Application of δ − Pre − I − Open Sets in Ideal Topological Spaces

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Abstract

In this paper we introduce some weak separation axioms by utilizing the notions of δ −pre- I-open sets and the δ −pre- I-closure operator. Also we define $(\delta, pI) - T_1{}^*$ Spaces, $(\delta, pI) - R_0{}^*$ spaces and $(\delta, pI) -$ symmetric spaces and show that $(\delta, pI) - T_1{}^*$ and $(\delta, pI) - R_0{}^*$ spaces are equivalent.

Keywords — ideal spaces, δ −pre- I-open, δ −pre- I-closed set, $(\delta, pI) - T_1{}^*$ $(\delta, pI) - R_0{}^*$ space and $(\delta, pI) -$ symmetric space.

I. INTRODUCTION AND PRELIMINARIES

By a space (X, τ), we always mean a topological space (X, τ) with no separation properties assumed. If $A \subseteq X$, $cl(A)$ and $int(A)$ will, respectively, denote the closure and interior of *A* in (X, τ). An ideal *I* [4] on a topological (X, τ) is a non empty collection of subsets of X which satisfies (i) $A \epsilon I$ and $B \subseteq A$ implies $B \epsilon I$ and (ii) $A \epsilon I$ and $B \epsilon I$ implies $A \cup B \epsilon I$. Given a space (X, τ) with an ideal *I* on *X* and if $\wp(X)$ is the set of all subsets of *X*, a set operator $(.)^* : \wp(X) \to \wp(X)$, called a local function [4] of *A* with respect to τ and I, is defined as follows: for $A \subseteq X$, $A^*(\tau, I) = \{x \epsilon X \,/ U \cap A \notin I\}$, for every $\{U \in \tau(x)\}$ where $\tau(x) = \{U \in \tau \,/\, x \in U\}$. A Kuratowski closure operator $cl^*(A)$ for a topology $\tau^*(X, \tau)$, called the *-topology, finer than τ is defined by $cl^*(A) = A \cup A^*(I, \tau)$ [5] when there is no chance for confusion, we will simply write A^* for $A^*(\tau, I)$ and τ^* for $\tau^*(I, \tau)$. If *I* is an ideal on *X*, then (X, τ, I) is called an ideal space. A subset *A* of a topological space (X, τ) is said to be δ −preopen[1] if $A \subset int(cl_\delta(A))$. A subset *A* of an ideal space (X, τ, I) is said to be pre-I-open[2] if $A \subset int(cl^*(A))$. The complement of pre-*I*-open set is called pre-*I*-closed.

The family of all pre-*I*-open sets in (X, τ, I) is denoted by *PIO* (X, τ, I) or simply *PIO(X)*. Clearly $\tau \subset PIO(X)$. The largest pre- *I*-open set contained in *A*, denoted by *pInt(A)*, called the pre-I-interior of *A*. The smallest pre-*I*-closed set containing *A* , denoted by *pIcl(A)*, is called the pre-*I*-closure of *A*.
A subset *A* of an ideal space (X, τ, I) is said to be R-*I*-open set [6] if $int(cl^*(A)) = A$.
A subset *A* of *X* is said to be *R-I*-closed if its complement is *R-I*-open. Let (X, τ, I) be an ideal space, *A* be a subset of *X* and *x* be a point of *X*. A point $x \in X$ is called a $\delta - I - cluster$ point of *A* if $A \cap V \neq \emptyset$ for every *regular-I-open* set *V* containing *x*. The set of all $\delta - I -$cluster point of *A* is called the $\delta - I -$closure of *A* and is denoted by $[A]_{\delta - I}$ but we denote it by $cl_{\delta I}{}^*(A)$. If $A = cl_{\delta I}{}^*(A)$, then *A* is $\delta - I -$closed. The complement of a $\delta - I -$closed set is said to be $\delta - I -$open.

