Γ-Semigroups in which Prime Γ-Ideals are Maximal

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ABSTRACT: *In this paper, the terms, Maximal Γ-ideal, primary Γ-semigroup and Prime Γ-ideal are introduced. It is proved that if S is a Γ- semigroup with identity and if (non zero, assume this if S has zero) proper prime Γ-ideals in S are maximal then S is primary Γ-semigroup. Also it is proved that if S is a right cancellative quasi commutative Γ-emigroup and if S is a primary Γ- semigroup or a Γ- semigroup in which semiprimary Γ- ideals are primary, then for any primary Γ-ideal Q, √ Q is non-maximal implies Q = √ Q is prime. It is proved that if S is a right cancellative quasi commutative Γ-semigroup with identity, then 1) Proper prime Γ-ideals in S are maximal. 2) S is a primary Γ-semigroup. 3) Semiprimary Γ-ideals in S are primary, 4) If x and y are not units in S, then there exists natural numbers n and m such that $(x\Gamma)^{n-1}x = y\Gamma s$ and $(y\Gamma)^{m-1}y = x\Gamma r$. For some s, r ∈ S are equivalent. Also it is proved that if S is a duo Γ–semigroup with identity, then 1) Proper prime Γ– ideals in S are maximal. 2) S is either a Γ– group and so Archimedian or S has a unique prime Γ–ideal P such that S = G∪ P, where G is the Γ–group of units in S and P is an Archimedian sub Γ–semi group of S are equivalent. In either case S is a primary Γ–semigroup and S has atmost one idempotent different from identity. It is proved that if S is a duo Γ-semigroup without identity, then S is a primary Γ-semigroup in which proper prime Γ-ideals are maximal if and only if S is an Archimedian Γ-semigroup. It is also proved that if S is a quasi commutative Γ-semigroup containing cancellable elements, then 1) The proper prime Γ-ideals in S are maximal. 2) S is a Γ-group or S is a cancellative Archimedian Γ-semigroup not containing identity or S is an extension of an Archimedian Γ-semigroup by a Γ-group S containing an identity are equivalent.*

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KEY WORDS: Γ-semigroup, Maximal Γ-ideal, primary Γ-semigroup, commutative Γ-semigroup, left (right) identity, identity, Zero element and Prime Γ-ideal.

1. INTRODUCTION:

Γ- semigroup was introduced by Sen and Saha [10] as a generalization of semigroup. Anjaneyulu. A [1], [2] and [3] initiated the study of pseudo symmetric ideals, radicals and semi pseudo symmetric ideals in semigroups. Giri and Wazalwar [4] intiated the study of prime radicals in semigroups. Madhusudhana Rao, Anjaneyulu and Gangadhara Rao [5], [6], [7] and [8] initiated the study of prime radicals and semi pseudo symmetric Γ-ideals in Γ-semigroups, primary and semiprimary Γ-ideals and pseudo symmetric Γ-ideals in Γ-semigroups. In this paper we characterize Quasi Commutative Γ-semigroup, semi pseudo symmetric Γ-semigroups and quasi commutative Γ-semigroups containing cancellable elements in which proper prime Γ-ideals are maximal. We first cite a wide class of primary Γ-semigroups.

2. PRELIMINARIES:

DEFINITION 2.1 : Let S and Γ be any two non-empty sets. Then S is said to be a **Γ-semigroup** if there exist a mapping from S × Γ × S to S which maps (a, γ ,b) ⟶ a γ b satisfying the condition : (aαb)βc = aα(bβc) for all a, b, c ∈ S and α,β, γ ∈ Γ.

NOTE2.2 : Let S be a Γ-semigroup. If A and B are two subsets of S, we shall denote the set { *aγb : a ∈ A , b ∈ B* and *γ* ∈ Γ } by AΓB.

DEFINITION 2.3 : A Γ-semigroup S is said to be ***commutative*** **Γ-semigroup** provided *aγb = bγa* for all *a, b* ∈ *S* and *γ* ∈ Γ.

NOTE 2.4 : If S is a commutative Γ-semigroup then *a Γb = b Γa* for all *a, b* ∈ *S.*

NOTE 2.5 : Let S be a Γ-semigroup and *a, b* ∈ S and *α* ∈ Γ. Then *aαaαb* is denoted by $(a\alpha)^2b$ and consequently *a α a α a α*…..(*n terms*)*b* is denoted by $(a\alpha)^n b$.

DEFINITION 2.6: A Γ-semigroup S is said to be ***quasi commutative*** provided for each *a, b* ∈ S, there exists a natural number *n* such that $a\gamma b = (b\gamma)^n a \ \forall \gamma \in \Gamma$.

NOTE 2.7 : If a Γ-semigroup S is ***quasi commutative*** then for each $a, b \in S$, there exists a natural number n such that, $a\Gamma b = (b\,\Gamma)^n a$.

DEFINITION 2.8 : An element ***a*** of a Γ- semigroup S is said to be a ***left identity*** of S provided $a\alpha s = s$ for all $s \in S$ and $\alpha \in \Gamma$.

DEFINITION 2.9: An element ***a*** of a Γ- semigroup S is said to be a ***right identity of S*** provided $s\alpha a = s$ for all $s \in S$ and $\alpha \in \Gamma$.

DEFINITION 2.10 : An element ***a*** of a Γ-semigroup S is said to be a ***two sided identity*** or an identity provided it is both a left identity and a right identity of S.

NOTATION 2.11 : Let S be a Γ- semigroup. If S has an identity , let S^1 = S and if S does not have an identity, let S^1 be the Γ- semigroup S with an identity adjoined, usually denoted by the symbol 1.

DEFINITION 2.12 : An element ***a*** **of** a Γ- semigroup S is said to be a ***left zero*** of S provided $a\Gamma s = a$ for all s belongs S.

