L-Covering Sets and L-Covering Polynomials of Chains

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ABSTRACT

Let P be a finite poset.Fora subset A ofP, the open lower cover set of A is defined as L(A)={x ∈P|x is covered by an a ∈A}. The closed lower coverset of A is defined as L[A]=L(A) ∪ A and A is called an L – covering set of P if L[A] = P. The L – covering number ⋀(P) is the minimum cardinality of aL-covering set. Let L_n^i be the family of all L-covering sets of a chain P_n with cardinality i. Similarly we can define U–covering and N-covering sets of P_n with cardinality i. $\ell(P_n,i) = |L_n^i|$, $u(P_n,i) = |U_n^i|$, $n(P_n,i) = |N_n^i|$. In this paper, we construct L_n^i, and obtain a recursive formula for $\ell(P_n,i)$. Using this recursive formula we construct the polynomial $L(P_n,x) = \sum_{i=\lceil n/2 \rceil}^{n} \ell(P_n,i)x^i$ called L-covering polynomial of P_n.

Keywords: *Poset, L-Covering set, L-Covering Polynomial.*

1. INTRODUCTION

A poset P is finite if it has finite number of elements. Let P be a finite poset. The open lowercover set of A is the set L(A) = {x∈P |x is coveredbyan a∈A}. The closed lower cover set of A is the set L[A] = L(A) ∪ A. We denote L({x}) as L(x). A set A⊆ P is a L-covering set of P ifL[A] = P. The L-covering number⋀(P) is the minimum cardinality of an L-covering set of P. A poset P is a chain if every pair of elements is comparable. Let P_n be the n element chain $x_1 < x_2 < \ldots < x_n$. Let L_n^i be the family of L-covering sets of P_n with cardinality i and let $\ell(P_n,i) = |L_n^i|$. The polynomial $L(P_n,x) = \sum_{i=\Lambda(P_n)}^{n} \ell(P_n,i)x^i$ is called the L-covering polynomial of P_n.

2. *L-covering sets of chains*

In this section we construct the family of L-covering sets of chains by a recursive method. We use ⌈x⌉, for the smallest integer greater than or equal to x. Let L_n^i be the family of L-covering sets of P_n with cardinality i. The following lemma follows from observation.

Lemma 2.1

$\Lambda(P_n) = \lceil \frac{n}{2} \rceil$.

By the definition of L-covering set and by lemma 2.1, we have the following lemma

Lemma 2.2

$L_j^i = \varphi$ if and only if i> j or i<$\lceil \frac{j}{2} \rceil$.

A chain connecting a and b where a < b is a simple chain if every element other than a and b in the chain has exactly one upper cover and lower cover.

The following lemma follows from observation.

Lemma 2.3

If a posetP contains a simple chain of length 2k-1, then every L-covering set of P must contain atleastk elements of the chain.

Tofind a L-covering set of $P_n$withcardinality i, we do not need to consider L-covering sets of P_{n-3} with cardinality i-1. We show this in lemma 2.4. So, we only need to consider $L_{n-1}^{i-1}$and L_{n-2}^{i-1}.

Lemma 2.4

If $D \in L_{n-3}^{i-1}$ and if there exist $x \in P_n$ suchthat $D \cup \{x\} \in L_n^i$ then $D \in L_{n-2}^{i-1}$.

Proof:

Suppose that $D \notin L_{n-2}^{i-1}$. Since $D \in L_{n-3}^{i-1}$, if $x_{n-2} \in D$,then $D \in L_{n-2}^{i-1}$, a contradiction.Hence $x_{n-2} \notin D$. Therefore,$D \cup \{x\} \notin L_n^i$ for any $x \in P_n$,a contradiction.

Lemma 2.5

(i) If $L_{n-1}^{i-1} = L_{n-3}^{i-1} = \varphi$ then $L_{n-2}^{i-1} = \varphi$.
(ii) If $L_{n-1}^{i-1} \neq \varphi$ and $L_{n-3}^{i-1} \neq \varphi$ then $L_{n-2}^{i-1} \neq \varphi$.
(iii) If $L_{n-1}^{i-1} = L_{n-2}^{i-1} = \varphi$ then $L_n^i = \varphi$.

Proof:

(i) Since $L_{n-1}^{i-1} = L_{n-3}^{i-1} = \varphi$ by lemma 2.2, i-1 > n-1or i-1 <$\lceil \frac{(n-3)}{2} \rceil$.
∴ i-1 > n-2 or i-1 <$\lceil \frac{(n-2)}{2} \rceil$ and hence$L_{n-2}^{i-1} = \varphi$
(ii) Suppose that $L_{n-2}^{i-1} = \varphi$, then by lemma 2.2,i-1 > n-2 ori-1 <$\lceil \frac{(n-2)}{2} \rceil$.
If i-1> n-2 then i-1 >n-3 and hence $L_{n-3}^{i-1} = \varphi$, a contradiction.

Therefore, i-1< $\lceil\frac{(n-2)}{2}\rceil \leq \lceil\frac{(n-1)}{2}\rceil$ andhence $L_{n-3}^{i-1}=\varphi$, a contradiction.

(iii) Suppose that $L_n^i \neq \varphi$. Let $D \in L_n^i$. Then $x_n \in D$. By lemma 2.3, atleast one of x_{n-1} or x_{n-2} is in D. If $x_{n-1} \in D$, then $D-\{x_n\} \in L_{n-1}^{i-1}$, a contradiction.

If $x_{n-2} \in D$, again $D-\{x_n\} \in L_{n-2}^{i-1}$, a contradiction.

Lemma 2.6

If $L_n^i \neq \varphi$, then

(i) $L_{n-1}^{i-1}=\varphi$ and $L_{n-2}^{i-1} \neq \varphi$ if and only if n=2k and i=k for some $k \in \mathbb{N}$.

(ii) $L_{n-1}^{i-1} \neq \varphi$ and $L_{n-2}^{i-1}=\varphi$ if and only if I =n.

(iii) $L_{n-1}^{i-1} \neq \varphi$ and $L_{n-2}^{i-1} \neq \varphi$ if and only if $\lceil\frac{(n-1)}{2}\rceil + 1 \leq i \leq n-1$.

Proof:

(i) (⟹) since $L_{n-1}^{i-1} \neq \varphi$, by lemma 2.2, i-1>n-1 or i-1 < $\lceil\frac{(n-1)}{2}\rceil$. If i-1 > n-1, then i>n and hence by lemma 2.2 $L_n^i=\varphi$, a contradiction.

Therefore, i-1 < $\lceil\frac{(n-1)}{2}\rceil$ and since $L_n^i \neq \varphi$ $\lceil\frac{n}{2}\rceil \leq i < \lceil\frac{(n-1)}{2}\rceil + 1$. This gives us n=2k and i=k for some $k \in \mathbb{N}$.

(⟸) If n=2k and i=k for some $k \in \mathbb{N}$, then by lemma2.2, $L_{n-1}^{i-1}=\varphi$ and $L_{n-2}^{i-1} \neq \varphi$.

(ii) (⟹) since $L_{n-2}^{i-1}=\varphi$, by lemma 2.2, i-1>n-2 or i-1< $\lceil\frac{(n-2)}{2}\rceil$. If i-1 < $\lceil\frac{(n-2)}{2}\rceil$ then i-1 < $\lceil\frac{(n-1)}{2}\rceil$ and hence $L_{n-1}^{i-1}=\varphi$, a contradiction. Therefore, i-1 > n-2 and so i>n-1. Also, since $L_n^i \neq \varphi$, i≤ n and hence i = n.

(⟸) If i=n, then by lemma 2.2, $L_{n-1}^{i-1} \neq \varphi$ and $L_{n-2}^{i-1}=\varphi$

(iii) (⟹) since $L_{n-1}^{i-1} \neq \varphi$ and $L_{n-2}^{i-1} \neq \varphi$, $\lceil\frac{(n-1)}{2}\rceil \leq i-1 \leq n-2$ and hence $\lceil\frac{(n-1)}{2}\rceil + 1 \leq i \leq n-1$.

(⟸) If $\lceil\frac{(n-1)}{2}\rceil + 1 \leq i \leq n-1$, by lemma 2.2, $L_{n-1}^{i-1} \neq \varphi$, and $L_{n-2}^{i-1} \neq \varphi$.

Theorem 2.7

For every $n \geq 3$ and $i \geq \lceil\frac{n}{2}\rceil$

(i) If $L_{n-1}^{i-1}=\varphi$ and $L_{n-2}^{i-1} \neq \varphi$, then $L_n^i = \{\{x_2, x_4, x_6, \ldots, x_n\}\}$.

(ii) If $L_{n-1}^{i-1} \neq \varphi$ and $L_{n-2}^{i-1}=\varphi$, then $L_n^i = \{\{x_1, x_2, x_3, \ldots, x_n\}\}$.

(iii) If $L_{n-1}^{i-1} \neq \varphi$ and $L_{n-2}^{i-1} \neq \varphi$, then $L_n^i = \{\{x_n\} \cup X \mid X \in L_{n-1}^{i-1}\} \cup \{\{x_n\} \cup X \mid X \in L_{n-2}^{i-1}\}$.

Proof:

(i) $L_{n-1}^{i-1}=\varphi$ and $L_{n-2}^{i-1} \neq \varphi$. So, by lemma 2.6 (i), n=2k and i=k for some $k \in N$.

Therefore, $L_n^i = L_n^{\frac{n}{2}} = \{\{x_2, x_4, x_6, \ldots, x_n\}\}$

(ii) $L_{n-1}^{i-1} \neq \varphi$ and $L_{n-2}^{i-1}=\varphi$. So, by lemma 2.6 (ii), i=n.

