Stability of n-type Cubic Functional Equation in Non- Archimedean Normed space: using direct and fixed point methods

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Abstract. *In this paper, the authors established the Stability for n- type of Cubic functional equation of the form*

$$3f\left(nx+n^2y+n^3z\right)+f\left(nx-n^2y+n^3z\right)+f\left(nx+n^2y-n^3z\right)$$
$$+f\left(-nx+n^2y+n^3z\right)=-4n^3f(x)-4n^6f(y)-4n^9f(z)$$
$$+4\left[f\left(nx+n^2y\right)+f\left(n^2y+n^3z\right)+f\left(n^3z+nx\right)\right]$$

in Non-Archimedean Normed spaces,using direct and fixed point methods ,where n is a positive integer with $n>0$.

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I. INTRODUCTION AND PRELIMINARIES

A classical question in the theory of functional equation is the following: when is it true that a function which approximately satisfies a functional equation ε must be close to an exact solution of ε ?”

If the problem accepts a solution, we say that the question ε is stable. The first stability problem concerning group homomorphisms was raised by Ulam [30],[31] in 1940. We are a group G and metric group G' with metric $d(.,.)$. Given $\varepsilon>0$, does there exist a $\delta>0$ such that $f:G\to G'$ satisfies $d\left(f(xy),f(x)f(y)\right)<\delta$, for all $x,y\in G$, then a homomorphism $h:G\to G'$; exists with $d\left(f(x),h(x)\right)<\varepsilon$ for all $x\in G$?

In the next year D. H. Hyers [13], gave a positive answer, to the above question for additive groups under the assumption that the groups are Banach spaces.

In 1978, Th. M. Rassias [25] proved a generalization of Hyer’s theorem for additive mappings in the following way.

Theorem 1.1. Let f be a approximately additive mapping from a normed vector space E into a Banach space E', ie., f satisfies the inequality

$$\left\|f(x+y)-f(x)-f(y)\right\|\le\varepsilon\left(\|x\|^r+\|y\|^r\right)$$

(1)

for all $x,y\in E$, where ε and r are constants with $\varepsilon>0$ and $0\le r<1$. Then there exists a unique additive mapping $T:E\to F$ such that for all $r\in E$

$$\frac{\left\|f(x)-T(x)\right\|}{\|x\|^r}\le\frac{2\varepsilon}{2-2^r}$$

(2)

for all $x\in E-\{0\}$.

The result of Th. Rassias[25] has influenced the development of what is now called the Hyers-Ulam-Rassias[13],[30],[25] stability theory for functional equations. In 1994, a generalization of Rassias,s theorem was obtained by Gavruta [11] by replacing the bound $\varepsilon\left(\|x\|^p+\|y\|^p\right)$ by a general control function $\phi(x,y)$.

Several stability results have been recently obtained for various equations, also for mappings with more general domains and ranges (see [1]-[6], [19]-[21]).

In 1987, Hensel [10] has introduced a normed space which does not have the Archimedean property. It turned out that non-Archimedean spaces have many nice applications (see[10], [18], [22]-[28]).

By a non-Archimedean field we mean a field k equipped with a function (valuation)

valuation $\|.\| : K \to [0,\infty)$ such that $|r| = 0$ if and only if $r = 0$, $|rs| = |r||s|$ and $|r+s| \le \max |r|,|s|$ for all $r,s \in K$.Clearly $|1| = |-1| = 1$ and $|n| \le 1$ for $n \in N$.

Then $X, \|.\|$ is called a non-Archimedean space.

Definition 1.2. Let X be a vector space over a scalar field k with non-Archimedean valuation $|.|$. A function $\|.\| : X \to [0,\infty)$ is said to be a non-Archimedean norm if satisfies the following conditions

(i) $\|x\| = 0$ if and only if $x = 0$.

(ii) $\|rx\| = |r|\|x\|$ $(r \in K, x \in X)$.

(iii) The strong triangle inequality (ultrametric); namely

$\|x+y\| \le \max \|x\|,\|y\|$ $: (x,y \in X)$

Then $X, \|.\|$ is called a non-Archimedean space. Due to the fact that,

$$\|x_n - x_m\| \le \max \|x_{j+1} - x_j\|; x \le j \le n-1 \ ;(n > m)$$

Definition 1.3. A sequence x_n is Cauchy if and only if x_{n+1}, x_n converges to zero in a non-Archimedean normed space. By a complete non-Archimedean space we mean one in which every Cauchy sequence is convergent.

Example 1.4. Fix a prime number p. For any non-zero rational number x, there exists a unique integer $n_x \in Z$ such that $x = \frac{a}{b} p^n x$, where a and b are integers not divisible by p. Then $|x|_p := p^{-n_x}$ defines a non-Archimedean norm on Q. The completion of Q with respect to the metric $d(x,y) = |x-y|_p$ is denoted by Q_p which is called the p-adic number field. Infact, Q_p is the set of all formal series $x = \sum_{k \ge n_x}^{\infty} a_k p^k$ where $|a_k| \le p-1$ are integers. The addition and multiplication between any two element of Q_p are defined naturally. The norm $\left|\sum_{k \ge n_x}^{\infty} a_k p^k\right|_p = p^{-n_x}$ is a non-Archimedean norm on Q_p and it makes Q_p a locally compact field.