DEFINITION 2.13 : An element ***a*** **of** a Γ-semigroup S is said to be a ***right zero*** of S provided $s\Gamma a = a$ for all s belongs S.

DEFINITION 2.14 : An element ***a*** **of** a Γ-semigroup S is said to be a ***zero*** of S provided it is both left and right zero of S.

DEFINITION 2.15 : A nonempty subset A of a Γ-semigroup S is said to be a ***left Γ-ideal*** of S if $s \in S, a \in A, \alpha \in \Gamma$ implies $s\alpha a \in A$.

NOTE 2.16: A nonempty subset A of a Γ-semigroup S is a ***left Γ- ideal*** of S iff $S\,\Gamma A \subseteq A$.

DEFINITION 2.17: A nonempty subset A of a Γ-semigroup S is said to be a ***right Γ-ideal*** of *S if* $s \in S, a \in A, \alpha \in \Gamma$ implies $a\alpha s \in A$.

NOTE 2.18 : A nonempty subset A of a Γ-semigroup S is a ***right Γ- ideal*** of S iff $A\Gamma S \subseteq A$.

DEFINITION 2.19 : A nonempty subset A of a Γ-semigroup S is said to be a ***two sided Γ-ideal*** or simply a ***Γ- ideal*** of S if $s \in S$, $a \in A$, $\alpha \in \Gamma$ imply $s\alpha a \in A$, $a\alpha s \in A$.

DEFINITION 2.20 :A Γ-ideal A of a Γ-semigroup S is said to be a ***maximal Γ-ideal*** provided A is a proper Γ-ideal of S and is not properly contained in any proper Γ-ideal of S.

DEFINITION 2.21: A Γ-ideal P of a Γ-semigroup S is said to be a ***completely prime Γ-ideal*** provided $x, y \in S$ and $x\Gamma y \subseteq P$ implies either $x \in P$ or $y \in P$.

DEFINITION 2.22: A Γ- ideal P of a Γ-semigroup S is said to be a ***prime Γ- ideal*** provided A, B are two Γ-ideals of S and $A\Gamma B \subseteq P \Rightarrow$ either $A \subseteq P$ or $B \subseteq P$.

DEFINITION 2.23: A Γ-ideal A of a Γ-semigroup S is said to be a ***completely semiprime Γ- ideal*** provided $x\Gamma x \subseteq A$; $x \in S$ implies $x \in A$.

DEFINITION 2.24: A Γ- ideal A of a Γ-semigroup S is said to be a ***semiprime Γ- ideal*** provided $x \in S$, $x\Gamma S^1\Gamma x \subseteq A$ implies $x \in A$.

DEFINITION 2.25: If A is a Γ-ideal of a Γ-semigroup S, then the intersection of all prime Γ-ideals of S containing A is called ***prime Γ-radical*** or simply ***Γ-radical*** of A and it is denoted by **√A** or ***rad A***.

THEOREM 2.26 [5]: If A is a Γ-ideal of a Γ-semigroup S then √A is a semiprime Γ-ideal of S.

THEOREM 2.27 [5]: A Γ- ideal Q of a Γ-semigroup S is a semiprime Γ- ideal of S iff ***√(Q) = Q*** implies $x\,\Gamma S^1\Gamma y \subseteq A$.

DEFINITION 2.28 : An element ***a*** of a Γ-semigroup S is said to be ***left cancellative*** provided $a\ \Gamma x = a\ \Gamma y$ for all $x, y \in S$ implies $x = y$.

DEFINITION 2.29: An element ***a*** of a Γ- semigroup S is said to be ***right cancellative*** provided $x\,\Gamma a = y\,\Gamma a$ for all $x, y \in S$ implies $x = y$.

DEFINITOIN 2.30: An element ***a*** of a Γ- semigroup S is said to be ***cancellative*** provided it is both left and right cancellative element.

DEFINITOIN 2.31: A Γ-ideal A of a Γ- semigroup S is said to be semiprimary provided √A is a prime Γ-ideal of S.

DEFINITOIN 2.32: A Γ- semigroup S is said to be a semiprimary Γ- semigroup provided every Γ- ideal of S is a semiprimary Γ- ideal.

DEFINITION 2.33 : A Γ-ideal A of a Γ- semigroup S is said be a ***left primary Γ-ideal*** provided

i) If X, Y are two Γ-ideals of S such that $X\,\Gamma Y \subseteq A$ and $Y \not\subseteq A$ then $X \subseteq \sqrt{A}$.

ii) √A is a prime Γ-ideal of S.

DEFINITION 2.34 : A Γ-ideal A of a Γ- semigroup S is said be a ***right primary Γ-ideal*** provided

i) If X, Y are two Γ-ideals of S such that $X\Gamma Y \subseteq A$ and $X \nsubseteq A$ then $Y \subseteq \sqrt{A}$.

ii) √A is a prime Γ-ideal of S.

EXAMPLE 2.35 : Let $S = \{a, b, c\}$ and $\Gamma = \{x, y, z\}$. Define a binary operation **.** in S as shown in the following table.

.	*a*	*b*	*c*
a	*a*	*a*	*a*
b	*a*	*a*	*a*
c	*a*	*b*	*c*

Define a maping $S X\Gamma X S \to S$ by $a\, \alpha b = ab$, for all $a, b \in S$ and $\alpha \in \Gamma$. It is easy to see that S is a Γ-semigroup. Now consider the Γ-ideal $<a> = S^1\Gamma a\, \Gamma S^1 = \{a\}$. Let $p\, \Gamma q \subseteq\ <a>$, $p \notin <a>$ $\Rightarrow\ q \in \sqrt{<a>} \Rightarrow\ (q\, \Gamma)^{n-1} q\ \subseteq\ <a>$ for some $n \in N$. Since $b\Gamma c \subseteq\ <a>$, $c \notin <a> \Rightarrow b \in\ <a>$. Therefore $<a>$ is left primary. If $b \notin <a>$ then $(c\ \Gamma)^{n-1} c \notin <a>$ for any $n \in N \Rightarrow c \notin \sqrt{<a>}$. Therefore $<a>$ is not right primary.

