Open Support of a Graph under Multiplication

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Abstract

In this paper, the open support of a vertex v under multiplication and open support of a graph G under multiplication is defined and studied. Also, we find the value of open support of some namely graphs like Dutch windmill graph, Butterfly graph and Ladder graph. Moreover, we generalized the value of open support under multiplication for any given graph G.

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I. INTRODUCTION

In this work we consider finite, undirected, simple graphs $G = (V, E)$ with n vertices and m edges. The *n*eighbourhood of a vertex $v \in V(G)$ is the set $N_G(v)$ of all the vertices adjacent to v in G. For a set $X \subseteq V(G)$, the *open neighbourhood*$N_G(X)$ is defined to be $\cup_{v\in X} N_G(v)$ and the *c*losed neighbourhood $N_G[X] = N_G(X) \cup X$. The *degree* of a vertex $v \in V(G)$ is the number of edges of G incident with v and is denoted by $deg_G(v)$ or $deg(v)$. The maximum and the minimum degrees of the vertices of G are respectively denoted by $\Delta(G)$ and $\delta(G)$. A vertex of a degree 0 in G is called an *isolated* vertex and a vertex of degree 1 is called a *pendent* vertex or an *end* vertex of G. A vertex of a graph G is said to be a vertex of full degree if it is adjacent to all other vertices in G. A graph G is said to be *regular of degree* r if every vertex of G has degree r. Such graphs are called *r-regular* graphs.

The *Dutch windmill graph*$D_n^{(m)}$, is the graph obtained by taking m copies of the cycle graph C_n with a vertex in common. The *Butterfly graph* (also called the bowtie graph and the hourglass graph) is a planar undirected graph with 5 vertices and 6 edges. It can be constructed by joining 2 copies of the cycle graph C_3 with a common vertex. It is denoted by B_n. The *ladder graph*L_n is a planar undirected graph with 2n vertices and n+2(n-1) edges. The Ladder graph obtained as the cartesian product of two graphs one of which has only one edge: $L_{n,1} = P_n \times P_1$.

An open support of a vertex v under multiplication is defined by $\prod_{u\in N(v)} deg(u)$ and is denoted by $mult(v)$. An open support of a graph G under multiplication is defined by $\prod_{u\in V(G)} mult(u)$ and it is denoted by $mult(G)$.

II. DEFINITIONS

Definition 2.1.*Let G=(V,E) be a graph. An open support of a vertex v under multiplication is defined by $\prod_{u\in N(v)} deg(u)$ and is denoted by $mult(v)$.*

Definition 2.2.*Let G=(V,E) be a graph. An open support of a graph G under multiplication is defined by $\prod_{u\in V(G)} mult(u)$ and it is denoted by $mult(G)$.*

III. RESULTS

Proposition 3.1.*For a Path P_n, $(n \geq 2)$, $mult(P_n) = 4^{n-2}$.*
Proof: Let G be a path on n vertices and let $V(G) = \{u_1, u_2, \dots, u_n\}$ where $deg(u_1) = deg(u_n) = 1$ and $deg(u_i) = 2$for $i = 2,3, \dots, n-1$.

If $n = 2, 3$, or 4, then clearly $mult(G) = 1, 4$ and 16 respectively.
Let $n \geq 5$. Then

$$mult(u_1) = mult(u_n) = 2,$$

$$mult(u_2) = mult(u_{n-1}) = 2$$

and

$$mult(u_i) = \Pi_{v \in N(u_i)} deg(v) = 4.$$

Therefore

$$mult(G) = \Pi_{u \in V(G)} (2^2) \times (2^2) \times (4^{(n-4)}) = 4^{n-2}.$$

Theorem 3.2 *For any r-regular connected graph G of order $n \geq 2$, $mult(G) = r^{rn}$.*
Proof: Let G be a r-regular connected graph on n vertices and let $V(G) = \{u_1, u_2, \dots, u_n\}$ where $deg(u_i) = r$ for all i. Let $n \geq 2$. Then

$$mult(u_i) = \Pi_{v \in N(u_i)} deg(v) = r \times r \times \dots \times r \quad (\mathrm{r-times}) = r^r.$$

Thus,$mult(u_i) = r^r$ and hence

$$mult(G) = r^r \times r^r \times \dots \times r^r = r^{rn}.$$

Corollary 3.3 *For a Cycle C_n, $(n \geq 3)$, $mult(C_n) = 4^n$.*

Corollary 3.4 *For a complete graph K_n, $(n \geq 2)$, $mult(K_n) = (n-1)^{n(n-1)}$.*

Corollary 3.5 *For a Petersen graph P, $mult(P) = 3^{30}$.*

Proposition 3.6.*For a complete bipartite graph $K_{m,n}$, $(m, n \geq 1)$,$mult(K_{m,n}) = (mn)^{mn}$.*
Proof: Let $G = K_{m,n}$ be a complete bipartite graph with bipartition (V_1, V_2) where $V_1 = \{u_1, u_2, \dots, u_m\}$ and $V_2 = \{v_1, v_2, \dots, v_n\}$. Then $deg(u_i) = n$ and $deg(v_j) = m$ for all i, j. Thus

$$mult(u_i) = \Pi_{v \in N(u_i)} deg(v) = \Pi_{v \in V_2} m = m^n \quad \mathrm{for } i = 1,2, \dots, m.$$

Similarly,

$$mult(v_j) = n^m \text{ for } j = 1,2, \dots, n.$$

Therefore

$$mult(G) = m^{nm} \times n^{mn} = (mn)^{mn}.$$

Proposition 3.7.*Let $G = L_{2n}$, $(n \geq 4)$, be a Ladder graph.*Then $mult(G) = 2^8 \times 3^{6(n-2)}$.
Proof: Let $G = L_{2n}$, $n \geq 4$. Let $V(G) = \{u_1, u_2, \dots, u_n, v_1, v_2, \dots, v_n\}$ where $deg(u_1) = deg(u_n) = deg(v_1) = deg(v_n) = 2$ and $deg(u_i) = deg(v_i) = 3$ for $i = 2,3, \dots, n-1$. Then

$$mult(u_1) = mult(v_1) = mult(u_n) = mult(v_n) = 6,$$

$$mult(u_2) = mult(v_2) = mult(u_{n-1}) = mult(v_{n-1}) = 18$$

and

$$mult(u_i) = \Pi_{v \in N(u_i)} deg(v) = 3 \times 3 \times 3 = 27.$$

Similarly,

$$mult(v_i) = 27.$$

Therefore,

$$\begin{aligned} mult(G) &= \Pi_{u \in V(G)} mult(u) \\ &= 6^4 \times (18)^4 \times (27)^{2n-8} \\ &= 2^4 \times 3^4 \times 2^4 \times 3^8 \times 3^{6(n-4)} \\ &= 2^8 \times 3^{12+6(n-4)} \\ &= 2^8 \times 3^{6(n-2)}. \end{aligned}$$

Proposition 3.8.*For a Fan F_n, $(n \geq 4)$,$mult(F_n) = 2^4 \times 3^{3(n-3)} \times (n-1)^{n-1}$.*
Proof: Let $G = F_n (n \geq 4)$. Let $V(G) = \{u, u_1, u_2, \dots, u_{n-1}\}$ where $deg(u) = n-1$, $deg(u_1) = deg(u_{n-1}) = 2$ and $deg(u_i) = 3$ for $i = 2,3, \dots, n-2$.
Then

$$mult(u) = 2^2 \times 3^{n-3},$$

$$mult(u_1) = mult(u_{n-1}) = 3 \times (n-1),$$

$$mult(u_2) = mult(u_{n-2}) = 6 \times (n-1)$$

and

$$mult(u_i) = \Pi_{v \in N(u_i)} deg(v) = 9 \times (n-1)$$

for $i = 2,3, \dots, n-3$.
Therefore

$$\begin{aligned} mult(G) &= \Pi_{v \in V(G)} mult(v) \\ &= 2^2 \times 3^{n-3} \times 3^2 \times (n-1)^2 \times 6^2 \times (n-1)^2 \times 9^{n-5} \times (n-1)^{n-5} \\ &= 2^4 \times 3^{n+1} \times (n-1)^4 \times 3^{2n-10} \times (n-1)^{n-5} \\ &= mult(G) = 2^4 \times 3^{3(n-3)} \times (n-1)^{n-1}. \end{aligned}$$

Theorem 3.9. *Let $G = L_{2n}$. Then $mult(G) = 6^4 \times 18^4 \times 27^{2(n-4)}$.*
Proof. Let $G = L_{2n}$ be a Ladder graph with $2n$ vertices. Let $V(G) = \{v_1, v_2, \dots, v_n, u_1, u_2, \dots, u_n\}$ and $deg(v_i) = deg(u_i) = 2$ for all $i = 1, n$; $deg(v_i) = deg(u_i) = 3$ for all $i = 2,3, \dots, n-1$. Then

$$mult(v_1) = \Pi_{v \in N(v_1)} deg(v)$$
$$= deg(u_1) \times deg(v_2)$$
$$mult(v_1) = 6$$

Similarly,

$$mult(v_n) = mult(u_1) = mult(u_n) = 6.$$

$$mult(v_2) = \Pi_{v \in N(v_2)} deg(v)$$
$$= deg(u_2) \times deg(v_1) \times deg(v_3)$$
$$mult(v_2) = 18$$

Similarly,

$$mult(u_2) = mult(u_{n-1}) = mult(v_{n-1}) = 18.$$

For each $i = 3,4, \dots, n-2$,

$$mult(v_i) = \Pi_{v \in N(v_i)} deg(v)$$
$$= deg(u_i) \times deg(v_{i-1}) \times deg(v_{i+1})$$
$$mult(v_i) = 27, \ foralli = 3,4, \dots, n-2.$$

Similarly,

$$mult(u_i) = 27, \ foralli = 3,4, \dots, n-2.$$

Now,

$$mult(G) = \Pi_{v \in V(G)} mult(v)$$
$$= \Pi_{i=1,n} mult(v_i) \times \Pi_{i=1,n} mult(u_i) \times \Pi_{i=2,n-1} mult(v_i) \times \Pi_{i=2,n-1} mult(u_i)$$
$$\times \Pi_{i=3}^{n-2} mult(v_i) \times \Pi_{i=3}^{n-2} mult(u_i)$$
$$= \sum_{i=1,n} 6 \times \Pi_{i=1,n} 6 \times \Pi_{i=2,n-1} 18 \times \Pi_{i=2,n-1} 18 \times \Pi_{i=3}^{n-2} 27 \times \Pi_{i=3}^{n-2} 27$$
$$mult(G) = 6^4 \times 18^4 \times 27^{2(n-4)}.$$

Theorem 3.10. *Let G=(m,m,...,m) be a caterpillar graph. Then*

$$mult(G) = (m+1)^{2(m+1)} (m+2)^{mn+2(m+n-2)}$$

Proof. Let G=$(m, m, \dots, m)$ be a caterpillar graph.

Let $V(G) = \{v_1, v_2, \dots, v_n, u_n, u_{11}, \dots, u_{1m}, \dots, u_{n1}, \dots, u_{nm}\}$ such that $deg(v_1) = deg(v_n) = m+1$; $deg(v_i) = m+2$, for all $i = 2,3, \dots, n-1$ and $deg(u_{ij}) = 1$ for all $i = 1,2, \dots, n$ and $j = 1,2, \dots, m$

$$mult(v_1) = \Pi_{i=1}^{n} deg(u_{1i}) \times deg(v_2)$$
$$= \Pi_{i=1}^{m} 1(m+2)$$
$$= m+2$$

Similarly,

$$mult(v_n) = m+2;$$

$$mult(v_2) = \Pi_{i=1}^{n} deg(u_{2i}) \times deg(v_1) \times deg(v_3)$$
$$= \Pi_{i=1}^{m} 1(m+1)(m+2)$$
$$= (m+1)(m+2)$$

Similarly,

$$mult(v_{n-1}) = (m+1)(m+2);$$

For $i = 3,4, \dots, n-2$

$$mult(v_i) = \Pi_{v\in N(v_i)} deg(v)$$
$$= deg(v_{i-1}) \times \Pi_{j=1}^{n} deg(u_{ij}) \times deg(v_{i+1})$$
$$= (m+2)\Pi_{j=1}^{n} 1 \times (m+2)$$
$$= (m+2)^2$$

For $j = 1,2, \dots, m$

$$mult(u_{1j}) = deg(v_1)$$
$$= m+1$$

Similarly,

$$mult(u_{nj}) = m+1;$$

For $i = 2,3, \dots, n-1$ and $j = 1,2, \dots, m$

$$supp(u_{ij}) = deg(v_i)$$
$$= m+2$$

Now,

$$mult(G) = \Pi_{v\in V(G)} mult(v)$$
$$= \Pi_{i=1}^{n} mult(v_i) \times \Pi_{i=1}^{n} \Pi_{j=1}^{m} mult(u_{ij})$$
$$= mult(v_1) \times mult(v_2) \times \Pi_{i=3}^{n-2} mult(v_i) \times mult(v_{n-1}) \times mult(v_n)$$
$$\times \Pi_{j=1}^{m} mult(u_{1j}) \times \Pi_{j=1}^{m} mult(u_{nj}) \times \Pi_{i=2}^{n-1} \Pi_{j=1}^{m} mult(u_{ij})$$
$$= (m+2)^2 (m+1)^2 (m+2)^2 \Pi_{i=3}^{n-2} (m+2)^2$$
$$\left(\Pi_{j=1}^{m} (m+1)\right)^2 \Pi_{i=2}^{n-1} \Pi_{j=1}^{m} (m+2)$$
$$= (m+1)^2 (m+2)^4 [(m+2)^2]^{(n-4)} (m+1)^{2m} (m+2)^{m(n-2)}$$
$$= (m+1)^{2(m+1)} (m+2)^{4+2n-8+mn-2m}$$
$$mult(G) = (m+1)^{2(m+1)} (m+2)^{mn+2(m+n-2)}.$$