Definition 1.5. Let X be a set. A function $d : X \times X \to [0,\infty)$ is called a generalized metric on X if d satisfies the following conditions:

(a) $d(x,y) = 0$ if and only if $x = y$ for all $x, y \in X$.

(b) $d(x,y) = d(y,x)$ for all $x, y \in X$.

(c) $d(x,z) = d(x,y) + d(y,z)$ for all $x, y, z \in X$.

Theorem 1.6 (Banach's contraction principal) Let (X,d) be a complete metric space and consider a mapping $T : x \to x$ which is strictly contractive mapping that is

(A_1) $d(Tx,Ty) \le Ld(x,y)$ for some (Lipscihitz constant) $L < 1$, then

(i) The mapping T has one and only fixed point $x^* = T(x^*)$;

(ii) The fixed point for each given element x^* is globally contractive that is

(A_2) $\lim_{n\to\infty} T^n x = x^*$ for any starting point $x \in X$

(iii) One has the following estimation inequalities.

$$(A_3) d\ T^n x, x^* \le \frac{1}{1-L} d(T^n x, T^{n+1} x) \ \forall\, n \ge 0, \forall\, x \in X$$

$$(A_4) d\ x, x^* \le \frac{1}{1-L} d(x, x^*) \ \forall\, x \in X.$$

Theorem 1.7 [27] (the alternative fixed point) Suppose that for a complete generalized metric space (X,d) and a strictly contractive mapping $T : X \to Y$ with Lipschitz constant L then for each element $x \in X$, either

$(B_1) d(T^n x, T^{n+1} x) = \infty \ \forall\, n \ge 0$

(B_2) there exists a natural number n_0 such that

(i) $d(T^n x, T^{n+1} x) < \infty \ \forall\, n \ge n_0$;

(ii) the sequence $(T^n x)$ is convergent to a fixed point y^* of T;

(iii) y^* is the unique fixed point of T in the set

$y = \ y \in X : T\ T^{n_0} x, y < \infty$;

(iv) $d(y^* y) \le \frac{1}{1-L} d(y, Ty) \ \forall\, y \in Y$.

In this paper, we prove the generalized Hyers-Ulam stability of the following functional equation

$$3f\left(nx+n^2y+n^3z\right)+f\left(nx-n^2y+n^3z\right)+f\left(nx+n^2y-n^3z\right)$$
$$+f\left(-nx+n^2y+n^3z\right)$$

$$=4\left[f\left(nx+n^2y\right)+f\left(n^2y+n^3z\right)+f\left(n^3z+nx\right)\right]-4n^3f(x)$$
$$-4n^6f(y)-4n^9f(z)$$

(4)

in Non-Archimedean normed spaces.

2. STABILITY OF FUNCTIONAL EQUATION (4):A FIXED POINT METHOD

In this section using the fixed point alternative approach, we prove the generalized Hyers-Ulam stability of functional equation (4) in Non-Archimedean spaces.

Theorem 2.1 Let $\Omega : X^3 \to [0,\infty)$ be a function such that there exists an $L<1$ with

$$\lim_{k\to\infty}\frac{1}{|n|^{3k}}\left(n^kx,n^ky,n^kz\right)$$

(5)

for all $x,y,z\in X$. Let $f:X\to Y$ be a mapping satisfying

$$\left\|3f\left(nx+n^2y+n^3z\right)+f\left(nx-n^2y+n^3z\right)+f\left(nx+n^2y-n^3z\right)\right.$$
$$+f\left(-nx+n^2y+n^3z\right)$$
$$-4\left[f\left(nx+n^2y\right)+f\left(n^2y+n^3z\right)+f\left(n^3z+nx\right)\right]+4n^3f(x)$$

$$\left.+4n^6f(y)+4n^9f(z)\right\|\le\Omega(x,y,z)$$

(6)

for all $x,y,z\in X$. Then there is a unique Cubic mapping $C:X\to Y$ such that

$$\left\|f(x)-C(x)\right\|\le\frac{1}{|n|^9}\Omega(x) \qquad (7)$$

for all $x\in X$.

Proof: Putting (x,y,z) by $(0,0,x)$ in (6), we have

$$\left\|4f\left(n^3x\right)-4n^9f(x)\right\|\le\Omega(0,0,x)$$

(8a)

for all $x\in X$ .Dividing by $4n^3$ in (8a),we get

$$\left\|\frac{f\left(n^3x\right)}{n^9}-f(x)\right\|\le\frac{\Omega}{4n^9}(0,0,x)$$

(8b)

for all $x\in X$ .From (8b) and rearranging ,we arrive

$$\left\|\frac{f\left(n^3x\right)}{n^9}-f(x)\right\|\le\frac{\Omega}{4|n|^9}(0,0,x)$$

(8c)

for all $x\in X$. Consider the set,

$$S:=\left\{g:X\to Y\right\}$$

(9)

and the generalized metric d in s defined by

$$d(f,g)=\inf\left\{\mu\in R^+:\left\|g(x)-h(x)\right\|\le\mu(0,0,x)\right.$$
$$\left.,\ \forall\, x\in X\right\}$$

(10)

where $\inf\phi=+\infty$. It is easy to show that (s,d) is complete (see[18] lemma 2.1.) Now, we consider a linear mapping $J:S\to S$ such that,

$$J\,h(x)=\frac{1}{4n^9}h\left(n^3x\right)$$

(11)

for all $x\in X$. Let $g,h\in S$ be such that $d(g,h)=\varepsilon$ then

$$\left\|g(x)-h(x)\right\|\le\varepsilon\,\Omega(0,0,x)$$

(12)

for all $x\in X$ and so

$$\left\|Jg(x)-Jh(x)\right\|=\left\|\frac{1}{n^9}g(n^3x)-\frac{1}{n^9}Jh(n^3x)\right\|$$

$$\le\frac{1}{4|n|^9}\in 4|n|^9\,L\Omega\left(0,0,x\right).$$

for all $x\in X$. Thus $d(g,h)=\varepsilon$ implies that $d(Jg,Jh)\le L\varepsilon$. This means that

$$d(Jg,Jh)\le Ld(g,h)$$

(13)

for all $g,h\in S$. It is follows from (8) that

$$d(f, Jf) \leq \frac{1}{4|n|^9} < +\infty$$

(14)

By a theorem (1.7), there exists a mapping $C : X \to Y$ satisfying the following:

(1) C is the fixed point of J, that is

$$C\left(n^3 x\right) = n^9 C(x)$$

(15)

for all $x \in X$. The mapping C is unique fixed point of J in the set

$$\phi = \{h \in S : d(g,h) < \infty\}$$

(16)

This implies that C is a unique mapping satisfying (15) such that there exist $\mu \in (0,\infty)$ satisfying

$$\| f(x) - C(x) \| \leq \mu\, \Omega(0,0,x)$$

(17)

for all $x \in X$.

(2) $d(J^n f, C) \to 0$ as $n \to \infty$ this implies the inequality

$$\lim_{n\to\infty} \frac{f\left(n^k x\right)}{4|n|^{9k}} = C(x)$$

(18)

for all $x \in X$.

(3) $d(f,C) \leq \frac{d(f, Jf)}{1-L}$ with $f \in \phi$ which implies the inequality

$$d(f,C) \leq \frac{1}{4|n|^{9k} - 4|n|^{9k} L}$$

(19)

This implies that the inequality (7) holds. By (5) and (6), we obtain

$$\Big\| 3f\left(n^k \cdot nx + n^k \cdot n^2 x + n^k \cdot n^3 x\right) + f\left(n^k \cdot nx - n^k \cdot n^2 x + n^k \cdot n^3 x\right) + f\left(n^k \cdot nx + n^k \cdot n^2 x - n^k \cdot n^3 x\right)$$
$$+ f\left(-n^k \cdot nx + n^k \cdot n^2 x + n^k \cdot n^3 x\right) - 4\left[f\left(n^k \cdot nx + n^k \cdot n^2 y\right) + f\left(n^k \cdot n^2 y + n^k \cdot n^3 z\right)\right]$$
$$-4\left[f\left(n^k \cdot n^3 z + n^k \cdot nx\right)\right] - 4n^3 f(n^k \cdot x) - 4n^6 f(n^k \cdot x) - 4n^9 f(n^k \cdot x) \Big\|$$
$$\leq \Omega(n^k \cdot x, n^k \cdot y, n^k \cdot z)$$

$$\leq \Big\| \frac{1}{n^{3k}} 3f\left(n \cdot n^k x + n^2 n^k \cdot x + n^3 \cdot n^k x\right) + \left(n \cdot n^k x - n^2 n^k \cdot x + n^3 \cdot n^k x\right) + f\left(n \cdot n^k x + n^2 n^k \cdot x - n^3 \cdot n^k x\right)$$
$$+ f\left(-n \cdot n^k x + n^2 n^k \cdot x + n^3 \cdot n^k x\right) - 4\left[f\left(n \cdot n^k x + n^2 n^k \cdot x\right) + f\left(n^2 n^k \cdot x + n^3 \cdot n^k x\right)\right]$$
$$-4\left[f\left(n^3 \cdot n^k x + n \cdot n^k x\right)\right] + 4n^3 f(n^k x) + 4n^6 f(n^k x) + 4n^9 f(n^k x) \Big\|$$
$$\leq \frac{1}{|n|^{3k}} L^{3k} \cdot |n|^{3k}\, \Omega(x,y,z)$$

for all $x, y, z \in X$ and $k \in N$. So

$$\Big\| 3C\left(nx + n^2 y + n^3 z\right) + C\left(nx - n^2 y + n^3 z\right) + C\left(nx + n^2 y - n^3 z\right) + C\left(-nx + n^2 y + n^3 z\right)$$
$$-4\left[C\left(nx + n^2 y\right) + C\left(n^2 y + n^3 z\right) + C\left(n^3 z + nx\right)\right] + 4n^3 C(x) + 4n^6 C(y) + 4n^9 C(z) \Big\| = 0$$

.

for all $x, y, z \in X$. Thus the mapping $C : X \to Y$ is Cubic.

Corollary 2.2. Let $\theta \geq 0$ and P be a real numbers with $P > 1$. Let $f : X \to Y$ be a mapping satisfying,

$$\Big\| f\left(nx + n^2 y + n^3 z\right) + f\left(nx - n^2 y + n^3 z\right) + f\left(nx + n^2 y - n^3 z\right) + f\left(-nx + n^2 y + n^3 z\right)$$
$$-4\left[f\left(nx + n^2 y\right) + f\left(n^2 y + n^3 z\right) + f\left(n^3 z + nx\right)\right] + 4n^3 f(x) + 4n^6 f(y) + 4n^9 f(z) \Big\|$$

$$\leq \theta \left(\|x\|^p + \|y\|^p + \|z\|^p \right)$$

(21)

for all $x, y, z \in X$. Then

$$C(x) = \lim_{k\to\infty} \frac{f\left(n^k x\right)}{4|n|^{9k}}$$

(22)

exists for all $x \in X$ and $C : X \to Y$ is a unique Cubic mapping such that

$$\| f(x)-C(x)\| \le \frac{\theta \|x\|^p \cdot n^3}{4\left[|n|^3-|n|^{3p}\right]}$$

for all $x \in X$.

Proof: The proof of the Theorem 2.1 by taking

$$\Omega(x,y,z) \le \theta \left(\|x\|^p + \|y\|^p + \|z\|^p \right)$$

(24)

for all $x, y, z \in X$.

Theorem 2.3. Let $\Omega : X^3 \to [0,\infty)$ be a function such that there exists an $L<1$ with

$$\Omega\left(\frac{x}{n^k},\frac{x}{n^k},\frac{x}{n^k}\right) \le \frac{L}{4|n|^{3k}} \phi(x,y,z)$$

(25)

for all $x, y, z \in X$. Let $f : X \to Y$ be a mapping satisfying

$$\Big\| f(nx+n^2y+n^3z)+f(nx-n^2y+n^3z)+f(nx+n^2y-n^3z) + f(-nx+n^2y+n^3z)$$

$$-4\left[f(nx+n^2y)+f(n^2y+n^3z)+f(n^3z+nx)\right]+4n^3f(x)$$

$$+4n^6f(y)+4n^9f(z)\Big\| \le \Omega(x,y,z)$$

(26)

for all $x, y, z \in X$. Then there is a unique Cubic mapping $C : X \to Y$ such that

$$\| f(x)-C(x)\| \le \frac{L}{4\left[|n|^{3k}-|n|^{3k}L\right]}\Omega(0,0,x)$$

(27)

$x \in X$.

Proof: Substituting (x,y,z) by $(0,0,x)$ in (26), we get

$$\left\|4f(n^3x)-4n^9f(x)\right\| \le \Omega(0,0,x)$$

(27a)

$x \in X$.Dividing 4 in (27a),we arrive

$$\left\|f(n^3x)-n^9\frac{f(x)}{4}\right\| \le \Omega(0,0,x)$$

(27b)

$x \in X$.Replacing x by $\frac{x}{n^3}$ in (27b),and rearranging we arrive

$$\left\|f(x)-n^9 f\left(\frac{x}{n^3}\right)\right\| \le \frac{\Omega}{4}\left(0,0,\frac{x}{n^3}\right)$$

(28c)

for all $x \in X$. Defining $d(f,g)$ as in the Theorem 2.1. Consider a linear mapping $J : S \to S$ such that,

$$Jh(x) = n^9 h\left(\frac{x}{n^3}\right)$$

(29)

for all $x \in X$. Let $g, h \in S$ be such that $d(g,h)=\varepsilon$, then

$$\| g(x)-h(x)\| \le \in \Omega(0,0,x)$$

(30)

for all $x \in X$ and so

$$\| Jg(x)-Jh(x)\| = \left\| n^9 g\left(\frac{x}{n^3}\right) - n^9 h\left(\frac{x}{n^3}\right)\right\|$$

$$\le |n|^{3k} \in \frac{L}{|n|^{9k}}\Omega(0,0,x)$$

for all $x \in X$. Thus $d(g,h)=\varepsilon$ implies that $d(Jg,Jh) \le L\varepsilon$. This means that

$$d(Jg,Jh) \le Ld(g,h)$$

(31)

for all $g, h \in S$. It follows from (28c) that

$$d(f,Jf) \le \frac{1}{|n|^{9k}} < +\infty$$

(32)

By Theorem 1.7, there exists a mapping $C : X \to Y$ satisfying the following:

(1) C is the fixed point of J, that is

$$C\left(\frac{x}{n^3}\right) = \frac{1}{n^9}C(x)$$

(33)

for all $x \in X$. The mapping C is unique fixed point of J in the set

$$\phi = \{h \in S : d(g,h) < \infty\}$$

(34)

This implies that C is a unique mapping satisfying (33) such that there exist $\mu \in (0,\infty)$ satisfying

$$\| f(x) - C(x) \| \le \mu\, \Omega(0,0,x)$$

(35)

for all $x \in X$.

(2) $d(J^n f, C) \to 0$ as $n \to \infty$ this implies the inequality

$$\lim_{k\to 0} n^{9k} f\left(\frac{x}{n^{3k}}\right) = C(x)$$

(36)

for all $x \in X$.

(3) $d(f,C) \le \dfrac{d(f, Jf)}{1-L}$ with $f \in \phi$ which implies the inequality

$$d(f,C) \le \frac{1}{4|n|^{9k} - 4|n|^{3k} L}$$

(37)

This implies that the inequality (27) holds. By (28) and (30), we obtain

$$\left\| \frac{1}{n^{9k}} \left[3f\left(n\cdot\frac{x}{n^k}+n^2\cdot\frac{x}{n^k}+n^3\cdot\frac{x}{n^k}\right)+f\left(n\cdot\frac{x}{n^k}-n^2\cdot\frac{x}{n^k}+n^3\cdot\frac{x}{n^k}\right) + f\left(n\cdot\frac{x}{n^k}+n^2\cdot\frac{x}{n^k}-n^3\cdot\frac{x}{n^k}\right) \right.\right.$$

$$+ f\left(-n\cdot\frac{x}{n^k}+n^2\cdot\frac{x}{n^k}+n^3\cdot\frac{x}{n^k}\right) - 4\left[f\left(n\cdot\frac{x}{n^k}+n^2\cdot\frac{x}{n^k}\right) + f\left(n^2\cdot\frac{x}{n^k}+n^3\cdot\frac{x}{n^k}\right)\right]$$

$$\left.\left. -4\left[f\left(n^3\cdot\frac{x}{n^k}+n\cdot\frac{x}{n^k}\right)\right] + 4n^3 f\left(\frac{x}{n^k}\right)+4n^6 f\left(\frac{x}{n^k}\right)+4n^9 f\left(\frac{x}{n^k}\right)\right]\right\|$$

$$\le \frac{1}{|n|^{3k}}\,\Omega\left(\frac{x}{n^k},\frac{y}{n^k},\frac{z}{n^k}\right)$$

$$\le \frac{1}{4|n|^{9k}} L^{9k} \cdot |n|^{9k}\, \Omega(x,y,z)$$

$$\le |n|^{9k} L^{9k} \frac{1}{|n|^{9k}}\, \Omega(x,y,z)$$

for all $x, y, z \in X$ and $k \in N$. So

$$\left\| 3C(nx+n^2y+n^3z)+C(nx-n^2y+n^3z) + C(nx+n^2y-n^3z)+C(-nx+n^2y+n^3z) -4\left[C(nx+n^2y)+C(n^2y+n^3z)+C(n^3z+nx)\right] +4n^3C(x)+4n^6C(y)+4n^9C(z)\right\| = 0$$

.

for all $x, y, z \in X$. Thus the mapping $C: X \to Y$ is Cubic.

Corollary 2.4. Let $\theta \ge 0$ and P be a real numbers with $P > 1$. Let $f: X \to Y$ be a mapping satisfying,

$$\| Df_C(x,y,z) \| \le \theta \left(\|x\|^P + \|y\|^P + \|z\|^P \right)$$

(38)

for all $x, y, z \in X$. Then

$$C(x) = \lim_{k\to\infty} n^{9k} f\left(\frac{x}{n^{3k}}\right)$$

(39)

exists for all $x \in X$ and $C: X \to Y$ is a unique Cubic mapping such that

$$\| f(x) - C(x) \| \le \frac{|n|^{9p}\, \theta \|x\|^p\, n^9}{4\left[|n|^3 - |n|^{9p}\right]}$$

(40)

for all $x \in X$.

Proof: The proof of the Theorem 2.3 by taking

$$\Omega(x,y,z) \le \theta \left(\|x\|^P + \|y\|^P + \|z\|^P \right)$$

(41)

for all $x, y, z \in X$.

3. STABILITY OF FUNCTIONAL EQUATION (4):A DIRECT METHOD

In this section, we prove the generalized HYERS-ULAM stability of the n-type cubic functional equation.Through this section assume that G is an Cubic semi group and X is a complete Non-Archimedean spaces.