DEFINITION 2.36: A Γ-ideal A of a Γ- semigroup S is said to be a **primary Γ-ideal** provided A is both left primary Γ-ideal and right primary Γ-ideal.

DEFINITION 2.37: A Γ-ideal A of a Γ- semigroup S is said to be a **principal Γ-ideal** provided A is a Γ-ideal generated by a single element **a**. It is denoted by J[a] = <a>.

DEFINITION 2.38: An element ***a*** of a Γ-semigroup S with 1 is said to be ***left invertible*** or ***left unit*** provided there is an element $b \in S$ such that $b\, \Gamma a = 1$.

DEFINITION 2.39: An element ***a*** of a Γ-semigroup S with 1 is said to be ***right invertible*** or ***right unit*** provided there is an element $b \in S$ such that $a\, \Gamma b = 1$.

DEFINITION 2.40: An element ***a*** of a Γ-semigroup S is said to be ***invertible*** or a ***Unit*** in S provided it is both left and right invertible element in S.

DEFINITION 2.41: A Γ-ideal A of a Γ- semigroup S is said to be ***pseudo symmetric*** provided $x, y \in S$, $x\, \Gamma y \subseteq A$ implies $x\, \Gamma s \Gamma y \subseteq A$, for all $s \in S$.

NOTE 2.42: A Γ-ideal A of a Γ- semigroup S is ***pseudo symmetric*** iff $x, y \in S$, $x\, \Gamma y \subseteq A$ implies $x\Gamma S^1\Gamma y \subseteq A$.

DEFINITION 2.43: A Γ- semigroup S is said to be ***pseudo symmetric*** provided every Γ-ideal is a pseudo symmetric Γ-ideal.

DEFINITION 2.44: A Γ-ideal A in a Γ- semigroup S is said to be a **semi pseudo symmetric Γ-ideal** provided for any natural number n, $x \in S$, $(x\, \Gamma)^{n-1} x \subseteq A \Rightarrow (<x>\, \Gamma)^{n-1} <x> \subseteq A$.

DEFINITION 2.45: A Γ- semigroup S is said to be an ***Archimedian* Γ- *semigroup*** provided for any $a, b \in S$, there exists a natural number n such that $(a\Gamma)^{n-1}a \subseteq\ < b >$.

DEFINITION 2.46: A Γ-semigroup S is said to be a ***strongly Archimedean* Γ*-semigroup*** provided for any $a, b \in$ S, there is a natural number n such that $(< a >\Gamma)^{n-1} < a > \subseteq\ < b >$.

DEFINITION 2.47: An element ***a* of** a Γ- semigroup S is said to be ***semisimple*** provided $a \in <a>\, \Gamma <a>$, that is, $<a>\, \Gamma <a>\ =\ <a>$.

DEFINITION 2.48: A Γ- semigroup S is said to be ***semisimple Γ- semigroup*** provided every element is a semisimple.

DEFINITOIN2.49: A Γ-semigroup S is said to be a ***simple Γ-semigroup*** provided S has no proper Γ- ideals.

DEFINITION 2.50: An element ***a*** of a Γ- semigroup S is said to be a ***Γ-idempotent*** provided $a \propto a = a$ for all $\propto \in \Gamma$.

NOTE 2.51: If an element ***a*** of a Γ- semigroup S is a **Γ-idempotent**, then $a\, \Gamma a = a$.

DEFINITION 2.52: A Γ- semigroup S is said to be an **idempotent Γ- semigroup** or a **band** provided every element in S is a Γ-idempotent.

DEFINITION 2.53: A Γ- semigroup S is said to be a **globally idempotent Γ- semigroup** provided $S\, \Gamma S = S$.

DEFINITION 2.54: A Γ- semigroup S is said to be a ***left duo* Γ- *semigroup*** provided every left Γ- ideal of S is a two sided Γ- ideal of S.

DEFINITION 2.55: A Γ- semigroup S is said to be a ***right duo* Γ- *semigroup*** provided every right Γ- ideal of S is a two sided Γ- ideal of S.

DEFINITION 2.56: A Γ- semigroup S is said to be a ***duo* Γ- *semigroup*** provided it is both a left duo Γ- semigroup and a right duo Γ- semigroup.

DEFINITION 2.57: A non empty subset P of a Γ-semigroup S is said to be an **Archimedian Γ-sub semigroup** of S provided P is itself an Archimedain Γ-semigroup.

DEFINITOIN 2.58: An element ***a*** of a Γ- semigroup S is said to be ***regular*** provided $a = a\alpha x\beta a$ for some $x \in S$, $\alpha, \beta \in \Gamma$. i.e, $a \in a\Gamma S\Gamma a$.

DEFINITION 2.59: A Γ- semigroup S is said to be a ***regular Γ- semigroup*** provided every element is regular.

DEFINITION 2.60: An element ***a*** of a Γ- semigroup S is said to be ***left regular*** provided $a = a\alpha a\beta x$, for some $x \in S$, $\alpha, \beta \in \Gamma$. i.e, $a \in a\Gamma a\Gamma S$.

DEFINITION 2.61: An element ***a*** of a Γ- semigroup S is said to be ***right regular*** provided $a = x\alpha a\beta a$, for some $x \in S$, $\alpha, \beta \in \Gamma$. i.e, $a \in S\Gamma a\Gamma a$.

