Fourier transform and Plancherel Theorem for Nilpotent Lie Group

Kahar El-Hussein
Department of Mathematics, Faculty of Science,
Al-Furat University, Deir-El-Zore, Syria
Department of Mathematics, Faculty of Arts and Science,
Al Quryyat, Kingdom of Saudi Arabia

Abstract

As will known the connected and simply connected nilpotent Lie group $N$ has an important role in quantum mechanics. In this paper we show how the Fourier transform on the $n$–dimensional vector Lie group $\mathbb{R}^n$ can be generalized to $N$ in order to obtain the Plancherel theorem. In addition we define the Fourier transform for the subgroup $NA = A \ltimes N$ of the real semi-simple Lie group $SL(n,\mathbb{R})$ to get also the Plancherel formula for $NA$

Keywords: Nilpotent Lie Group, Semi-simple Lie Group, Fourier Transform and Plancherel Theorem

AMS 2000 Subject Classification: 43A30 & 35D 05.

1 Notations and Results.

1.1. The fine structure of the nilpotent Lie groups will help us to do the Fourier transform on a nilpotent Lie groups $N$. As well known any group connected and simply connected $N$ has the following form

$$
N = \begin{pmatrix}
1 & x_1 & x_2 & x_3 & \ldots & x_{n-2} & x_{n-1} & x_n \\
0 & 1 & x_2 & x_3 & \ldots & x_{n-2} & x_{n-1} & x_n \\
0 & 0 & 1 & x_3 & \ldots & x_{n-2} & x_{n-1} & x_n \\
0 & 0 & 0 & 1 & \ldots & x_{n-2} & x_{n-1} & x_n \\
& & & & & \ddots & \ddots & \ddots \\
& & & & & & & \ddots & \ddots \\
& & & & & & & & \ddots & \ddots \\
& & & & & & & & & \ddots & \ddots \\
0 & 0 & 0 & 0 & \ldots & 1 & x_{n-1} & x_n \\
0 & 0 & 0 & 0 & \ldots & 0 & 1 & x_n \\
0 & 0 & 0 & 0 & \ldots & 0 & 0 & 1
\end{pmatrix}
$$

As shown, the matrix (1) is formed by the subgroup $\mathbb{R}, \mathbb{R}^2, \ldots, \mathbb{R}^{n-1}$, and $\mathbb{R}^n$
Each $R^i$ is a subgroup of $N$ of dimension $i$, $1 \leq i \leq n$, put $d = n + (n - 1) + \ldots + 2 + 1$, which is the dimension of $N$. According to [6,7], the group $N$ is isomorphic onto the following group

$$
(((\mathbb{R}^n \times \rho_n)R^{n-1}) \times \rho_{n-1} \ldots) \times \rho_2 \mathbb{R} \times \rho_1 \mathbb{R}
$$

That means

$$
N; (((\mathbb{R}^n \times \rho_n)R^{n-1}) \times \rho_{n-1} \ldots) \times \rho_3 \mathbb{R}^3 \times \rho_2 \mathbb{R}^2 \times \rho_1 \mathbb{R}
$$

1.2. Denote by $L^1(N)$ the Banach algebra that consists of all complex valued functions on the group $N$, which are integrable with respect to the Haar measure of $N$ and multiplication is defined by convolution on $N$ as follows:

$$
g \ast f(X) = \int_N f(Y^{-1}X) g(Y) dY
$$

for any $f \in L^1(N)$ and $g \in L^1(N)$, where $X = (X^1, X^2, X^3, \ldots, X^{n-2}, X^{n-1}, X^n)$, $X^1 = x_1^n$, $X^2 = (x_1, x_2^n)$, $X^3 = (x_1^n, x_2, x_3)$, $\ldots$, $X^{n-1} = (x_1^{n-1}, x_2, x_3, \ldots, x_{n-2}, x_{n-1})$, $X^n = (x_1^n, x_2, x_3, \ldots, x_{n-2}, x_{n-1}, x_n^n)$, $Y = (Y^1, Y^2, Y^3, \ldots, Y^{n-1}, Y^n)$, $Y^1 = y_1^n, Y^2 = (y_1, y_2^n)$, $\ldots$, $Y^{n-1} = (y_1, y_2, y_3^n, y_4, \ldots, y_{n-2}, y_{n-1})$, $Y^n = (y_1, y_2, y_3, y_4^n, y_5, \ldots, y_{n-2}, y_{n-1}, y_n^n)$, and $dY = dy^1 dy^2 dy^3 \ldots dy^{n-2} dy^n$ is the Haar measure on $N$ and $\ast$ denotes the convolution product on $N$. We denote by $L^2(N)$ its Hilbert space

Let $M = \mathbb{R}^n \times \mathbb{R}^{n-1} \times \mathbb{R}^{n-2} \times \ldots \times \mathbb{R}^3 \times \mathbb{R}^2 \times \mathbb{R} = \mathbb{R}^d$ be the Lie group, which is the direct product of $\mathbb{R}^n, \mathbb{R}^{n-1}, \mathbb{R}^{n-2}, \ldots, \mathbb{R}^3, \mathbb{R}^2$ and $\mathbb{R}$. Denote by $L^1(M)$ the Banach algebra consists of all complex valued functions on the group $M$, which are integrable with respect to the Lebesgue measure on $M$ and multiplication is defined by convolution on $M$ as:

$$
g \ast f(X) = \int_M f(X - Y) g(Y) dY
$$
by any $f \in L^1(M), \ g \in L^1(M)$, where $*$ signifies the convolution product on the abelian group $M$. In this paper, we use the methods in [3,4,6,7] to show the powerful of the Fourier transform on $\mathbb{R}^d$ which can be generalized on $N$ in order to obtain the Plancherel theorem.

2 Fourier Transform and Plancherel Formula for $N$.

**Definition 2.1.** For $1 \leq i \leq n$, let $\mathcal{F}^i$ be the classical Fourier transform on $\mathbb{R}^d$, we can define the Fourier transform on $N$ as

$$\mathcal{F}f(\lambda) = \int_N f(X)e^{-i(\lambda,X)}\,dX$$

(7)

for any $f \in L^1(N)$, where $X = (X^1, X^2, X^3, \ldots, X^{n-2}, X^{n-1}, X^n)$, $dX = dx^1dx^2\ldots dx^{n-2}dx^{n-1}dx^n$ and $\lambda = (\lambda^1, \lambda^2, \lambda^3, \ldots, \lambda^{n-2}, \lambda^{n-1}, \lambda^n)$, where $F$ is the commutative Fourier transform on $\mathbb{R}^d$.

**Plancherel formula (Theorem 2.1).** For every function $f \in L^1(N)$, we have

$$\int_{\mathbb{R}^d} \left|\mathcal{F}f(\lambda)\right|^2\,d\lambda = \int_N |f(X)|^2\,dX$$

(8)

where $d\lambda$ is the lebesgue measure on $\mathbb{R}^d$.

**Proof:** For each $1 \leq j \leq n$, let $\mathcal{F}^j$ be the Fourier transform on $\mathbb{R}^d$. If we denote $T^j_n = \mathcal{F}^n \mathcal{F}^{n-1} \mathcal{F}^{n-2} \ldots \mathcal{F}^j$, $R^n = \mathbb{R}^n \times \mathbb{R}^{n+1} \times \mathbb{R}^{n+2} \times \ldots \mathbb{R}^d$, $X^j = X^1 \times X^2 \times X^3 \ldots X^n$, $dX^j = dx^1dx^2\ldots dx^{n-2}dx^{n-1}dx^n$, then for any function $f \in L^1(N)$, we get

$$\int_{\mathbb{R}^d} \left|\mathcal{F}f(\lambda)\right|^2\,d\lambda = \int_N |f(X)|^2\,dX$$

(9)

and

$$\int_{\mathbb{R}^d} \left|\mathcal{F}f(\lambda)\right|^2\,d\lambda = \int_N |f(X)|^2\,dX$$

(10)

By indiction we get

$$\int\int\ldots\int_{\mathbb{R}^n} \left|\mathcal{F}f(X^n, X^{n-1}, X^{n-2}, \ldots, X^3, X^2, X^1)\right|^2\,dX^n dX^{n-1} dX^{n-2} \ldots dX^3 dX^2 dX^1$$

$$= \int\int\ldots\int_{\mathbb{R}^n} \left|\mathcal{F}f(\lambda^n, \lambda^{n-1}, \lambda^{n-2}, \ldots, \lambda^3, \lambda^2, \lambda^1)\right|^2\,d\lambda^n d\lambda^{n-1} d\lambda^{n-2} \ldots d\lambda^3 d\lambda^2 d\lambda^1$$

(11)
Hence the proof of our theorem 2.1.

3 Plancherel Formula for the Solvable Lie Group $AN$

3.1. Let $G = SL(n, \mathbb{R})$ be the real semi-simple Lie group and let $G = KAN$ be the Iwasawa decomposition of $G$, where $K = SO(n, \mathbb{R})$,

$$N = \begin{bmatrix}
1 & * & \cdots & * \\
0 & 1 & * & \cdots \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & 0
\end{bmatrix}
$$

and

$$A = \begin{bmatrix}
a_1 & 0 & 0 & 0 \\
0 & a_2 & 0 & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & a_n
\end{bmatrix}
$$

where $a_1, a_2, \ldots, a_n = 1$ and $a_i \in \mathbb{R}_{+}^*$.

The product $AN$ is a closed subgroup of $G$ and is isomorphic (algebraically and topologically) to the semi-direct product of $A$ and $N$ with $N$ normal in $AN$. Then the group $AN$ is nothing but the group $= SN \times \rho_{\mathbb{R}}$, where $\rho: A \rightarrow Aut(N)$ the group homomorphism from $A$ into $Aut(N)$ of all automorphisms of $N$, which is defined by

$$\rho(a)(m) = ama^{-1}$$

So the product of two elements $X$ and $Y$ by

$$(x, a)(m, b) = (x, \rho(a)m, ab) = (ama^{-1}, ab)$$

for any $X = (x, a) \in S$ and $Y = (m, b) \in S$. Let $dnda$ be the right haar measure on $S$ and let $L^2(S)$ be the Hilbert space of the group $S$. Let $L(S)$ be the Banach algebra that consists of all complex valued functions on the group $S$, which are integrable with respect to the Haar measure of $S$ and multiplication is defined by convolution on $S$ as

$$g * f = \int_S f((m, b)^{-1}, (n, a))g(m, b)dmb$$

Where

$$m = (m_1, m_2, \ldots, m_r, b), b = (b_1, b_2, \ldots, b_{r-1}, b_n)$$

and

$$dmb = dm_1dm_2\ldots dm_rdb_1\ldots db_{r-1}\ldots db_{n-1}$$

In the following we prove the Plancherel theorem for $NA$. Therefore let $T = N \times A$ be the Lie group which is the direct product of the two Lie groups $N$ and $A$, and let $H = N \times A \times A$ be the Lie group, with multiplication
for all \((n,t,r) \in H\) and \((m,s,q) \in H\). In this case the group \(S\) can be identified with the closed subgroup \(N \times \{0\} \times \rho\ A\) of \(H\) and \(T\) with the subgroup \(N \times A \times \{0\}\) of \(H\).

**Definition 3.1.** For every function \(f\) defined on \(S\), one can define a function \(\tilde{f}\) on \(L\) as follows:

\[
\tilde{f}(n,a,b) = \tilde{f}(\rho(a)n,ab)
\]

for all \((n,a,b) \in H\). So every function \(\psi(n,a)\) on \(S\) extends uniquely as an invariant function \(\tilde{\psi}(n,b,a)\) on \(L\).

**Remark 3.1.** The function \(\tilde{f}\) is invariant in the following sense:

\[
\tilde{f}(\rho(s)n,as^{-1},sb) = \tilde{f}(n,a,b)
\]

for any \((n,a,b) \in H\) and \(s \in A\).

**Lemma 3.1.** For every function \(f \in L^1(S)\) and for every \(g \in L^1(S)\), we have

\[
g \ast \tilde{f}(n,a,b) = \int g(n,c^{-1})(n,a,b) \, dmdc
\]

for every \((n,a,b) \in H\), where \(\ast\) signifies the convolution product on \(S\) with respect the variables \((n,b)\) and \(\ast\) signifies the commutative convolution product on \(B\) with respect the variables \((n,a)\).

**Proof:** In fact we have

\[
g \ast \tilde{f}(n,a,b) = \int f((m,c^{-1})(n,a,b))g(m,s) \, dmdc
\]

\[
= \int \int f(\rho(c^{-1})(m^{-1}),a^{-1}(n,a,b))g(m,s) \, dmdc
\]

\[
= \int \int f(\rho(c^{-1})(m^{-1}n),a^{-1}b)g(m,c) \, dmdc
\]

\[
= \int \int m^{-1}n,a^{-1}b \, dmdc = g \ast \tilde{f}(n,a,b)
\]

**Definition 3.2.** If \(f \in L^1(S)\), one can define its Fourier transform \(\mathfrak{F}f\) by:

\[
\mathfrak{F}f(\xi,\lambda) = \int f(n,a)e^{-i\xi(n,a)}a^{-\lambda} \, dnda
\]

for any \(\xi = (\xi_1, \xi_2, ..., \xi_n) \in \mathbb{R}^n\), \(n = (x_1, x_2, ..., x_n) \in \mathbb{R}^n\), \(\lambda = (\lambda_1, \lambda_2, ..., \lambda_{n-1}) \in \mathbb{R}^{n-1}\) and \(a = (a_1, a_2, ..., a_{n-1})\), \(a_i \in \mathbb{R}^n\), \(1 \leq i \leq n\), \(a_1a_2...a_{n-1}a_n = 1\), where

\[
\langle \xi, x \rangle = \xi_1x_1 + \xi_2x_2 + ... + \xi_n x_n
\]

\[
dn = dx_1 dx_2 ... dx_n
\]

\[
da = da_1 da_2 ... da_{n-1}
\]

and \(a^{-\lambda} = a_1^{-\lambda_1}a_2^{-\lambda_2}...a_{n-1}^{-\lambda_{n-1}}\). Denote by \(\mathcal{S}(S)\) the Schwartz space of the group \(S = N \times \rho\ A\), it is clear that if \(f \in \mathcal{S}(S)\), then \(\mathfrak{F}f \in \mathcal{S}(S)\) and the mapping \(f \rightarrow \mathfrak{F}f\) is topological isomorphism of the topological vector space \(\mathcal{S}(S)\) onto \(\mathcal{S}(\mathbb{R}^{n+1})\).

**Definition 3.3.** If \(f \in L^1(S)\), we define the Fourier transform of its invariant \(\tilde{f}\) as follows

\[
\mathfrak{F}(\tilde{f})(\xi,\lambda,0) = \int_{H \times \mathbb{R}^{n-1}} \tilde{f}(n,a,b)e^{-i\xi(n,a)}a^{-\lambda}b^{-\mu} \, dnda\, dbd\mu
\]

where \(\mu = (\mu_1, \mu_2, ..., \mu_{n-1}) \in \mathbb{R}^{n-1}\) and \(b = (b_1, b_2, ..., b_{n-1}) \in \mathbb{R}^{n-1}\).
Theorem 3.1. For every $g \in \mathcal{L}^1(S)$, and $f \in \mathcal{L}^1(S)$, we have
\[
\int_{\mathbb{R}^{n-1}} (g \ast \check{f})(\xi, \mu, \lambda) d\mu = \int_{\mathbb{R}^{n-1}} \mathcal{F}(g)(\xi, \lambda) d\lambda = \mathcal{F}(\check{f})(\xi, \lambda, 0) \mathcal{F}(g)(\xi, \lambda)
\] (25)

Proof: By equation (21) we get immediately
\[
\int_{\mathbb{R}^{n-1}} \mathcal{F}(g \ast \check{f})(\xi, \lambda, \mu) d\mu = \int_{\mathbb{R}^{n-1}} \mathcal{F}(g)(\xi, \lambda) d\lambda = \mathcal{F}(\check{f})(\xi, \lambda, 0) \mathcal{F}(g)(\xi, \lambda)
\] (26)

Plancherel’s theorem 3.2. For any $f \in \mathcal{L}^1(S) \cap \mathcal{L}^2(S)$, we have
\[
\int_S |f(n, a)|^2 d\mu = \int_{\mathbb{R}^n} |\mathcal{F}(f)(\xi, \lambda)|^2 d\xi d\lambda
\] (27)

Proof: First, let $\check{\tilde{f}}$ be the function defined by
\[
\check{\tilde{f}}(n, a, b) = f((\rho(a)n, ab)^{-1})
\] (28)
then first we have
\[
f \ast \check{\tilde{f}}(I_N, I_A, I_A) = \int_S \check{\tilde{f}}(n, a)^{-1}(I_N, I_A, I_A)f(n, a) d\mu
\]
\[
= \int_S \check{\tilde{f}}((\rho^{-1})(n, a^{-1}))(I_N, I_A, I_A)f(n, a) d\mu
\]
\[
= \int_S \check{\tilde{f}}((\rho^{-1})(n, a^{-1}))(I_N, I_A, a^{-1})f(n, a) d\mu
\]
\[
= \int_S \check{\tilde{f}}((\rho^{-1})(n, a^{-1}))(I_N, a^{-1})f(n, a) d\mu
\]
\[
= \int_S \check{\tilde{f}}((\rho^{-1})(n^{-1}, a^{-1}))f(n, a) d\mu = \int_S \check{\tilde{f}}((n, a)f(n, a) d\mu
\]
\[
= \int_S |f(n, a)|^2 d\mu
\] (29)

Secondly by (27), we obtain
\[
f \ast \check{\tilde{f}}(I_N, I_A, I_A)
\]
\[
= \int_{\mathbb{R}^{2(n-1)}} \mathcal{F}(f \ast \check{\tilde{f}})(\xi, \lambda, \mu) d\xi d\lambda d\mu = \int_{\mathbb{R}^{2(n-1)}} \mathcal{F}(f \ast \check{\tilde{f}})(\xi, \lambda, \mu) d\xi d\lambda d\mu
\]
\[
= \int_{\mathbb{R}^{n-1}} \mathcal{F}(\check{\tilde{f}})(\xi, \lambda, 0) d\xi d\lambda = \int_{\mathbb{R}^{n-1}} \mathcal{F}(\check{\tilde{f}})(\xi, \lambda, 0) d\xi d\lambda
\]
\[
= \int_{\mathbb{R}^{n-1}} (\mathcal{F}(f)(\xi, \lambda) \mathcal{F}(f)(\xi, \lambda) d\xi d\lambda = \int_{\mathbb{R}^{n-1}} |\mathcal{F}(f)(\xi, \lambda)|^2 d\xi d\lambda = \int_S |f(n, a)|^2 d\mu
\]
which is the Plancherel’s formula on $S$. So the Fourier transform can be extended to an isometry of $\mathcal{L}^1(S)$ onto $\mathcal{L}^2(\mathbb{R}^{n-1})$. 

ISSN: 2231-5373  http://www.ijmttjournal.org  Page 293
Corollary 2.2. In equation (29), replace the first \( f \) by \( g \), we obtain

\[
\int_{S} f(x,t)g(x,t)dxdt = \int_{\mathbb{R}^{n+1}} \mathcal{F}(f)(\xi,\lambda)\mathcal{F}(g)(\xi,\lambda)d\xi d\lambda
\]

which is the Parseval formula on \( S \).

### 4 References