Theorem 3.1 Let $\psi: G^3 \to [0,+\infty)$ be a function such that

$$\lim_{k\to\infty} \frac{\psi(n^k x, n^k y)}{|n|^{3k}} = 0$$

(42)

for all $x, y \in G$. Let for each $x \in G$ the limit

$$\psi(x) = \lim_{k\to\infty} \max\left\{ \frac{\psi\left(n^k x, 0, 0\right)}{|n|^{3k}} ; 0 \le n < k \right\}$$

(43)

exists. Suppose that $f : G \to X$ be a mapping satisfying the inequality

$$\Big\| 3f\left(nx+n^2y+n^3z\right) + f\left(nx-n^2y+n^3z\right) + f\left(nx+n^2y-n^3z\right) + f\left(-nx+n^2y+n^3z\right)$$

$$-4\left[f\left(nx+n^2y\right)+f\left(n^2y+n^3z\right)+f\left(n^3z+nx\right)\right]+24n^3 f(x)$$

$$+4n^6 f(y) + 4n^9 f(z)\Big\| \le \psi(x,y,z)$$

(44)

for all $x, y, z \in G$. Then the limit

$$T(x) := \lim_{k\to\infty} \frac{f\left(n^k x\right)}{n^{3k}} = 0$$

(45)

exists for all $x \in G$ and $T : G \to X$ is a cubic mapping satisfying

$$\| f(x) - T(x)\| \le \frac{1}{n^{3k}} \psi(x) \ \forall \ x \in G$$

(46)

moreover, if

$$\lim_{j\to\infty}\lim_{k\to\infty} \max\left\{ \frac{\psi(n^k x, 0, 0)}{|n|^{3k}} ; \quad j \le n \le k+j \right\} = 0$$

(47)

Then T is a unique mapping satisfying (46).

Proof: Putting x, y, z by $x, 0, 0$ in (44), that

$$\left\| 4n^3 f(x) - 4f\left(nx\right)\right\| \le \psi(0,0,x)$$

(47a)

$x \in G$.Dividing $4n^3$ in (47a),we get

$$\left\| f(x) - \frac{f\left(nx\right)}{n^3} \right\| \le \frac{\psi}{4n^3}(x,0,0)$$

(47b)

$x \in G$.From (47b),remodifying we arrive

$$\left\| f(x) - \frac{f\left(nx\right)}{n^3} \right\| \le \frac{\psi}{|4||n|^3}(x,0,0)$$

(48)

Replacing x by $n^k x$ in (48), we get

$$\left\| f(n^k x) - \frac{f\left(n \cdot n^k x\right)}{n^3} \right\| \le \frac{\psi}{|4||n|^3}(n^k x, 0, 0)$$

$$\left\| \frac{f(n^k x)}{n^{3k}} - \frac{f\left(n^{k+1} x\right)}{n^{3(k+1)}} \right\| \le \frac{\psi}{|4||n|^3 |n|^{3k}}(n^k x, 0, 0)$$

(49)

$x \in G$.It is follows from (42) and (49) ,that the sequence $\left\{ \frac{f(n^k x)}{n^{3k}} \right\}_{k=1}^{\infty}$ is a Cauchy sequence. Since X is complete. So $\left\{ \frac{f(n^k x)}{n^{3k}} \right\}$ is convergent. Set

$$T(x) := \lim_{k\to\infty} \frac{f(n^k x)}{n^{3k}}$$

using induction ,we see that

$$\left\| \frac{f(n^k x)}{n^{3k}} - f(x) \right\| \le \frac{1}{|4||n|^3} \max\left\{ \frac{\psi(n^k x, 0, 0)}{|n|^{3k}} ; \quad 0 \le n < k \right\} = 0$$

(50)

Indeed, (50) holds for $k = 1$ by (48). Let, (50) holds for k , so by (49), we obtain,

$$\left\| \frac{f(n^{k+1} x)}{n^{3(k+1)}} - f(x) \right\| = \frac{1}{|4|} \left\| \frac{f(n^{k+1} x)}{n^{3(k+1)}} \pm \frac{f(n^k x)}{n^{3k}} - f(x) \right\|$$

$$\le \frac{1}{|4|} \max\left\{ \left\| \frac{f(n^{k+1} x)}{n^{3(k+1)}} - \frac{f(n^k x)}{n^{3k}} \right\|, \left\| \frac{f(n^k x)}{n^{3k}} - f(x) \right\| \right\}$$

$$\leq \frac{1}{|4||n|^3}\max\left\{\frac{\psi(n^k x)}{|n|^{3k}}, \max\left\{\frac{\psi(n^l x,0,0)}{|n|^{3l}}; 0 \leq k < l\right\}\right\}$$

$$\leq \frac{1}{|4||n|^3}\max\left\{\frac{\psi(n^l x,0,0)}{|n|^{3l}}; 0 \leq l < n+1\right\}$$ so for all $k \in N$ and all $x \in G$, (50) holds. By taking k to approach infinity in (51) on obtains (43).If S is another mapping satisfies (46), then for $x \in G$, we get

$$\|T(x)-S(x)\| = \lim_{n\to\infty}\left\|\frac{T(n^l x)}{n^{3l}} - \frac{S(n^l x)}{n^{3l}}\right\|$$

$$\leq \lim_{n\to\infty}\left\|\frac{T(n^l x)}{n^{3l}} \pm \frac{f(n^l x)}{n^{3l}} - \frac{S(n^l x)}{n^{3l}}\right\|$$

$$\leq \lim_{n\to\infty}\max\left\{\left\|\frac{T(n^l x) - f(n^l x)}{n^{3l}}\right\|, \left\|\frac{f(n^l x) - S(n^l x)}{n^{3l}}\right\|\right\}$$

$$\leq \frac{1}{|4||n|^3}\lim_{j\to\infty}\lim_{k\to\infty}\max\left\{\frac{\psi(n^l x,0,0)}{|n|^{3l}}; \quad j \leq l \leq k+j\right\} = 0$$

$\therefore$ Therefore $T = S$.

Corollary 3.2. Let $\xi:[0,\infty)\to[0,\infty)$ be a function satisfying

$$\xi(|n|t) \leq \xi(|n|)\lambda(t), \qquad \xi(|n|) < |n|^3$$

(52)

for all $t \geq 0$. Let $\delta > 0$ and $f : G \to X$ is mapping satisfying the inequality

$$\|Df_C(x,y,z)\| \leq \delta(\xi(\|x\|) + \xi(\|y\|) + \xi(\|z\|))$$

(53)

for all $x,y,z \in G$. Then there exists a unique cubic mapping $T : G \to X$ such that

$$\|f(x)-T(x)\| \leq \frac{1}{|4||n|^3}\delta\xi(\|x\|); \quad x \in G.$$

(54)

Proof: Defining $\psi : G^3 \to [0,\infty)$ by $\psi(x,y,z) := \delta(\xi(\|x\|) + \xi(\|y\|) + \xi(\|z\|))$. Since $\frac{\xi(|n|)}{|n|^3} < 1$. We have

$$\lim_{k\to\infty}\frac{\psi(n^k x, n^k y, n^k z)}{|n|^{3k}} \leq \lim_{k\to\infty}\left(\frac{\xi(|n|)}{|n|^3}\right)^k \psi(x,y,z) = 0$$

(55)

for all $x,y,z \in Z$. Also for all $x \in G$.

$$\psi(x) = \lim_{k\to\infty}\frac{1}{|4||n|^3}\left\{\frac{\psi(n^k x,0,0)}{|n|^{3k}}; \quad 0 \leq k < n\right\}$$

$$= \psi(x,0,0) = \delta\xi(\|x\|)$$

exists for all $x \in G$. On the other hand

$$\frac{1}{|4||n|^3}\lim_{j\to\infty}\lim_{k\to\infty}\max\left\{\frac{\psi(n^k x,0,0)}{|n|^{3k}}; \quad j \leq l < k+j\right\} = 0$$

(54)

$x \in G$.Applying Theorem 3.1,that

$$\|f(x)-T(x)\| \leq \frac{1}{|2||n|^3}(\delta(\xi(\|x\|)) + 0 + 0)e^{i\theta}$$

$$\|f(x)-T(x)\| \leq \frac{\delta\xi(\|x\|)}{|4||n|^3}$$

(55)

Theorem 3.3 Let $\psi : G^3 \to [0,\infty)$ be a function such that

$$\lim_{k\to\infty}|n|^{3k}\psi\left(\frac{x}{n^k},\frac{y}{n^k},\frac{z}{n^k}\right) = 0$$

(56)

for all $x, y \in G$. Let for each $x \in G$ the limit

$$\psi(x) = \lim_{k\to\infty}\max\left\{|n|^{3(k+1)}\psi\left(\frac{x}{n^{k+1}},0,0\right); 0 \leq k < n\right\}$$

(57)

exists. Suppose that $f : G \to X$ be a mapping satisfying the inequality

$$\|Df_C(x,y,z)\| \leq \psi(x,y,z)$$

(58)

for all $x,y,z \in G$. Then the limit

$$T(x) := \lim_{k\to\infty} n^{3k} f\left(\frac{x}{n^k}\right)$$

(59)

exists for all $x \in G$ and $T : G \to X$ is a cubic mapping satisfying

$$\left\| f(x) - T(x) \right\| \le \frac{1}{|4||n|^3} \delta\xi \ \|x\| \ ; \quad x \in G$$

(60)

for all $x \in G$. Moreover, if

$$\lim_{j\to\infty} \lim_{k\to\infty} \max\left\{ |n|^{3\ l+1} \ \psi\left(\frac{x}{n^{l+1}}, 0, 0 \right); \quad j \le l \le n+j \right\} = 0$$

(61)

Then T is a unique mapping satisfying (60).

Proof: Replacing x, y, z by $x, 0, 0$,we get

$$\left\| 4f \ \ nx \ \ - 4n^3 f(x) \right\| \le \psi(x, 0, 0)$$

(61a)

for all $x \in G$.Dividing by 4 in and setting x by $\frac{x}{n}$ in (61a),we obtain

$$\left\| f \ \ x \ \ - n^3 f\left(\frac{x}{n} \right) \right\| \le \frac{\psi}{4}\left(\frac{x}{n}, 0, 0 \right)$$

(62)

for all $x \in G$. Replacing x by $\left(\frac{x}{n^k} \right)$ in (62), we get

$$\left\| n^{3k} f\left(\frac{x}{n^k} \right) - n^{3\ k+1} \ f\left(\frac{x}{n^{k+1}} \right) \right\| \le \frac{|n|^{3k}}{|4|} \psi\left(\frac{x}{n^{k+1}}, 0, 0 \right)$$

