_2}{m_1} \right) \right] \]

\[ + a^2 \left( 2K_2 - \frac{3}{2} K_1^2 + \frac{3K_2 m_2}{m_1} - \frac{K_1 K_2 m_2}{m_1} \right) - 12\epsilon_0 M_1 \left( \frac{K_2 m_2}{m_1} + \frac{K_1}{2} \right) \cos \alpha - M_2 \left( \frac{K_2 m_2}{m_1} + \frac{K_1}{2} \right) \sin \alpha \]
\[ b = \frac{1}{4(w_1^2 - 1)}, \quad b_1 = \frac{3er_0}{2(w_1^2 - 1)}, \quad E_1 = K_2 + 3K_2^3 - \frac{6K_2m_2}{m_1} - \frac{K_2^2m_2}{m_1} \]
\[ E_2 = 2K_2 - \frac{3}{2}K_1^2 + \frac{3K_1m_2}{m_1} - \frac{K_2K_1^2m_2}{m_1}, \quad E_3 = 2K_2 + 3K_1^2K_2 + \frac{6K_1m_2}{m_1} - \frac{K_2^2m_2}{m_1} \]
\[ E_4 = 2K_1^2 - \frac{3}{2}K_1K_2^2 + \frac{6K_2m_2}{m_1} - \frac{K_3m_2}{m_1}, \quad \text{and} \quad E_5 = 2K_1^2 + \frac{3K_1m_2}{m_1} - \frac{K_2K_1^2m_2}{m_1} \]
\[
\begin{array}{c}
E_1 = \sin \alpha, \\
E_2 = v + \eta, \\
E_3 = R_1M_1 + \frac{M_1a}{m_1}, \\
E_4 = \cos \alpha, \\
E_5 = \cos \alpha - \cos \alpha - \cos \alpha
\end{array}
\]

Here \( R_1, R_2 \) and \( R_3 \) stand for functional notations.

Now let us examine the system of equations [4.11] we see that the solutions given in [4.3] can be stable only when the given conditions are satisfied:
\[ R_1[M_1, M_2, a] = 0, \quad R_2[M_1, M_2, a] = 0, \quad R_3[M_1, M_2, a] = 0 \]

If the conditions mentioned in (4.13) are satisfied then the values of \( M_1, M_2 \) and \( a \) will remain constants and in that case the value of \( \phi = \omega^* \) will also be a constant quantity. Therefore, we shall have a new frequency \( \omega^* \) in place of \( \omega_1 \). But it will not affect the stability. Thus, in the first approximation, the stationary solutions of the system of equations [4.2] for non-linear oscillation can be written in the form:
\[ \eta_1 = a^{**} \sin \phi + M_1^{**} \cos(v - \alpha) + M_2^{**} \sin(v - \alpha) \]
\[ \eta_2 = a^{**} \sin \phi - M_1^{**} \cos(v - \alpha) + M_2^{**} \cos(v - \alpha) \]

Where \( \theta = \omega^{**} v + \alpha_1 \) being arbitrary constant and \( a^{**}, M_1^{**}, M_2^{**} \) are the roots of the system of equations.

\[ b\{M_1^2E_1 - M_2^2E_2 - a^2E_3\}M_2 + b_1\left\{M_1^2\left(K_2 + \frac{K_1m_2}{m_1}\right)\sin \alpha \right\} \\
- M_2^2\left\{K_2 + \frac{K_1m_2}{m_1}\right\}\sin \alpha + \frac{2K_1m_2}{m_1}M_1M_2 \cos \alpha = 0 \\
+ b_1\left\{M_2^2\left(K_2 + \frac{K_1m_2}{m_1}\right)\cos \alpha - M_1^2\left(K_2 + \frac{K_1m_2}{m_1}\right)\cos \alpha + \frac{2K_1m_2}{m_1}M_1M_2 \sin \alpha \right\} = 0 \]

and \( [M_1 \sin \alpha + M_2 \cos \alpha]a = 0 \)

Hence, we finally come to the conclusion that the stationary solution [4.14] can be stable for 
\[ M_1 = M_1^{**}, \quad M_2 = M_2^{**} \quad \text{and} \quad a = a^{**} \]

Only when the roots of the following characteristic equation
\[
\begin{vmatrix}
\frac{\partial R_1}{\partial m_1} - \eta & \frac{\partial R_1}{\partial m_2} & \frac{\partial R_1}{\partial \alpha} \\
\frac{\partial R_2}{\partial m_1} & \frac{\partial R_2}{\partial m_2} & \frac{\partial R_2}{\partial \alpha} \\
\frac{\partial R_3}{\partial m_1} & \frac{\partial R_3}{\partial m_2} & \frac{\partial R_3}{\partial \alpha} - \eta \\
\end{vmatrix} = 0 \]

have negative real parts.

From what we have discussed above in this paper, it follows that the stationary solution in the non-resonance case is stable in the first approximation in elliptic motion of the system where as the stationary solution in the resonance case exists only when the roots of the characteristic equation [4.16] have negative real parts.
References:


