International Journal of Mathematics Trends and Technology
ISSN: 2231 – 5373 /doi:10.14445/22315373/IJMTT-V66I12P503
Volume 66 Issue 12, 17-22, December, 2020

An Integrated Approach of The Brachistochrone And Tautocrone Curve

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Abstract- *The Brachistochrone and Tautocrone problem is not new but it was one of the fundamental problems given to the best mathematical minds in the world like Bernoulli brothers, L-hopital, Newton and Gottfried Leibnitz. As an enlightening fact the problem was both given and cracked by Johann Bernoulli. This was not just the foremost mathematical problem in the world but also the instigator of the branch of mathematics called as "Calculus of variation" and also proved many physical prodigies so, in essence this problem marked a breakthrough in the world of science. I have developed an approach based on the parity of both the Brachistochrone and Tautocrone curve that syndicates both of them which can further lead to mathematical progressions or can make new progresses in the field of physics.*

Keywords: *Brachistochrone Cycloid, Descend, Gravity, Shortest path, Tautochrone.*

I. INTRODUCTION

Brachistochrone problem was unravelled by Johann Bernoulli, he got the clue using Galileo's principle of classical mechanics which stated that the quickest path to descend is the path with an inclined angle. The Tautocrone problem was solved by Christiaan Huygens he demonstrated that if a cycloidal path has an array of particles lined to plunge down the route at different time period, they will reach the terminal point at the same time.

Brachistochrone problem proves that the if a particle has gravity acting on and it has travel from a point 'A' to 'B' then quickest path is not the hypotenuesical straight line but a cycloid between the two point.

Tautocrone curve attests that if an arrangement of particles with the effect of gravity on them are placed at a cycloidal declined path then all of them reach the end point at the same time.

On careful observation one can see the correlation between these 2 curves, the apparent ones being that they are both cycloids and both of them reflect on the principles of physics and that is how I had an idea of amalgamating both the problem and creating a unified approach to analyse and use both of them which can prove different propositions in the field of mathematics and physics.

These curves find their application in multiple marvel's in universe of science such as design of roller coasters, construction of high-speed roads, design of gears and making of accurate pendulums clocks with unvarying oscillations.

II. MATHEMATICAL METHODOLGY

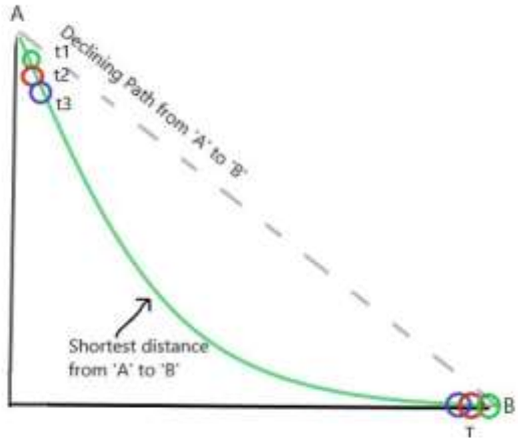

Figure 1. Illustration of the cohesive approach of both the curves.

If we can observe the figure, it is perceptible that there are three particles which are placed linearly ready to descend towards the point 'B' from starting point 'A' all of them will reach at the same time and they are on cycloidal path which is the fastest path as it's the brachistochrone curve. If we assume the time of the three particles at initial stage, we can call it as t1, t2 and t3 respectively and as mentioned they will all reach the point 'B' at same time so their final time can be called as 'T'.

So, we can say that t1=t2=t3=T at the end point and let us call the cycloidal path as ''x'. The time taken for particle to travel from point 'A' to 'B' can be written as:

$\mathrm{T} = \int \frac{\mathrm{ds}}{\mathrm{v}}$ Where 'ds' is the arc length and 'v' is the speed.

We have to compare the kinetic energy of the particles to the gravity that acts on the particle.

$$\frac{\mathrm{mv}^2}{2} = \mathrm{mgy} \text{ where 'y' is the height of the curve}$$

$$\frac{\mathrm{v}^2}{2} = \mathrm{gy} \quad \therefore \mathrm{v}^2 = 2\mathrm{gy} \text{ Finaly, } \mathrm{v} = \sqrt{2\mathrm{gy}}$$

The arc length differential can be written as:

$$\mathrm{ds} = \sqrt{\mathrm{dx}^2 + \mathrm{dy}^2} = \sqrt{1 + \mathrm{y}'^2}$$

$$\text{So, } \mathrm{t_{AB}} = \int \frac{\sqrt{1+\mathrm{y}^2}}{\sqrt{2\mathrm{gy}}}$$

We can write this in an alternate way as: $\mathrm{t_{AB}} = \int_A^B \frac{\sqrt{\mathrm{d}_x^2 + \mathrm{d}_y^2}}{\sqrt{2\mathrm{gy}}}$