DEFINITION 2.62: An element ***a*** of a Γ - semigroup S is said to be ***completely regular*** provided there exists an element $x \in S$ such that $a = a\alpha x\beta a$ for some $\alpha, \beta \in \Gamma$ and $a\alpha x = x\beta a$, for all $\alpha, \beta \in \Gamma$. i.e, $a \in a\Gamma x\Gamma a$ and $a\Gamma x = x\Gamma a$.

DEFINITION 2.63: A Γ- semigroup S is said to be a ***completely regular* Γ- *Semigroup*** provided every element is completely regular.

DEFINITION 2 .64: An element ***a*** of a Γ-semigroup S is said to be ***intra regular*** provided $a = x\alpha a\beta a\gamma y$ for some $x, y \in S$ and $\alpha, \beta, \gamma \in \Gamma$.

THEOREM 2.65 [5] : If A and B are any two Γ-ideals of a Γ-semigroup S, then

(1) $A \subseteq B \Rightarrow \sqrt{A} \subseteq \sqrt{B}$.
(2) $\sqrt{A\Gamma B} = \sqrt{A \cap B} = \sqrt{A} \cap \sqrt{B}$.
(3) $\sqrt{\sqrt{A}} = \sqrt{A}$.

THEOREM 2.66[6]: Let A **be a semi pseudo symmetric** Γ**-ideal in a** Γ**-semigroup** S. **Then the following are equivalent.**

1) A_1 **= The intersection of all completely prime Γ-ideals in S containing A.**
2) A_1' **= The intersection of all minimal completely prime Γ- ideals in S containing A.**
3) A_1'' **= The minimal completely semiprime Γ- ideal relative to containing A.**
4) $A_2 = \{ x \in S : (x\Gamma)^{n-1}x \subseteq A$ **for some natural number** $n \}$.
5) A_3 **= The intersectin of all prime Γ -ideals containing A.**
6) A_3' **= The intersection of all minimal prime Γ- ideals containing A.**
7) A_3'' **= The minimal semiprime Γ- ideal relative to containing A.**
8) $A_4 = \{ x \in S : (\langle x\rangle \Gamma)^{n-1} \subseteq A$ **for some natural number** $n \}$.

THEOREM 2.67 [6]: If S is a semi pseudo symmetric Γ-semigroup then the following are equivalent .

1) **S is strongly Archimedian Γ-semigroup.**
2) **S is an Archimedain Γ-semigroup.**
3) **S has no proper completely prime Γ-ideals.**
4) **S has no proper completely semi prime Γ-ideals.**
5) **S has no proper prime Γ-ideals.**
6) **S has no proper semiprime Γ-ideals.**

THEOREM 2.68 [7]: Let S be a Γ-semigroup with identity and let M be the unique maximal Γ-ideal of S. If $\sqrt{A} = M$ **for some Γ-ideal of S. Then A is a primary Γ-ideal.**

THEOREM 2.69 [7]: A Γ-semigroup S is semiprimary iff prime Γ-ideals of S form a chain under set inclusion.

THEOREM 2.70 [7]: If S is a duo Γ-semigroup, then the following are equivalent for any element $a \in S$.

1) ***a* is completely regular.**
2) ***a* is regular.**
3) ***a* is left regular.**
4) ***a* is right regular.**
5) ***a* is intra regular.**
6) ***a* is semisimple .**

THEOREM 2.71 [8]: Every duo Γ-semigroup is a pseudo symmetric Γ-semigroup.

THEOREM 2.72 [6]: Every pseudo symmetric Γ-semigroup is a semi pseudo symmetric Γ-semigroup.

THEOREM 2.73 [8]: Every commutative Γ-semigroup is a pseudo symmetric Γ-semigroup.

DEFINTION 2.74: A Γ-semigroup S is said to be **a Γ-*group*** provided S has no left and right Γ-ideals.

3.PRIME Γ-IDEALS ARE MAXIMAL Γ-IDEALS:

THEOREM 3.1: Let S be a Γ- semigroup with identity. If (non zero, assume this if S has zero) proper prime Γ-ideals in S are maximal then S is primary Γ-semigroup.

Proof: Since S contains identity, S has a unique maximal Γ- ideal M, which is the union of all proper Γ- ideals in S. If A is a (non zero) proper Γ- ideal in S, then $\sqrt{A} = M$ and hence by theorem 2.68, A is primary Γ- ideal. If S has zero and if $\langle 0\rangle$ is a prime Γ-ideal, then $\langle 0\rangle$ is primary and hence S is primary. If $\langle 0\rangle$ is not a prime Γ-ideal, then $\sqrt{\langle 0\rangle} = M$ and hence by theorem 2.68, $\langle 0\rangle$ is a primary Γ- ideal. Therefore S is a primary Γ- semigroup.

Note 3.2: If the Γ-semigroup S has no identity, then from example 2.35 we remark that theorem 3.1 is not true even if the Γ-semigroup has a unique maximal Γ-ideal. The converse of the above theorem is not true even if the Γ-semigroup is commutative.

Example 3.3: Let S = {a, b, 1} and Γ = S. Define a binary operation . in S as shown in the following table.

.	a	b	1
a	a	a	a
b	a	b	b
1	a	b	1

Now S is a primary Γ-semigroup in which the prime Γ-ideal <a> is not a maximal Γ-ideal.

THEOREM 3.4: Let S be a right cancellative quasi commutative Γ-semigroup. If S is a primary Γ- semigroup or a Γ- semigroup in which semiprimary Γ- ideals are primary, then for any primary Γ-ideal Q, √ Q is non-maximal implies Q = √ Q is prime.