II. $\delta - pre - I - open\ sets$

Definition 2.1. A subset *A* of an ideal space is said to be $\delta - pre - I -$open if $A \subset int(cl_{\delta I}{}^*(A))$, $cl_{\delta I}{}^*(A)$ is the family of all $\delta - I -$cluster point of *A*. The complement of a $\delta - pre - I -$open set is said to be $\delta - pre - I -$closed. The family of all $\delta - pre - I -$open (resp. $\delta - pre - I -$closed) sets in a topological space *X* is denoted by $\delta IPO(X, \tau, I)$ (resp. $\delta IPC(X, \tau, I)$). The intersection of all $\delta - pre - I -$closed sets containing *A* is called the $\delta - pre - I -$closure of *A* and is denoted by $pcl_{\delta I}{}^*(A)$.

Definition 2.2. A subset *U* of an ideal space (X,τ,I) is called a $(\delta,PI^*)-$neighbourhood of a point $x\in X$ if there exists a $\delta-pre-I-$open set V such that $x\in V\subset U$.

Lemma 2.3. For the $\delta-pre-I-$closure subsets of *A* and *B* in an ideal topological space (X,τ,I) the following properties hold:

(a) A is $\delta-pre-I-$closed in (X,τ,I) if and only if $A=pcl_{\delta I}{}^*(A)$.

(b) If $A\subset B$, then $pcl_{\delta I}{}^*(A)\subset pcl_{\delta I}{}^*(B)$.

(c) $pcl_{\delta I}{}^*(A)$ is $\delta-pre-I-$closed, that is $pcl_{\delta I}{}^*(pcl_{\delta I}{}^*(A))=pcl_{\delta I}{}^*(A)$.

(d) $x\in pcl_{\delta I}{}^*(A)$ if and only if $A\cap V\neq\emptyset$ for every $V\in\delta IPO(X,\tau,I)$ containing *x*.

Lemma 2.4. For a family $\{A_\alpha \mid \alpha\in\Delta\}$ of subsets in an ideal space (X,τ,I), the following properties hold:

(a) $pcl_{\delta I}{}^*(\cap\{A_\alpha \mid \alpha\in\Delta\})\subset\cap\{pcl_{\delta I}{}^*(A_\alpha) \mid \alpha\in\Delta\}$.

(b) $pcl_{\delta I}{}^*(\cup\{A_\alpha \mid \alpha\in\Delta\})\supset\cup\{pcl_{\delta I}{}^*(A_\alpha) \mid \alpha\in\Delta\}$.

Proof. (a) Since $\cap_{\alpha\in\Delta}\{A_\alpha\}\subset A_\alpha$ for each $\alpha\in\Delta$ by Lemma 2.3, we have $pcl_{\delta I}{}^*(\cap\{A_\alpha \mid \alpha\in\Delta\})\subset pcl_{\delta I}{}^*(A_\alpha)$ for each $\alpha\in\Delta$ and hence $pcl_{\delta I}{}^*(\cap\{A_\alpha \mid \alpha\in\Delta\})\subset\cap\{pcl_{\delta I}{}^*(A_\alpha)|\alpha\in\Delta\}$

(b) Since $A_\alpha\subset\cup_{\alpha\in\Delta}\{A_\alpha\}$ for each $\alpha\in\Delta$ by Lemma 2.3 we have $pcl_{\delta I}{}^*(A_\alpha)\subset pcl_{\delta I}{}^*(\cup_{\alpha\in\Delta}(A_\alpha))$ and hence $pcl_{\delta I}{}^*(\cup\{A_\alpha \mid \alpha\in\Delta\})\supset\cup\{pcl_{\delta I}{}^*(A_\alpha) \mid \alpha\in\Delta\}$.

Lemma 2.5. Let *A* be an ideal space (X,τ,I). If *A* is $\delta-pre$open in *X*, then it is $\delta-pre-I-$open .

Lemma 2.6. Let (X,τ,I) be an ideal space. For each point $x\in X$, $\{x\}$ is $\delta-pre-I-$open or $\delta-pre-I-$closed.