Therefore, $L_n^i = L_n^n = \{\{x_1, x_2, x_3, \ldots, x_{n-1}, x_n\}\}$

(iii) $L_{n-1}^{i-1} \neq \varphi$ and $L_{n-2}^{i-1} \neq \varphi$. Let $X_1 \in L_{n-1}^{i-1}$. Then $X_1 \cup \{x_n\} \in L_n^i$. Let $X_2 \in L_{n-2}^{i-1}$. Again, $X_2 \cup \{x_n\} \in L_n^i$. Hence, we have $\{\{x_n\} \cup X \mid X \in L_{n-1}^{i-1}\} \cup \{\{x_n\} \cup X \mid X \in L_{n-2}^{i-1}\} \subseteq L_n^i$ (1)

Conversely, let $Y \in L_n^i$. Then $x_n \in Y$. By lemma 2.3, atleast one of x_{n-1} or x_{n-2} is in Y. If $x_{n-1} \in Y$, then $Y = X \cup \{x_n\}$ for some $X \in L_{n-1}^{i-1}$. If $x_{n-2} \in Y$, then $Y = X \cup \{x_n\}$ for some $X \in L_{n-2}^{i-1}$. Therefore,

$$L_n^i \subseteq x_n\} \cup X \mid X \in L_{n-1}^{i-1}\} \cup \{\{x_n\} \cup X \mid X \in L_{n-2}^{i-1}\} \quad (2)$$

From (1) and (2), we get (iii)

Table.1 $\ell(P_n,j)$ the number of L-Covering sets of P_n with cardinality j.

j / n	1	2	3	4	5	6	7	8	9	10
1	1									
2	1	1								
3	0	2	1							
4	0	1	3	1						
5	0	0	3	4	1					
6	0	0	1	6	5	1				
7	0	0	0	4	10	6	1			
8	0	0	0	1	10	15	7	1		
9	0	0	0	0	5	20	21	8	1	
10	0	0	0	0	1	15	35	28	9	1

3. *L-covering polynomial of a chain*

Let $L(P_n,x) = \sum_{i=\lceil\frac{n}{2}\rceil}^{n} \ell(P_n,i)\, x^i$ be the L-covering polynomial of a chain P_n. In this section we study this polynomial.

Theorem 3.1

(i) If L_n^i is the family of L-covering sets with cardinality i of P_n, then $|L_n^i| = |L_{n-1}^{i-1}| + |L_{n-2}^{i-1}|$.

(ii) For every $n \geq 3$, $L(P_n, x) = x\,[L(P_{n-1}, x) + L(P_{n-2}, x)]$ with initial values $L(P_1,x)=x$ and $L(P_2, x)=x^2+x$.

Proof

(i) It follows from Theorem 2.7

(ii) It follows from part (i) and the definition of the L-covering Polynomial.

REFERENCES

[1] M. Bayer and J. Billera, Counting chains and Faces in Polytopes andPosets, Contemporary Mathematics, 34, 207-252 (1984).

[2] P. Crawley and R.P.Dilworth, Algebraic theory of Lattices, Prentice-Hall, Inc. Englewod, Cliffs, New Jersey. 1973.

[3] B.A Davey and H.A.Priestley, Introduction to Lattices and Order, Second Edition, Cambridge University Press, 2002.

[4] Garrett Birkhoff, Lattice Theory, American Mathematical Society Colloquim Publications, Vol.XXV, 1961.

[5] G.Grätzer, General Lattice Theory, BirkhauserVerlag, Basel, 1978.

[6] Greene C. On the Mobius algebra of a partially ordered set, Advances in math. 10, 177-187(1973).

[7] Gunter M. Ziegler, Lectures on Polytopes, Springer – Verlag, New York, Inc., 1995.

[8] Paffenholz and Andreas, Construction for posets, Lattices, and Polytopes, Doctoral Dissertation, School of Mathematical and Natural Sciences, Technical University of Berlin, 2005.

[9] SaeidAlikhani and Yee-Hock Peng, Dominating Sets and Domination Polynomials of Paths, International Journal of Mathematics and Mathematical Sciences, Vol.2009, PP.1-10.

[10] R.P.Stanley, Enumerative Combinatorics, Volume 1, Wordsworth and Brooks / Cole, 1986.

[11] R.Subbarayan andA.Vethamanickam, On the Lattice of Convex Sub lattices, Elixir Dis. Math. 50 (2012), 10471-10474.

[12] A.Vethamanickam, Topics in Universal Algebra, Ph.D. Thesis, Madurai Kamaraj University, 1994.

[13] A.Vethamanickam,and R.Subbarayan, Simple extensions of Eulerian Lattices, Acta Math. Univ. Comenianae, Vol. LXXIX, I(2010), PP.47-54.

[14] A.Vethamanickam and R.Subbarayan, Some Properties of Eulerian Lattices, CommentationesMathematicaeUniversitatitsCarolinae, Vol.55 (2014), (4) PP.499-507.

[15] A.Vethamanickam and K.M.Thirunavukkarasu, U-covering sets and U-covering polynomials of chains, International Journal of Mathematical Archive, Vol.8 (8), 2017, 41-44.