Proof: Since √Q is non maximal, there exists a Γ- ideal A in S such that √Q ⊆ A ⊆ S. Let a ∈ A\√Q and b ∈ √Q. Now Q ⊆ Q ∪ <a $γ$ b> ⊆ √Q for $γ$ ∈ Γ. This implies by theorem 2.65, √Q ⊆ √Q ∪ <aΓb> ⊆ √√Q = √Q. Hence √Q ∪ <a $γ$ b> = √Q, for some $γ$ ∈ Γ. Thus by hypothesis Q ∪ <aΓb> is a primary Γ-ideal. Let s ∈ S\A. Then aΓsΓb ⊆ Q ∪ <aΓb>. Since a ∉ √Q = √Q ∪ <aΓb> and Q ∪ <aΓb> is a primary Γ-ideal, sΓ b ⊆ Q ∪ <aΓb>. If $sγb$ ∈ <aΓb>, then $sαb$ = $rβaγb$ for some r ∈ S and $α$, $β$, $γ$ ∈ Γ and hence by right cancellative property, we have s = $rβa$ ∈ A, implies s ∈ A it is a contradiction. Thus $sγb$ ∈ Q, which implies, since s ∉ √Q, b ∈ Q and hence √Q = Q. Therefore Q = √Q and so Q is prime.

THEOREM 3.5: Let S be a right cancellative quasi commutative Γ- semigroup. If S is either a primary Γ- semigroup or a Γ- semigroup in which semiprimary Γ- ideals are primary, then proper prime Γ- ideals in S are maximal.

Proof: First we show that if P is a minimal prime Γ- ideal containing a principal Γ- ideal <d>, then P is a maximal Γ- ideal. Suppose P is not a maximal Γ- ideal. Write M = S\P and A = { x ∈ S ; $xαm$ ∈ <d>, for some m ∈ M, $α$ ∈ Γ}. Clearly A is a Γ- ideal in S. If x ∈ A then $xαm$ ∈ <d> ⊆ P and hence x ∈ P. So A ⊆ P. Let b ∈ P and suppose N = {$(bα)^{K-1}bβm$ such that m ∈ M, $α$, $β$ ∈ Γ and k is a nonnegative integer}. Now N is a Γ-sub semigroup containing M properly. Since bam ∈ N and bam ∉ M. Since P is a minimal prime Γ- ideal containing <d>, M is a maximal Γ-sub semigroup not meeting <d>. Since N contains M properly we have N∩<d> ≠ ∅. So there exists a natural number k such that $(bγ)^k$ m ∈ <d> this implies $(bα)^{k-1}$ b ∈ A implies b ∈ √A therefore P ⊆ √A ⊆ √P = P. So P = √A. Now by hypothesis , A is a primary Γ – ideal . Since P is not a maximal Γ–ideal, we have by theorem 3.4, P = A. Now P is also minimal prime Γ-ideal containing <$dγd$>. Let B = { y ∈ S : $yαm$ ∈ <$dγd$> for some m ∈ M, $α$ ∈ Γ, $γ$ ∈ Γ}. As before we have P = B. Since d ∈ P = A = B we have $dγm$ = $sαdδd$ for some s ∈ S^1, $α$, $δ$ ∈ Γ. Since S is a quasi commutative Γ-semigroup, $dαm$ = $(mα)^n d$ for some natural number n implies $(mΓ)^n d$ = s ΓdΓd implies $(m\Gamma)^{n-1}m\Gamma d$ = sΓdΓd implies $(m\Gamma)^{n-1}m$ = s Γd ⊆ <d>. This is a contradiction. So P is a maximal Γ- ideal. Now if P is any proper prime Γ–ideal, then for any d ∈ P, <d> is contained in a minimal prime Γ-ideal, which is maximal by the above and hence p is a maximal Γ-ideal.

THEOREM 3.6: If S is a cancellative commutative Γ-semigroup such that S is a primary Γ-semigroup or in S a Γ-ideal A is primary if and only if √A is a prime Γ-ideal, then the proper Γ-ideals in S are maximal.

Proof: The proof of this theorem is a direct consequence of theorem 3.5.

THEOREM 3.7: Let S be a right cancellative quasi commutative Γ-semigroup with identity. Then the following are equivalent.

1) Proper prime Γ-ideals in S are maximal.
2) S is a primary Γ-semigroup.
3) Semiprimary Γ-ideals in S are primary.
4) If x and y are not units in S, then there exists natural numbers n and m such that $(x\ \Gamma)^{n-1}$ x = yΓs and $(y\Gamma)^{m-1}$ y = xΓr. For some s, r ∈ S.

Proof: Combining theorem 3.1 and 3.5 we have 1), 2) and 3) are equivalent; Assume 1). Since S contains identity, then S has a unique maximal Γ-ideal M which is the only prime Γ-ideal in S. If x and y are not units, then x, y ∈ M implies √<x> = √<y> = M implies x ∈ √<y>, y ∈ √<x>, implies $(x\ α)^{n-1}$ x ∈ <y>, $(y\ β)^{m-1}y$ ∈ <x>, $α$, $β$ ∈ Γ implies $(x\ \Gamma)^{n-1}$ x = yΓs, $(\ y\ \Gamma)^{m-1}y$ = xΓr, r, s ∈ S. Assume 4). Let xΓy ⊆ A, A is semiprimary. If x is a unit in S then y ∈ A. If y is a unit in S then x ∈ A. If y ∉ A and $(x\ γ)^{n-1}$ x = $yγs$ implies $(x\ γ)^n x$ = $xγyγs$ ∈ A implies x ∈ √A. Therefore if $xγy$ ∈ A, y ∉ A implies x ∈ √A. Therefore A is left primary.

Similarly A is right primary, Hence A is primary. Therefore S is primary.

THEOREM 3.8: Let S be a right cancellative quasi commutative Γ-Semigroup not containing identity. Then the following are equivalent.