III. $D^*_{\delta pI}$ - SETS AND ASSOCIATED SEPARATION AXIOMS

Definition 3.1. A subset *A* of an ideal space (X,τ,I) is called a $D^*_{\delta pI}$ -set if there are two $\delta-pre-I-$open sets *U*, *V* such that $U\neq X$ and $A=U-V$. If *A*= *U* and $V=\emptyset$, Then it follows that every $\delta-pre-I-$open set *U* different from *X* is a $D^*_{\delta pI}$ -set.

Definition 3.2. An ideal space (X,τ,I) is called (δ,pI) - D^*_0 if every pair of distinct points *x* and *y* of *X*, there exists a $D^*_{\delta pI}$ -set of *X* containing *y* but not *x* or a $D^*_{\delta pI}$ -set of *X* containing *x* but not *y*.

Definition 3.3. An ideal space (X,τ,I) is called (δ,pI) - D^*_1 if every pair of distinct points *x* and *y* of *X*, there exists a $D^*_{\delta pI}$ -set of *X* containing *x* but not *y* and a $D^*_{\delta pI}$ -set of *X* containing *y* but not *x*.

Definition 3.4. An ideal space (X,τ,I) is called (δ,pI) - D^*_2 if every pair of distinct points *x* and *y* of *X*, there exists a $D^*_{\delta pI}$ -set of *X* containing *G* and *E* of *X* containing *x* and *y* respectively.

Definition 3.5. An ideal space (X,τ,I) is called (δ,pI) -T^*_0 if for every pair of distinct points of *X*, there is a $\delta-pre-I-$open set containing one of the points but not the other.

Definition 3.6. An ideal space (X,τ,I) is called (δ,pI) -T^*_1 if for every pair of distinct points of *X*, there is a $\delta-pre-I-$open set *U* in *X* containing *x* but not *y* and a $\delta-pre-I-$open set *V* in *X* containing *y* but not *x*.

Definition 3.7. An ideal space (X,τ,I) is called (δ,pI) - T_2^* if for every pair of distinct points x and y of X, there is a $\delta-pre-I-$open set U and V in X containing x and y respectively such that $U\cap V=\emptyset$.

Remark 3.8. (a) If (X,τ,I) is (δ,pI) - T_i^*, then it is (δ,pI) - T_{i-1}^* , i= 1,2.

(b) If (X,τ,I) is (δ,pI) - T_i^*, then it is (δ,pI) - D_i^* , i = 0,1,2.

(c) If (X,τ,I) is (δ,pI) - D_i^*, then it is (δ,pI) - D_{i-1}^* , i = 1,2.

Theorem 3.9 An ideal space (X,τ,I) is (δ,pI) - D_1^* if and only if it is (δ,pI) - D_2^* .

Proof. Let (X,τ,I) is (δ,pI) - D_1^*. Let $x, y\in X$ such that $x\neq y$. Since X is (δ,pI) - D_1^*, there exist a $D_{\delta pI}^*$ -set G_1 and G_2 such that $x\in G_1$ and $y\notin G_1$ and $y\in G_2$ and $x\notin G_2$.

Let $G_1=(U_1-U_2)$ and $G_2=(U_3-U_4)$. From $x\notin G_2$, we have either $x\notin G_3$ or $x\in G_3$ and $x\in G_4$. We discuss the two cases separately.

(1) Suppose $x\notin G_3$. From $y\notin G_1$, we obtain the following two sub cases. (a) $y\notin U_1$

From $x\in(U_1-U_2)$ we have $x\in U_1-(U_2\cup U_3)$ and $y\in(U_3-U_4)$ we have $y\in U_3-(U_1\cup U_4)$ it is easy to see that $\left(U_1-(U_2\cup U_3)\right)\cap\left(U_3-(U_1\cup U_4)\right)=\emptyset$.