Theorem 3.11 *Let G be a butterfly graph. Then* $mult(G) = 2^{16}$

Proof. Let G be a butterfly graph. Let $V(G) = \{x, v_1, v_2, v_3, v_4\}$ such that $deg(x) = 4; deg(v_i) = 2$,for all $i = 1,2,3,4$;

$$mult(x) = \Pi_{v\in N(x)} deg(v)$$
$$= deg(v_1) \times deg(v_2) \times deg(v_3) \times deg(v_4)$$
$$= 16$$

For $i = 1,2,3,4$

$$mult(v_i) = \Pi_{v\in N(v_i)} deg(v)$$
$$= 2 \times 4$$
$$= 8$$

Now,

$$mult(G) = \Pi_{v\in V(G)} mult(v)$$
$$= mult(v_1) \times mult(v_2) \times mult(v_3) \times mult(v_4) \times mult(x)$$
$$= 2^{16}$$

Theorem 3.12.*Let* $G = D_{(n)}^{(m)}$ *be a Dutch windmill graph. Then mult(G)=*$\left[m2^{(2n-1)}\right]^m$

Proof. Let $G = D_{(n)}^{(m)}$ be a Dutch windmill graph. Let $V(G) = \{x, v_1^i, v_2^i, v_3^i, \dots, v_{n-1}^i\}$ for $i = 1,2, \dots, m$ such that $deg(x) = 2m; deg(v_j^i) = 2$,for all $i = 1,2, \dots, m$ and $j = 1,2, \dots, n-1$;

$$mult(x) = \Pi_{v\in N(x)} deg(v)$$
$$= \Pi_{i=1}^{m} (deg(v_1^i) \times deg(v_{n-1}^i))$$
$$= \Pi_{i=1}^{m} (4)$$
$$= 4^m$$

For $i = 1,2, \dots, m$,

$$mult(v_1^i) = \Pi_{v\in N(v_1^i)} deg\,(v)$$
$$= 2m \times 2$$

Similarly,

$$mult(v_{n-1}^i) = 4m$$

For $j = 2,3, \dots, n-2$,

$$mult\left(v_j^i\right) = \Pi_{v\in N(v_j^i)} deg\,(v)$$
$$= 2 \times 2 = 4$$

Now,

$$\begin{aligned}
mult\,(G) &= \Pi_{v\in V(G)}\, mult\,(v). \\
&= mult(x) \times \Pi_{i=1}^{m}\{\Pi_{j=2}^{n-2} mult\big(v_j^i\big) \times mult\big(v_1^i\big) \times mult(v_{n-1}^i)\} \\
&= 4^m \times \Pi_{i=1}^{m}\{\Pi_{j=2}^{n-2} 4 \times 8m\} \\
&= 4^m \times \Pi_{i=1}^{m}\{4^{n-3} \times 8m\} \\
&= 4^m \times [4^{n-3} \times 8m]^m \\
&= [8m \times 4^{n-2}]^m \\
&= [m \times 2^3 \times 2^{2n-4}]^m \\
&= [m \times 2^{2n-1}]^m
\end{aligned}$$

Theorem 3.13. *Let* $G = K_m(a_1, a_2, \ldots, a_m)$. Then $mult(G) = n^{n^2}$.

Proof. Let $G = K_m(a_1, a_2, \ldots, a_m)$ be a multistar graph of order $m + a_1 + a_2 + \cdots + a_m$. Let $V(G) = \{v_1, v_2, \ldots, v_n, u_1, u_2, \ldots, u_n\}$ such that $deg(v_i) = n$ and $deg(u_i) = 1$ for all $i = 1,2, \ldots, n$.

$$mult(u_i) = \Pi_{v\in N(u_i)} deg(v)$$
$$mult(u_i) = n$$

$$mult(v_i) = \Pi_{v\in N(v_i)} deg(v)$$
$$mult(v_i) = n \times n \times \ldots \times n[(n-1)times]$$
$$mult(v_i) = n^{n-1}$$

Now,

$$\begin{aligned}
&mult(G) = \Pi_{v\in V(G)} mult(v) \\
&= \Pi_{i=1}^{n} mult(v_i) \times \Pi_{i=1}^{n} mult(u_i) \\
&= \Pi_{i=1}^{n} n^{n-1} \times \Pi_{i=1}^{n} n \\
&= n^{n(n-1)} \times n^n \\
&mult(G) = n^{n^2}.
\end{aligned}$$

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