1) S is a primary Γ-semigroup.
2) Semiprimary Γ-ideals in S are primary Γ-ideals.
3) S has no proper prime Γ-ideals.
4) If $x, y \in S$, there exists natural numbers n, m such that $(x\,\Gamma)^{n-1}x = y\,\Gamma\,s$, $(y\,\Gamma)^{m-1}y = x\Gamma r$, for some $s, r \in S$.

Proof : 1) implies 2) is clear. Assume 2), by theorem 3.5, proper prime Γ-ideals in S are maximal and hence if P is any prime Γ-ideal, P is maximal. Now S\P is a Γ- group. Let *e* be the identity of the Γ- group S\P. Now *e* is an idempotent in S and since S is a right cancellative Γ-semigroup, *e* is a right identity of S. Since S is a quasi commutative Γ–semigroup idempotent in S are commutative with every element of S and hence *e* is the identity of S. It is a contradiction. So S has no proper prime Γ-ideals. Therefore 2) implies 3). Assume 3). Since S has no proper prime Γ-ideals, we have for any Γ-ideal A of S, √A = S. Let $x, y \in$ S. Now √<x> = √<y> = S, implies $x \in$ √<y>, $y \in$ √<x> implies $(x\,\alpha)^{n-1}x = y\beta s, (y\,\gamma)^{m-1}y = x\,\delta\,s,$ for some $s, r \in$ S and $\alpha, \beta, \gamma, \delta \in \Gamma$. Therefore 3) implies 4). Assume 4). Let A be any Γ-ideal in S. Let $x\,\alpha\,y \in$ A, for some $\alpha \in \Gamma$, $y \notin$ A, $x, y \in$ S, implies $(x\,\Gamma)^{n-1}x = y\,\Gamma s$ implies $(x\,\gamma)^n x = x\gamma y\gamma s \in$ A implies $x \in$ √A. Therefore if $x\,\gamma\,y \in$ A, $y \notin$ A implies $x \in$ √A. Therefore A is left primary. Similarly A is right primary. Hence A is primary. Therefore S is primary .

THEOREM 3.9: Let S be a right cancellative quasi commutative Γ– semigroup. Then the following are equivalent.
1) S is a primary Γ–semigroup.
2) Semiprimary Γ–ideals in S are primary.
3) Proper prime Γ–ideals in S are maximal.

Proof: 1) ⇒ 2): If S has identity by Theorem 3.7, semiprimary Γ-ideals in S are primary. If S has no identity by Theorem 3.8, Semiprimary Γ-ideals in S are primary. 2) implies 3); by theorem 3.5. Assume 3): If S has identity, by theorem 3.7, 3) ⇒ 1). If S has no proper Γ- ideals, by Theorem 3.8, (3) implies (1).

NOTE: Furthermore S has no idempotents except identity if it exists.

THEOREM 3.10: Let S be a cancellative commutative Γ– semigroup. Then S is a primary Γ– semigroup if and only if proper prime Γ– ideals in S are maximal. Furthermore S has no idempotents except identity, if it exists.

Proof: The proof is a direct consequence of Theorem 3.9.

THEOREM 3.11: Let S be a semi pseudo symmetric Γ–semigroup with identity. Then the following are equivalent.
1) Proper prime Γ–ideals in S are maximal.
2) S is either a simple Γ–semigroup or S has a unique prime Γ–ideal P such that S is a 0-simple extension of the Archimedian sub Γ–semigroup P.
In either case S is a primary Γ–semigroup and S has atmost one globally idempotent principal Γ–ideal.

Proof: Suppose proper prime Γ-ideals in S are maximal. If S is a simple Γ-semigroup then S has no proper Γ-ideals implies S has no proper prime Γ- ideals implies S is an Archimedian semigroup by Theorem 2.67. If S is not a simple Γ–semigroup, then S has a Unique maximal Γ–ideal P, which is also the unique prime Γ–ideal. Since P is a maximal Γ–ideal in S, we have S\P is a 0-simple Γ–semigroup. Let $a, b \in$ P. Then √<a> = √<b> = P. So by theorem 2.66, $(<a>\,\Gamma)^{n-1}<a> \subseteq <b>$, for some natural number n implies $(a\,\gamma)^{n+1}a \in P\gamma a\,\gamma P$, $\gamma \in \Gamma$. So P is an Archimedian Γ-sub semigroup of S.Therefore 1) implies 2).

Assume 2):

Case i): If S is simple then S has no proper prime Γ–ideals and hence 1) is true.

Case ii): Suppose S is not simple, then S has a unique proper prime Γ–ideal P such that S is a 0-simple extension of P implies S\P is 0-simple. Therefore there exists no proper Γ–ideals of S containing P. Therefore P is maximal. Therefore 1) holds. By theorem 3.1, S is a primary Γ–semigroup. Suppose <a> and <b> be two proper globally idempotent principal Γ–ideals. Then √<a> = √<b> = P by theorem 2.66, $(<a>P)^{n-1}<a> \subseteq <b>$ for some n. Since <a> is globally idempotent, $<a> \subseteq <b>$. Similarly we can prove that $<b> \subseteq <a>$. Therefore <a> = <b>.

THEOREM 3.12: Let S be a duo Γ–semigroup with identity. Then the following are equivalent.
1) Proper prime Γ– ideals in S are maximal.
2) S is either a Γ– group and so Archimedian or S has a unique prime Γ–ideal P such that S = G ∪ P, where G is the Γ–group of units in S and P is an Archimedian sub Γ–semigroup of S.

In either case S is a primary Γ–semigroup and S has atmost one idempotent different from identity.

Proof: If S is a duo Γ– semigroup which is not a Γ– group, then S has a unique maximal Γ– ideal M and hence M is the unique prime Γ– ideal, by assuming 1). Now S\M is the Γ– group of units in S. By the theorem 3.11, 1) and 2) are equivalent. Clearly S is a primary Γ– semigroup. If *e* and *f* are two proper idempotents in S, then $<e>$ and $<f>$ are two globally idempotent principal Γ-ideals. So by theorem 3.11, *e* = *f*.