(b)$y\in U_1$ and $\in U_2$, we have $x\in(U_1-U_2)$ and $y\in U_2$, $(U_1-U_2)\cap U_2=\emptyset$.

(2) $x\in U_3$ and $x\in U_4$, we have $y\in(U_3-U_4)$, $x\in U_4$ and $(U_3-U_4)\cap U_4=\emptyset$. Hence X is (δ,pI) - D_2^*.

Conversely, Suppose X is (δ,pI) - D_2^* Let $x, y\in X$ such that $x\neq y$. Since X is (δ,pI) - D_2^*, there exist a $D_{\delta pI}^*$ -set U and V containing x and y respectively such that $U\cap V=\emptyset$. Hence X is (δ,pI) - D_1^*

Definition 3.10. A point $x\in X$ which has only X as the (δ,pI^*)-neighbourhood is called a (δ,pI^*)-neat point.

Theorem 3.11. For the ideal space (X,τ,I) the following are equivalent.

(a) (X,τ,I) is (δ,pI) - D_1^*.

(b) (X,τ,I) has no (δ,pI^*)-neat point.

Proof. $(a)\Rightarrow(b)$. Since X is (δ,pI) - D_1^* So each point X is contained in a $D_{\delta pI}^*$ -set $O=U-V$ and $U\neq X$. This implies that x is not a (δ,pI^*)-neat point.

$(b)\Rightarrow(a)$. By Lemma 2.6, for each pair of points x, y $\in X$, at least one of them say x has a $(\delta,pI^*)-$neighbourhood U containing x and not y . Thus U which is different from X is a $D_{\delta pI}^*$ -set, If X has no (δ,pI^*)-neat point, then y is not a (δ,pI^*)-neat point. This means that there exists (δ,pI^*)-neighbourhood V of y such that $V\neq X$. Thus $y\in(U-V)$ but not x and $(V-U)$ is a $D_{\delta pI}^*$ -set. Hence (δ,pI) - D_1^*

Definition 3.12. An ideal space (X,τ,I) is said to be (δ,pI^*)- symmetric if for each point $x,y\in X$ $x\in pcl_{\delta I}^*(\{y\})$ implies $y\in pcl_{\delta I}^*(\{x\})$.

Theorem 3.13. For the ideal space (X,τ,I), the following are equivalent.

(a) (X,τ,I) is (δ,pI^*)- symmetric.

(b) For each $x\in X$, {x} is $\delta-pre-I-$closed.

(c) (X,τ,I) is (δ,pI) - T_1^*

Proof. $(a)\Rightarrow(b)$. Let x be any point of X. Let y by any distinct point from x. By Lemma 2.6, {y} is $\delta - pre - I - open$ or $\delta - pre - I -$closed in (X, τ, I).

(i) In case when {y} is $\delta - pre - I - open$, let $V_y = \{y\}$. Then $V_y \in \delta IPO(X, \tau, I)$.

(ii) In case when {y} is $\delta - pre - I - closed$, $x \notin \{y\} = pcl^*_{\delta I}(\{y\})$.

By (a) , $y \notin pcl^*_{\delta I}(\{x\})$. Now put $V_y = X - pcl^*_{\delta I}(\{x\})$. Then $x \notin V_y$, $y \in V_y$ and $V_y \in \delta IPO(X, \tau, I)$. Hence $X - \{x\} = \cup V_y \in \delta IPO(X, \tau, I)$, $y \in X - \{x\}$. This shows that {x} is $-pre - I - closed$ (X, τ, I).

$(b) \Rightarrow (c)$. Suppose {p} is $\delta - pre - I - closed$ for every $p \in X$. Let x, y $\in X$ with $x \neq y$. Now $x \neq y$ implies that $y \in X - \{x\}$. Hence $X - \{x\}$ is a $\delta - pre - I - open$ setcontaining y but not x. Similarly $X - \{y\}$ $is\ a\ \delta - pre - I - open$ set containing x but not containing y. Accordingly (X, τ, I) is (δ, pI) - T_1^*

$(c) \Rightarrow (a)$. Suppose that $y \notin pcl^*_{\delta I}(\{x\})$. Since $x \neq y$, by (c), there exists a $\delta - pre - I - open$ set U containing x such that $y \notin U$ and hence $x \notin pcl^*_{\delta I}(\{y\})$.

This shows that (X, τ, I) is (δ, pI^*)- symmetric.

Definition 3.14. A function $f : (X, \tau, I) \to (Y, \sigma)$ is said to be $\delta - pre - I - continuous$ if for each $x \in X$ and each $\delta - pre - I - open$ set U in X containing x such that $f(u) \subset V$.

Theorem 3. 15. If $f : (X, \tau, I) \to (Y, \sigma)$ is a $\delta - pre - I - continuous$ surjective function and E is a $D^*_{\delta pI}$ -set in Y, then the inverse image of E is a $D^*_{\delta pI}$ -set in X.

Proof. Let E is a $D^*_{\delta pI}$ -set in Y. Then there exists a $\delta - pre - I - open$ sets U_1 and U_2 in Y such that $E = U_1 - U_2$ and $U_1 \neq Y$. By the $\delta - pre - I - continuity$ of f, $f^{-1}(U_1)$ and $f^{-1}(U_2)$ are $\delta - pre - I - open$ sets in X . Since $U_1 \neq Y$, we have $f^{-1}(U_1) \neq X$. Hence $f^{-1}(U_1) - f^{-1}(U_2)$ is a $D^*_{\delta pI}$ -set.

Theorem 3. 16. If (Y, σ) is a (δ, pI) - D_1^* and $f : (X, \tau, I) \to (Y, \sigma)$ is a $\delta - pre - I - continuous$ bijective function, then (X, τ, I) is (δ, pI) - D_1^*.

Proof. Suppose (Y, σ) is a (δ, pI) - D_1^* space . Let x, y be any pair of distinct points in X. Since f is injective and Y is (δ, pI) - D_1^* there exist a $D^*_{\delta pI}$ -sets G_x and G_y of Y containing $f(x)$, f(y) respectively, such that $f(y) \notin G_x$ and $f(x) \notin G_y$. Therefore, by Theorem 3.15, $f^{-1}(G_x)$ and $f^{-1}(G_y)$ are $D^*_{\delta pI}$ -sets in X containing x and y respectively, such that $y \notin f^{-1}(G_x)$ and $x \notin f^{-1}(y)$. Hence X is (δ, pI) - D_1^*.

Theorem 3.17. An ideal space (X, τ, I) is (δ, pI) - D_1^* if and only if for each pair of distinct points $x, y \in X$, there exists a $\delta - pre - I - continuous$ surjective function $f : (X, \tau, I) \to (Y, \sigma)$ such that $f(x)$ and $f(y)$ are distinct where (Y, σ) is a (δ, pI) - D_1^* space.

Proof. For every pair of distinct points of X, it suffices to take the identity function on X.

Conversely, let x, y be distinct points in X. By hypothesis, there is a $\delta - pre - I - continuous$ surjective function f of a space X onto a (δ, pI) - D_1^* space Y such that $f(x) \neq f(y)$. By Theorem 3.9, there exist disjoint $D^*_{\delta pI}$ -sets G_x and G_y in Y such that $f(x) \in G_x$ and $f(y) \in G_y$. Since f is $\delta - pre - I - continuous$ surjective, by Theorem 3.15,

$f^{-1}(G_x)$ and $f^{-1}(G_y)$ are disjoint $D^*_{\delta pI}$ -sets in X containing x and y respectively. Hence again by Theorem 3.9, (δ, pI) - D_1^* space.

IV. (δ, pI) - R_0^* SPACE AND (δ, pI) - R_1^* SPACES

Definition 4.1. Let A be a subset of an ideal space (X,τ,I). The $\delta - pre - I - kernel$ of A, denoted by $pIker_\delta(A)$, is defined to be the set $pIker_\delta(A) = \cap\{U \in \delta IPO(X,\tau,I) \mid A \subset U\}$.