(63)

for all $x \in G$.It is follows from (56) and (63) that the sequence $\left\{ n^{3k} f\left(\frac{x}{n^k} \right) \right\}_{k=1}^{\infty}$ is a Cauchy sequence. Since X is complete. So $\left\{ n^{3k} f\left(\frac{x}{n^k} \right) \right\}_{k=1}^{\infty}$ is convergent.It follows from (63) that,

$$\left\| n^{3k} f\left(\frac{x}{n^k} \right) - n^{3p} f\left(\frac{x}{n^p} \right) \right\| = \left\| \sum_{r=p}^{k} \begin{matrix} n^{3\ r+1} \ f\left(\frac{x}{n^{r+1}} \right) \\ -n^{3r} f\left(\frac{x}{n^r} \right) \end{matrix} \right\|$$

$$\le \max\left\{ \left\| n^{3\ r+1} \ f\left(\frac{x}{n^{r+1}} \right) - n^{3r} f\left(\frac{x}{n^r} \right) \right\| ; p \le r < n \right\}$$

$$\le \frac{1}{|4|} \max\left\{ |n|^{3\ r+1} \ \psi\left(\frac{x}{n^{k+1}}, 0, 0 \right); p \le r < n \right\}$$

for all $x \in G$,for all non-negative integer n, p with $n > p > 0$. Letting $p = 0$ and passing the limit in the last inequality, we obtain (60).If S is another mapping satisfies (60), then for $x \in G$, we get

$$\left\| T(x) - S(x) \right\| = \lim_{k\to\infty} \left\| n^{3k} T\left(\frac{x}{n^k} \right) - n^{3k} S\left(\frac{x}{n^k} \right) \right\|$$

$$\le \lim_{k\to\infty} \left\| n^{3k} T\left(\frac{x}{n^k} \right) \pm n^{3k} T\left(\frac{x}{n^k} \right) - n^{3k} S\left(\frac{x}{n^k} \right) \right\|$$

$$\le \lim_{n\to\infty} \max\left\{ \begin{matrix} \left\| n^{3k} \left[T\left(\frac{x}{n^k} \right) - f\left(\frac{x}{n^k} \right) \right] \right\| \\ , \left\| n^{3k} \left[f\left(\frac{x}{n^k} \right) - S\left(\frac{x}{n^k} \right) \right] \right\| \end{matrix} \right\}$$

$$\le \frac{1}{|4|} \lim_{j\to\infty} \lim_{k\to\infty} \max\left\{ n^{3k} \psi\left(\frac{x}{n^k}, 0, 0 \right); \ j \le k < n+j \right\} = 0$$

$\therefore$ Therefore $T = S$.

Corollary 3.4. Let $\xi : [0,\infty) \to [0,\infty)$ be a function satisfying

$$\xi \ \ |n|^{-1} t \ \ \le \xi \ \ |n|^{-1} \ \ \lambda(t), \qquad \xi \ \ |n|^{-1} < |n|^{-3}$$

(65)

for all $t \ge 0$. Let $\delta > 0$ and $f : G \to X$ is mapping satisfying the inequality

$$\left\| Df_C(x, y, z) \right\| \le \delta \ \ \xi \ \ \|x\| \ \ + \xi \ \ \|y\| \ \ + \xi \ \ \|z\|$$

(66)

for all $x, y, z \in G$. Then there exists a cubic mapping $T : G \to X$ such that

$$\left\| f(x) - T(x) \right\| \le \frac{1}{|4||n|^3} \delta\xi \ \ \|x\| \ ; \forall x \in G.$$

(67

Proof: Defining $\psi : G^3 \to [0,\infty)$ by

$$\psi(x, y, z) := \delta \ \ \xi \ \ \|x\| \ \ + \xi \ \ \|y\| \ \ + \xi \ \ \|z\|$$

.Since $\xi\left(\left\| \frac{x}{n} \right\| \|x\|^3 \right) < 1$, we have

$$\lim_{k\to\infty} |n|^{3k} \psi\left(\frac{x}{n^k}, \frac{x}{n^k}, \frac{x}{n^k} \right) \le \lim_{k\to\infty} \ \xi |n| |n|^{3 \ \ k} \ \ \psi(x, y, z) = 0$$

(68)

for all $x, y, z \in G$. Also for all $x \in G$,then

$$\psi(x) = \lim_{k\to\infty} \frac{1}{|4||n|^3} \max\left\{ n^{3k}\psi\left(\frac{x}{n^k},0,0\right); \qquad 0 \le k < n \right\}$$

$$= \psi(x,0,0) = \delta\xi \; \|x\|$$

for all $x \in G$. On the other hand

$$\frac{1}{|4||n|^3} \lim_{j\to\infty} \lim_{k\to\infty} \max\left\{ n^{3\rho}\psi\left(\frac{x}{n^{\rho}},0,0\right); j \le l < k+j \right\} = 0$$

for all $x \in G$.The rest of the proof applying the Theorem 3.3.

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