THEOREM 3.13: Let S be a commutative Γ– semigroup with identity. Then the following are equivalent.
1) Prime Γ-ideals are maximal.
2) S is either a Γ–group and so Archimedian or S has a unique prime Γ- ideal P such that S = G∪P, where G is the Γ–group of units of S and P is an Archimedian Γ–sub semigroup of S.

In either case S is a primary Γ–semigroup and S has atmost one idempotent from identity.

Proof: The proof of this theorem is a direct consequence of theorem 3.12.

THEOREM 3.14: Let S be a semi pseudo symmetric Γ–semigroup without identity. Then the following are equivalent.

1) Proper prime Γ-ideals in S are maximal and globally idempotent principal Γ-ideals form a chain.
2) S is an Archimedian Γ–semigroup or there exists a unique prime Γ-ideal P in S and S is a 0-simple extention of the Archimedian sub Γ–semigroup P.
3) Proper prime Γ-ideals in S are maximal and S has atmost two distinct globally idempotent principal Γ-ideals with one of its Γ–radical is S itself.

Proof: If S has no proper Prime Γ-ideals then by theorem 2.67, S is an Archimedian Γ–semigroup. Suppose S has proper prime Γ-ideals. Let M and N be two proper prime Γ-ideals in S, by assumption M and N are maximal Γ-ideals in S and every element in S\N and S\M is semisimple. Let *a* ∈ S\M and *b* ∈ S\N. Now *a* and *b* are semisimple elements implies $<a>$ and $<b>$ are globally idempotent Principal Γ-ideals. By hypothesis either $<a> \subseteq <b>$ or $<b> \subseteq <a>$. Suppose $<a> \subseteq <b>$. If *b* ∈ M then *a* ∈ M, this is a contradiction. So *b* ∉ M implies *b* ∈ S\M and thus $<b> = <a>$. Similarly we can show that if $<b> \subseteq <a>$ then also $<a> = <b>$. From this we can conclude that S\M = S\N and hence M = N. Thus S has a unique prime Γ-ideal. By an argument similar to theorem 3.11, we can prove that P is an Archimedian sub Γ–semigroup of S. Therefore 1) implies 2). If S is an Archimedian Γ–semigroup then clearly S has no proper prime Γ–ideals, by theorem 2.67. Let $<a>$ and $<b>$ be two globally idempotent principal Γ-ideals. Now since S has no proper prime Γ-ideals, we have $\sqrt{<a>} = \sqrt{<b>}$ = S. By theorem 2.66, $<a>^{n-1}\ \Gamma<a> \subseteq <b>$ and $<b>^{m-1}\ \Gamma<b> \subseteq <a>$ for some natural numbers *n* and *m*. Thus we have $<a> \subseteq <b>$ and $<b> \subseteq <a>$. So $<a> = <b>$. Suppose S has a unique prime Γ-ideal P such that S is a 0-simple extension of the Archimedian Γ–sub semigroup P. Since S\P is a 0– simple Γ– semigroup, We have P is a maximal Γ-ideal. Now for every *a*, *b* ∈ S\P, we have $<a> = <b>$ and $\sqrt{<a>} = \sqrt{<b>}$ = S. Let $<a>$ and $<b>$ be two globally idempotent principal Γ-ideals. Since a, b ∈ P. Now $\sqrt{<a>} = \sqrt{<b>}$ = P and hence $<a> = <b>$. Thus S has atmost two proper globally idempotent principal Γ-ideals one of its Γ-radicals is S itself. Therefore 2) implies 3).

Let $<a>$ and $<b>$ be two globally idempotent principal Γ-ideals in S. Let $\sqrt{<b>}$ = S, then *a* ∈ $\sqrt{<b>}$. Therefore $<a> \subseteq \sqrt{<b>}$ implies $(<a>\ p)^{n-1}<a> \subseteq <b>$ implies $<a> \subseteq <b>$. Therefore globally idempotent principal Γ-ideals in S form a chain. Therefore 3) implies 1).

THEOREM 3.15: Let S be a duo Γ–semigroup without identity. Then the following are equivalent.

1) Proper prime Γ-ideals in S are maximal.

2) S is Archimedian or there exists only one prime Γ-ideal P in S and S = P ∪ (S\P) where P is an Archimedian Γ–semigroup and S\P is a Γ–group.
3) Proper prime Γ-ideals in S are maximal and S has atmost two idempotents.
Proof: The proof of this theorem follows from theorem 3.14.

NOTE: Every commutative Γ–semigroup is a duo Γ–semigroup.

THEOREM 3.16: Let S be a commutative Γ–semigroup without identity (1), (2), (3) of theorem 3.15 are equivalent.
Proof: The proof this theorem is an immediate consequence of theorem 3.15.

THEOREM 3.17: Let S be a semi pseudo symmetric Γ–semigroup with S ≠ S ΓS. Then S is a primary Γ-semigroup in which proper prime Γ-ideals in S are maximal if and only if S is an Archimedian Γ-semigroup.

Proof: Let S be an Archimedian Γ-semigroup. Then by theorem 2.67, S has no proper prime Γ-ideals. Hence it is trivially true that proper prime Γ-ideals in S are maximal. Let A be any Γ-ideal in S such that $<x> \Gamma<y> \subseteq A$ and $y \notin A$. Since S is an Archimedian Γ-semigroup there exists a natural number n such that $(x\gamma)^{n-1} x \in S\Gamma y\Gamma S$, for $\gamma \in \Gamma$. Now $(x\gamma)^n x \in <x>\Gamma<y> \subseteq A$. So by theorem 2.66, $x \in \sqrt{A}$. Thus A is left primary. Similarly we can show that A is right primary. Therefore S is a primary Γ-semigroup in which proper prime Γ-ideals in S are maximal.