Theorem 4.2. Let (X,τ,I) be an ideal space and $A \subset X$.

Then $pIker_\delta(A) = \{x \in X \mid \delta cl_{pI}{}^*(\{x\}) \cap A \neq \emptyset\}$.

Proof. Let $x \in pIker_\delta(A)$ and $\delta cl_{pI}{}^*(\{x\}) \cap A \neq \emptyset$. Now $\{x\} \subset \delta cl_{pI}{}^*(\{x\})$. Hence $\{x\} \notin (X - \delta cl_{pI}{}^*(\{x\}))$ which is a $\delta - pre - I - open$ set containing A. This is absurd, since $x \in pIker_\delta(A)$. Consequently $pcl_{\delta I}{}^*(\{x\}) \cap A \neq \emptyset$.
Let x be such that $pcl_{\delta I}{}^*(\{X\}) \cap A \neq \emptyset$ and suppose that $x \notin pIker_\delta(A)$. Then there exists a $\delta - pre - I - open$ set U containing A and $x \notin U$. Let $y \in pcl_{\delta I}{}^*(\{X\}) \cap A$. Hence, U is a $(\delta, pI^*) -$ neighbourhood of y which does not contains x. This is a contradiction to $x \in pIker_\delta(A)$. Hence $pIker_\delta(A) = \{x \in X \mid \delta cl_{pI}{}^*(\{x\}) \cap A \neq \emptyset\}$.

Definition 4.3. An ideal space (X,τ,I) is said to be (δ, pI) - R_0^* if every $\delta - pre - I - open$ set contains the $\delta - pre - I -$ closure of each of its singletons.

Theorem 4.4. An ideal space (X,τ,I) is (δ, pI) - R_0^* if and only if it is (δ, pI) - T_1^*

Proof. Let x and y be two distinct points of X. For $x \in X$, {x} is $\delta - pre - I - open$ or $\delta - pre - I -$ closed by lemma 2.6. (i) When {x} is $\delta - pre - I - open$, let V = {x}, then $x \in V, y \notin V$ and $V \in \delta IPO(X,\tau,I)$. Moreover, since (X,τ,I) is (δ, pI) - R_0^*
We have $pcl_{\delta I}{}^*(\{x\}) \subset V$. Hence $x \notin X - V$, $y \in X - V$ and $X - V \in \delta IPO(X,\tau,I)$.
(i) When {x} is $\delta - pre - I -$closed, $y \in X - \{x\}$ and $X - \{x\} \in \delta IPO(X,\tau,I)$.

Hence $pcl_{\delta I}{}^*(\{y\}) \subset X - \{x\}$, since (X,τ,I) is (δ, pI) - R_0^* .

Now, let $V = X - pcl_{\delta I}{}^*(\{y\})$, then $x \in V, y \notin V$ and $V \in \delta IPO(X,\tau,I)$. Then, we obtain (X,τ,I) is (δ, pI) - T_1^* Conversely, Let V be any $\delta - pre - I - open$ set of X and $x \in V$. For each $y \in X - V$, there exists $V_y \in \delta IPO(X,\tau,I)$ such that $x \notin V_y$ and $y \in V_y$. Therefore, we have $pcl_{\delta I}{}^*(\{x\}) \cap (\cup V_y) = \emptyset$. Since $\in V_y$, $X - V \subset (\cup V_y)$ and hence $pcl_{\delta I}{}^*(\{x\}) \cap (X - V) = \emptyset$. This implies that $pcl_{\delta I}{}^*(\{x\}) \subset V$.

Hence (X,τ,I) is (δ, pI) - R_0^*.

Theorem 4.5. For an ideal space (X,τ,I) the following are equivalent.

(a) (X,τ,I) is (δ, pI) - R_0^*

(b) (X,τ,I) is (δ, pI) - T_1^*

(c) (X,τ,I) is $(\delta, pI^*) - symmetric$.

