Hungarian Method to Solve Travelling Salesman Problem with Fuzzy Cost

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Abstract

The Travelling Salesman problem is one of the most common problem in O.R. But in real life situation the problem is often not useful due to its crisp nature. Use of fuzzy can make the problem more useful in day to day life. We consider a Travelling Salesman Problem with costs of travelling as trapezoidal fuzzy numbers. Fuzzy ranking function is used to handel trapezoidal fuzzy numbers. Hungering Algorithm is used to solve the problem. An example illustrates the use of the proposed method.

Keywords: *Travelling Salesman Problem, Fuzzy .*

1 Introduction

The travelling salesman problem was mathematically formulated in the 1800s by the Irish mathematician W.R. Hamilton and by the British mathematician Thomas Kirkman. Hamiltons Icosian Game was a recreational puzzle based on finding a Hamiltonian cycle[2]. The general form of the TSP appears to have been first studied by mathematicians during the 1930s in Vienna and at Harvard.Hassler Whitney[6]at Princeton University introduced the name travelling salesman problem.

Sarhadi and Ghoseiri[7] extended Travelling salesman problem with time windows to model fuzzy travelling salesman problem with time windows. basic advantage of this new modelling is its flexibility which enables us to optimize the delivery process in the network of customers with different satisfaction patterns and different priorities. Foldesi and Botzheim[4]stated that practical application in road transportation and supply chains are often fuzzy. The risk attitude depends on the features of the given operation. The model presented handles the fuzzy, time dependent nature of the TSP and also gives solution for the asymmetric loss aversion by embedding the risk attitude into the fitness function of the bacterial memetic algorithm. Chaudhuri and De[3] defined Fuzzy Multi-Objective Linear Programming which is used for solving TSP with vague and imprecise parameters. Objectives and imprecise parameters are also discussed. Kumar and Gupta[5] proposed methods for solving fuzzy assignment problems and fuzzy travelling salesman problems which cannot be solved by using existing methods.

The General Travelling salesman problem is stated as follows:

Suppose a salesman has to visit n cites. He wishes to start from a particular city, visit each city once and return to starting point.when the cost of travelling from each city to other are given. The problem is to find out a particular path out of (n-1)! paths so as to minimize total cost (or time). But in real life situation the cost of travellings are not fixed. We represent the cost as trapezoidal fuzzy number, convert them to crisp number by using ranking method[1] and then solve by Hungarian Algorithm

2 Basic Preliminary

1) 2.1 Trapezoidal fuzzy number

A fuzzy number $\tilde{B}$ is a trapezoidal fuzzy number denoted by $(\alpha,\beta,\gamma,\delta)$ where α,β,γ and δ are real number and its membership function $\mu_B(x)$ is given below

$$\mu_B(x)=\begin{cases} 0 \ for\ x\le\alpha \\ \dfrac{(x-\alpha)}{(\beta-\alpha)}\ for\ \alpha\le x\le\beta \\ 1\ for\ \beta\le x\le\gamma \\ \dfrac{(x-\gamma)}{(\delta-\gamma)}\ for\ \gamma\le x\le\delta \\ 0\ for\ x\ge\delta \end{cases} \quad (1)$$

According to the above mentioned definition of a Trapezoidal fuzzy number , let B = ($\underline{B}$ (r), $\overline{B}$ (r)), $(0\le r\le 1)$ be a fuzzy number, then the value M(B), is assigned to B is calculated as follows[1];

$$M_O(B)=\frac{1}{2}\int_0^1\{\underline{B}(r)+\overline{B}(r)\}dr=\frac{1}{4}(\alpha+\beta+\gamma+\delta) \quad (2)$$

Which is very convenient for calculation.

3 Mathematical Model

3.0.1 Mathematical model of travelling Salesman Problem

We can state the travelling salesman problem as an Assignment problem as follows:

Suppose C_{ij} is the cost of travelling from city i to city j.

Let

$$\sum_{j=1}^{n}X_{ij}=1,\ for\ i=1,2,\ldots,n \quad (3)$$

$$\sum_{i=1}^{n} X_{ij} = 1, \; for \; j = 1,2,\ldots,n \qquad (4)$$

$$X_{ij} = \begin{cases} 1, if \text{ the salesman travels from city } i \text{ to city } j \\ 0 \; otherwise \end{cases}$$

(5)

Then the problem is to minimize total cost

$$Z = \sum_{i=1}^{m}\sum_{j=1}^{n} C_{ij} X_{ij} \qquad (6)$$

3.0.2 Mathematical model of a travelling salesman problem with fuzzy cost values

We can state a Travelling salesman Problem as a Fuzzy Travelling Salesman Problem if the cost of travelling from each city to other are fuzzy numbers $\tilde{C}_{ij}$. Then the mathematical formulation of the problem is.

$$\min : Z = \sum_{i=1}^{n}\sum_{j=1}^{n} \tilde{C}_{ij} X_{ij} \qquad (7)$$

Subject to

$$\sum_{j=1}^{n} X_{ij} = 1, \; fori = 1,2,\ldots,n \qquad (8)$$

$$\sum_{i=1}^{n} X_{ij} = 1, \; forj = 1,2,\ldots,n \qquad (9)$$

$$X_{ij} = \begin{cases} 1, \; if \text{ travels from city } i \text{ to city } j \\ 0 \quad otherwise \end{cases} \qquad (10)$$

Let the cost coefficient $\tilde{C}_{ij}$ are trapezoidal fuzzy numbers. i.e. $\tilde{C}_{ij} = (C_{ij}^{(1)}, C_{ij}^{(2)}, C_{ij}^{(3)}, C_{ij}^{(4)})$

We use fuzzy ranking function for each $\tilde{C}_{ij} = (C_{ij}^{(1)}, C_{ij}^{(2)}, C_{ij}^{(3)}, C_{ij}^{(4)})$ to find a crisp equivalent.

We use

$$M_O(C_{ij}) = \frac{1}{2}\int_0^1 \{\underline{C_{ij}}(r) + C_{ij}(r)\}dr$$
$$= \frac{1}{4}(C_{ij}^{(1)} + C_{ij}^{(2)} + C_{ij}^{(3)} + C_{ij}^{(4)}) \qquad (11)$$

By using this conversion the problem reduced to.

$$\min : Z = \sum_{i=1}^{m}\sum_{j=1}^{n} T_O(C_{ij}) X_{ij} \qquad (12)$$

Subject to

$$\sum_{j=1}^{n} X_{ij} = 1, \; fori = 1,2,\ldots,m \qquad (13)$$

$$\sum_{i=1}^{m} X_{ij} = 1, \; forj = 1,2,\ldots,n \qquad (14)$$

$$X_{ij} = \begin{cases} 1, \; if \text{ travels from city } i \text{ to city } j \\ 0 \quad otherwise \end{cases} \qquad (15)$$

Which is a crisp travelling salesman problem can be solve using any method. the most commonly used method is Hungarian Algorithm.

3.0.3 Hungarian Algorithm

step-1 Check whether the cost matrix is square, if not make it square by adding suitable number of dummy row (or column) with cost value 0.

Step-2 Locate the smallest cost elements in each row of the cost matrix. Subtract this element from every other elements of corresponding row. As a result there is at least one zero in each row of the reduced cost matrix.

Step-3 In the reduced cost matrix obtained, locate the smallest element of each column and subtract that from each elements of the corresponding column. As a result there should be at least one zero in each rows and columns of the second reduced cost matrix.

Step-4 In the modified cost matrix obtained in step 4, search for an optimal assignment as follows.

(a) Examine the row successively until a row with single zero is found. Enrectangle (W) this zero and cross off (X) all other zeros in its column. Continue this process until all the rows have been taken care of.

(b) Repeat the procedure for each column of the reduced matrix.

(c) If a row and (or) a column has two or more zeros and one cannot be chosen by inspection then assign arbitrary any one of these zeros and cross off all other zeros of that row / column.

(d)Repeat (a) through (c) above successively until the chain of assigning (W) or (X) ends.

Step-5 If the number of assignment is less then n (order of the matrix) an optimal solution is reached. If the numbers of assignments are less then n, go to next step.

Step-6 Draw the minimum number of horizontal and / or vertical lines to cover all the zeros of the reduced cost matrix. This can be conveniently done by the procedure.

(a) Mark (√) rows that do not have any assigned zero.

(b) Mark (√) column that have zero in marked rows.

(c) Mark (√) rows that have assigned zero in marked column.

(d) Repeat (b) and (c) above until the chain of marking completed.

(e) Draw line through all the unmarked rows and marked column. This gives us the desired minimum number of lines.

Step-7 Develop the new revised cost matrix as follows.

(a) Find the smallest element of the reduced cost matrix not covered by any of the lines.

(b) Subtract this element from all the uncovered elements and add the same to all the elements lying at the intersection of any two lines.

Step-8 Go to step 6 and repeat the procedure until an optimum solution is obtained. Using the above algorithm we can solve the given problem.

4 Numerical Example

A travelling salesman planned to visit to 5 cities. He likes to start his visit from his home city A and visits all other cities once and return to his home city. The costs of travel is given below in form of trapezoidal fuzzy numbers

The table given below show the fuzzy cost value of travelling from each city to other. The costs are in the form of trapezoidal fuzzy numbers.

–	(5,6,8,9)	(3,4,6,7)	(0,2,4,6)	(3,4,6,7)
(5,6,8,9)	–	(5,6,10,11	(2,3,5,6)	(0,2,4,6)
(5,6,8,9)	(5,6,10,11	–	(4,5,7,8)	(0,1,3,4)
(0,2,4,6)	(2,3,5,6)	(4,5,7,8)	–	(0,1,3,4)
(3,4,6,7)	(0,2,4,6)	(0,1,3,4)	(0,1,3,4)	–

Solution

We first convert all trapezoidal fuzzy numbers to crisp number by using 16

$$M_O(B)=\frac{1}{4}(\alpha+\beta+\gamma+\delta) \quad (16)$$

$$M_O(C_{12})=7 \quad (17)$$
$$M_O(C_{13})=5 \quad (18)$$
$$M_O(C_{14})=3 \quad (19)$$
$$M_O(C_{15})=5 \quad (20)$$

$$M_O(C_{21})=7 \quad (21)$$
$$M_O(C_{23})=8 \quad (22)$$
$$M_O(C_{24})=4 \quad (23)$$
$$M_O(C_{25})=3 \quad (24)$$

$$M_O(C_{31})=7 \quad (25)$$
$$M_O(C_{32})=8 \quad (26)$$
$$M_O(C_{34})=6 \quad (27)$$
$$M_O(C_{35})=2 \quad (28)$$

$$M_O(C_{41})=3 \quad (29)$$
$$M_O(C_{42})=4 \quad (30)$$
$$M_O(C_{43})=6 \quad (31)$$
$$M_O(C_{45})=2 \quad (32)$$

$$M_O(C_{51})=5 \quad (33)$$
$$M_O(C_{52})=3 \quad (34)$$
$$M_O(C_{53})=2 \quad (35)$$
$$M_O(C_{54})=2 \quad (36)$$

thus the table of cost coefficient is reduced to.

–	7	5	3	5
7	–	8	4	3
7	8	–	6	2
3	4	6	–	2
5	3	2	2	–

Subtracting lowest element of each row from all entries of the row we have.

–	4	2	0	2
4	–	5	1	0
3	6	–	4	0
1	2	4	–	0
3	1	0	0	–

Subtracting lowest element of each column from all entries of the column we have.

–	3	2	0	2
3	–	5	1	0
2	5	–	4	0
0	1	4	–	0
2	0	0	0	–

By moving with zero assignments we get

–	3	2	0	2
3	–	5	1	0
2	5	–	4	0
0	1	4	–	0
2	0	0	0	–

Drawing lines through row-1, row-4 and column 5 we found that the schedule is not completed, we deduce minimum uncovered value (i.e.1) out of all uncovered values and adding it at the intersection of lines we get the new reduced cost matrix. Making assignments by the same procedure, and drawing parallel lines through unmarked rows and marked columns, we have.

–	3	2	0	3
2	–	4	0	0
1	4	–	3	0
0	1	4	–	0
2	0	0	0	–

Drawing lines through row-4, row-5 and column 4 and column 5 we found that The schedule is still not complete. Again let us deduct the minimum value 1 out of all uncovered elements and add it at the intersection of lines we get the improved matrix.

–	1	0	0	3
1	–	2	0	0
0	3	–	2	0
0	0	3	–	1
2	0	0	1	–

Now we can assign. Two cases arises.

Case-1 :

$A \to C \to E \to B \to D \to A$

Total cost of travelling

= 20 + 20 + 7.5 + 40 + 15 = 102.5

Or

Case-2 :

A $\to D \to B \to E \to C \to A$

Total cost of travelling

= 30 + 10 + 15 + 8 + 35 = 98 So we consider the 2nd path .

A $\to D \to B \to E \to C \to A$ with approximate cost of travel = 98.

5 Result Discussion And Conclusion

In the above paper we considered a Travelling Salesman Problem with fuzzy cost coefficient, the uncertainties factors in real life situation are represented trough fuzzy numbers (Trapezoidal fuzzy numbers). We convert the value to a crisp equivalent by using ranking function and then use Hungarian Algorithm to solve the problem. Further research can help to improve the solution by using other ranking functions. Methods other then Hungarian can also be used to solve the problem. Use of Fuzzy Variables can be more effective to find a solution in fuzzy form.

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