Conversly suppose that S is a primary Γ-semigroup in which proper prime Γ-ideals in S are maximal. Now S is semiprimary Γ-semigroup and hence by theorem 2.69, prime Γ-ideals in S form a chain. Since prime Γ-ideals are maximal if S has proper prime Γ-ideals, then S has a unique proper prime Γ-ideal which is also the unique maximal Γ-ideal.

Now every element of S\P is semi simple and hence $S\backslash P \subseteq S\Gamma S$. Let $a \in S\backslash P$ and $x \in P$. If $<a>\Gamma<x> \neq <x>$ then since S is a primary Γ-semigroup and $x \notin <a>\Gamma<x>$, we have $a \in \sqrt{<a>\Gamma<x>} = P$. This is a contradiction. So $<a>\Gamma<x> = <x>$ *for all* $x \in P$ and hence $P \subseteq S\Gamma S$. Thus $S\Gamma S = S$. So S has no proper prime Γ-ideals and hence by theorem 2.67, S is an Archimedian Γ-semigroup.

THEOREM 3.18: Let S be a duo Γ-semigroup without identity. Then S is a primary Γ-semigroup in which proper prime Γ-ideals are maximal if and only if S is an Archimedian Γ-semigroup.

Proof: If S is an Archimedian Γ-semigroup, then clearly S is a primary Γ-semigroup in which proper prime Γ-ideals are maximal, by theorem 2.67 and by theorem 2.71, S is pseudo symmetric Γ-semigroup, by theorem 2.72, every pseudo symmetric Γ-semigroup is semi pseudo symmetric Γ-semigroup. Therefore duo Γ-semigroup is semi pseudo symmetric Γ-semigroup and by theorem 3.17.

Conversely if S is a primary Γ-semigroup in which proper prime Γ-ideals are maximal, then similar to the theorem 3.17, if S is not an Archimedian Γ-semigroup then S has a unique proper prime Γ-ideal P which is also the unique maximal Γ-ideal. Then S\P is a Γ-group and if e is the identity of S\P then as in the above theorem we can show that $<e> \Gamma<x> = <x>$ for all $x \in P$. Since S is a duo Γ-semigroup, we have e is the left identity of S. Similarly we get e is the right identity of S, this is a contradiction. So S is an Archimedian Γ-semigroup.

THEOREM 3.19: Let S be a commutative Γ-semigroup without identity. Then S is a primary Γ-semigroup in which proper prime Γ-ideals are maximal if and only if S is an Archimedian Γ-semigroup.

Proof: The proof of this theorem is an immediate consequence of theorem 3.18.

THEOREM 3.20: Let S be a quasi commutative Γ-semigroup containing cancellable elements. Then the following are equivalent.

1) The proper prime Γ-ideals in S are maximal.

2) S is a Γ-group or S is a cancellative Archimedian Γ-semigroup not containing identity or S is an extension of an Archimedian Γ-semigroup by a Γ-group S containing an identity.

Proof: Suppose proper prime Γ-ideals in S are maximal. If S contains an identity and S is not a Γ-group, then S contains a unique maximal Γ-ideal, which is also the unique prime Γ-ideal by virtue of the hypothesis. Then $\sqrt{<a>} = M$ for every $a \in M$. So for any $b \in M$, $<b>^{n-1} \Gamma<b> \subseteq <a>$ for some natural number n. Thus $(b\gamma)^{n+1} b \in M\Gamma a\Gamma M$ and hence M is an Archimedian Γ-sub semigroup, Clearly S\M is a Γ-group.

Assume that S does not contain an identity. If Z, the set of all non cancellable elements is not empty, then Z is a prime Γ-ideal. Hence Z is a maximal Γ-ideal in S. Since S contains cancellable elements. Now for any $b \in S\backslash Z$, we have $S = Z \cup <b> = Z \cup (<b>\Gamma<b>)$. From this we obtain b is semisimple and hence by theorem 2.70, b is completely regular. So there is an element $x \in S$ such that $b = b\alpha x\gamma b$ and $b\alpha x = x\gamma b$ is an idempotent . If $x \in Z$ then since Z is a Γ-ideal, we have $b \in Z$. So $x \notin Z$ and hence $x\gamma b$ is a cancellable idempotent. Therefore S contains an identity. This is a contradiction. So $Z = \phi$ and hence S is a cancellable Γ-semigroup, by theorem 3.9, S is a primary Γ-semigroup and hence by theorem 3.18, S is an Archimedian Γ-semigroup.

Conversely if S is either a Γ-group or a cancellative Archimedian Γ-semigroup then S has no proper prime Γ-ideals and hence it is trivially true that proper prime Γ-ideals are maximal. Suppose S contains identity and S is an extension of an Archimedian Γ-sub semigroup M by a Γ-group. Since S has identity, M is the unique maximal Γ-ideal. The Archimedian property of M forces that M is the unique prime Γ-ideal of S. Let P be any proper prime Γ-ideal of S. Since M is union of all proper Γ-ideals of S, then $P \subseteq M$. Let $a \in M$, $b \in P$ implies $a, b \in M$. Since M is Archimedian Γ-sub semigroup implies $(a\gamma)^{n-1} a \in M\Gamma b\Gamma M \subseteq S\Gamma b\Gamma S \subseteq P$ implies $a \in P$. Therefore $M \subseteq P$. Therefore M = P. Therefore M is the unique maximal Γ-ideal.

THEOREM 3.21: Let S be a commutative Γ-semigroup containing cancellative elements. Then 1) and 2) of theorem 3.20 are equivalent.

Proof: The proof of this theorem is an immediate consequence of theorem 3.20.

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