Proof. The proof follows from Theorems 3.13 and 4.4.

Theorem 4.6. For an ideal space (X,τ,I) the following are equivalent.

(a) (X,τ,I) is (δ, pI) - R_0^* space.

(b) For any nonempty set A and $G \in \delta IPO(X,\tau,I)$ such that $A \cap G \neq \emptyset$, there exist $F \in \delta IPC(X,\tau,I)$ such that $A \cap F \neq \emptyset$ and $F \subset G$.

(c) For any $G \in \delta IPO(X,\tau,I)$, $G = \cup\{F \in \delta IPC(X,\tau,I) | F \subset G\}$.

(d) For any $F \in \delta IPC(X,\tau,I)$, $F = \cap\{G \in \delta IPO(X,\tau,I) | F \subset G\}$.

(e) For any $x \in X$, $pcl_{\delta I}{}^*(\{x\}) \subset pIker_\delta(\{x\})$.

Proof. $(a) \Rightarrow (b)$. Let A be any non empty subset of X and $G \in \delta IPO(X,\tau,I)$ such that $A \cap G \neq \emptyset$. Which implies $x \in G \in \delta IPO(X,\tau,I)$. Since X is (δ,pI) - R_0^* every $\delta - pre - I - open$ contains the closure of each of its singletons.
Therefore, $pcl_{\delta I}{}^*(\{x\}) \subset G$. Set $F = pcl_{\delta I}{}^*(\{x\})$. Then $A \cap F \neq \emptyset$ and $F \subset G$.
$(b) \Rightarrow (c)$ Let $G \in \delta IPO(X,\tau,I)$ then $G \supset \cup \{F \in \delta IPC(X,\tau,I) \ | F \subset G\}$. Let x be any point of G. Then there exists $F \in \delta IPC(X,\tau,I)$ such that $x \in F$ and $F \subset G$. Therefore, we have $x \in F \subset \cup\{F \in \delta IPC(X,\tau,I) \ | F \subset G\}$.

$(c) \Rightarrow (d)$. The proof is clear.
$(d) \Rightarrow (e)$ Let x be any point of X and $y \notin pIker_\delta(\{x\})$. Then there exists $V \in \delta IPO(X,\tau,I)$ such $x \in V$ and $y \notin V$ and hence $pcl_{\delta I}{}^*(\{y\}) \cap V = \emptyset$. By (d), $pIker_\delta(pcl_{\delta I}(\{y\}) \cap V) = \emptyset$ and there exists $G \in \delta IPO(X,\tau,I)$ such that $x \notin G$ and $pcl_{\delta I}{}^*(\{y\}) \subset G$.
Therefore, $pcl_{\delta I}{}^*(\{x\}) \cap G = \emptyset$ and thus by Lemma 1.2, (c) and (d), $y \notin pcl_{\delta I}{}^*(pcl_{\delta I}{}^*(\{x\})) = pcl_{\delta I}{}^*(\{x\})$. Consequently, $pcl_{\delta I}{}^*(\{x\}) \subset pIker_\delta(\{x\})$.
$(d) \Rightarrow (e)$. The proof is clear.

Theorem 4.7. For an ideal space the following properties are equivalent,

(a) (X,τ,I) is (δ,pI) - R_0^* space.

(b) If F is $\delta - pre - I - closed$ and $x \in F$, then $pIker_\delta(\{x\}) \subset F$.

(c) If $x \in X$, then $pIker_\delta(\{x\}) \subset pcl_{\delta I}{}^*(\{x\})$.

Proof. $(a) \Rightarrow (b)$. Let F be $\delta - pre - I - closed$ and $x \in F$. Then $\{x\} \subset F$ which implies that $pIker_\delta(\{x\}) \subset pIker_\delta(F)$. By (a), it follows from Theorem 4.6, $pIker_\delta(F) = F$.
Thus $pIker_\delta(\{x\}) \subset F$.
$(b) \Rightarrow (c)$. Since $x \in pcl_{\delta I}{}^*(\{x\})$ and $pcl_{\delta I}{}^*(\{x\})$ is $\delta - pre - I - closed$, by (b) $pIker_\delta(\{x\}) \subset pcl_{\delta I}{}^*(\{x\})$.
$(c) \Rightarrow (a)$. Let $x \in pcl_{\delta I}{}^*(\{y\})$. then $y \in pIker_\delta(\{x\})$. By (c), $y \in pcl_{\delta I}{}^*(\{x\})$. Therefore, $x \in pcl_{\delta I}{}^*(\{y\})$ implies that $y \in pcl_{\delta I}{}^*(\{x\})$. Hence by Theorem 4.5, (X,τ,I) is (δ,pI) - R_0^* .

Definition 4.8. An ideal space (X,τ,I) is said to be (δ,pI) - R_1^* space if for each $x, y \in X$ $pcl_{\delta I}{}^*(\{x\}) \neq pcl_{\delta I}{}^*(\{y\})$, there exists disjoint $\delta - pre - I - open$ sets U and V such that $pcl_{\delta I}{}^*(\{x\})$ is a subset of U and $pcl_{\delta I}{}^*(\{y\})$ is a subset of V.

Theorem 4.9. An ideal space (X,τ,I) is (δ,pI) - R_1^* space if and only if X is (δ,pI) - T_2^*

Proof. Let x and y be any distinct points of X. By Lemma 2.5, each point x of X is $\delta - pre - I - open$ or $\delta - pre - I -$closed.

(i) When {x} is $\delta - pre - I - open$, since $\{x\} \cap \{y\} = \emptyset$, $\{x\} \cap pcl_{\delta I}{}^*(\{y\}) \subset \emptyset$, and hence $pcl_{\delta I}{}^*(\{x\}) \neq pcl_{\delta I}{}^*(\{y\})$.

(ii) When {x} is $\delta - pre - I - closed$, $pcl_{\delta I}{}^*(\{x\} \cap \{y\}) \subset pcl_{\delta I}{}^*(\{x\}) \cap \{y\} \subset \emptyset$, and hence $pcl_{\delta I}{}^*(\{x\}) \neq pcl_{\delta I}{}^*(\{y\})$. Since X is (δ,pI) - R_1^* , there exists disjoint $\delta - pre - I - open$ sets U and V such that $x \in pcl_{\delta I}{}^*(\{y\}) \subset U$ and $y \in pcl_{\delta I}{}^*(\{y\}) \subset V$. This shows that X is (δ,pI) - T_2^*

Conversely, let x and y be any points of X such that $pcl_{\delta I}{}^*(\{x\}) \neq pcl_{\delta I}{}^*(\{y\})$. By Remark 3.8, every (δ, pI) - T_2^* space is (δ, pI) - T_1^*. Therefore, by Theorem 3.13, $pcl_{\delta I}{}^*(\{x\}) = \{x\}$ and $pcl_{\delta I}{}^*(\{y\}) = \{y\}$ and hence $x \neq y$. Since X is (δ, pI) - T_2^* there exists disjoint $\delta - pre - I - open$ set U and V such that $pcl_{\delta I}{}^*(\{x\}) = \{x\} \subset U$ and $pcl_{\delta I}{}^*(\{y\}) = \{y\} \subset V$. This shows that (X, τ, I) is (δ, pI) - R_1^*

CONCUSION

In this paper, we define a new class of sets $\delta - pre - I - open$ set and $\delta - pre - I - closed$ set and some weak separation axiom by utilizing the notion of $\delta - pre - I - open$ set and $\delta - pre - I - closed$ set. Also we define (δ, pI) - R_0^* space, (δ, pI) - T_1^* ,$(\delta, pI^*) - symmetric$, (δ, pI) - T_2^* space, (δ, pI) - R_1^* We characterize these sets and study some of their fundamental properties.

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