So, $a\sqrt{\frac{\sqrt{2(1-\cos\theta)d\theta}}{\sqrt{2g[a(1-cos\theta)]}}} = \mathrm{a}\sqrt{\frac{(1-\cos\theta)\mathrm{d}\theta}{\mathrm{g}[\mathrm{a}(1-\cos\theta)]}} = \sqrt{\frac{\mathrm{a}}{\mathrm{g}}}\mathrm{d}\theta$ So, $\mathrm{dt} = \sqrt{\frac{\mathrm{a}}{\mathrm{g}}}\mathrm{d}\theta$

We know that $\mathrm{T} = \int \frac{\mathrm{ds}}{\frac{\mathrm{ds}}{\mathrm{dt}}} = \int \mathrm{dt}$ therefore, $\mathrm{T} = \int_0^1 \sqrt{\frac{\mathrm{a}}{\mathrm{g}}}$

$v = \frac{ds}{dt} = \sqrt{2gy}$ The diffrential change can be inserted as : $\frac{ds}{dt} = \sqrt{2g(y - yo)}$

Finally, we can write 'T' as $T = \int dt = \int \sqrt{\frac{2a^2(1-\cos\theta)}{2ag(\cos\theta o-\cos\theta)}} = \sqrt{\frac{a}{g}} \int_0^{\pi} \sqrt{\frac{1-\cos\theta}{\cos\theta o-\cos\theta}} * (\frac{2}{2})$

Using half angle formula of trigonometry, we can balance the equation as its in radical form.

The half angle formula is: $\sin\frac{x}{2} = \sqrt{\frac{1-\cos\theta}{2}}$, $\cos\frac{x}{2} = \sqrt{\frac{1+\cos\theta}{2}}$ *hence 'cosθ' is* $2\cos\theta^2$-1

$T = \sqrt{\frac{a}{g}} \frac{\sin\left(\frac{\theta}{2}\right) d\theta}{\sqrt{\cos^2\left(\frac{\theta o}{2}\right) - \cos^2\left(\frac{\theta}{2}\right)}}$ We can use a variable 'Z' for the denominator coefficients

$So, Z = \frac{\cos\left(\frac{\theta}{2}\right)}{\cos\left(\frac{\theta o}{2}\right)}$ $dz = \frac{\sin\left(\frac{\theta}{2}\right)}{Z\cos\left(\frac{\theta}{2}\right)}$ substituting this in the integeral we get

$T = -2\sqrt{\frac{a}{g}} \int_1^0 \frac{dz}{\sqrt{1-Z^2}}$ The limit of integration has changed after substitution as 1to 0 from 0 to π.

$T = 2\sqrt{\frac{a}{g}} [\sin^{-1}\ 0 \text{ to } 1]$ The limit of integeration is reversed so we have cahnged the sign of integeration.

$T = 2\sqrt{\frac{a}{g}} \sin^{-1}(0 \text{ to } 1) = 2\sqrt{\frac{a}{g}} * \frac{\pi}{2}$ hence, $T = \pi\sqrt{\frac{a}{g}}$ -----Equation (1)

The equation no.1 proves that the amount of time or the time period will be equivalent for any starting point or time of the particle descending.

As we discussed formerly that the arc differential can be written in an alternate way and we have solved it lets now take the actual form and elucidate it.

The actual arc diffrential integeral form was $t_{AB} = \int_A^B \frac{\sqrt{1+y'^2}}{\sqrt{2gy}}$

Consider the variable content inside the integral in a power conversion form and we will write it as a function. So, it will be depicted as 'f':

$f = (1 + y')^{\frac{1}{2}}.(2gy)^{\frac{-1}{2}}$ Further use euler lagrange formula in this equation

$\frac{\partial f}{\partial x} - \frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right) = 0$; $\frac{\partial f}{\partial x}$ is zero as there is no value for x pressent

$\therefore$ we can rewrite is as : $f - y'\left(\frac{\partial f}{\partial y'}\right) = c$, were c is contant

So, $\frac{\partial f}{\partial y'} = y'((1 + y')^{\frac{-1}{2}}.\left(2gy\right)^{\frac{-1}{2}}\right)$ ultimately I will bring it in orignal form $\frac{1}{\sqrt{2gy}\sqrt{1+y'^2}} = c$

Taking square on both sides , $\left[1 + \left(\frac{dy}{dx}\right)^2 y\right] = \frac{1}{2gC^2} = k^2$ *where k is new constant*

We solve the above equation using the formula of parametric equation so the variable x & y will be: $x = \frac{k^2}{2}(\theta - \sin\theta)$ & $y = \frac{k^2}{2}(1 - \cos\theta)$ which are the equations of cycloid

So, the array of particles will arrive at the identical time following a cycloidal path. To establish the notion that the transversal path is the shortest displacement point between the 2 points but not the fastest which is the cycloid will require the analysis through prospective of principles from classical mechanics.

When we analyse it using classical mechanics, we detect that because of the curvature of the cycloid and push-acceleration caused by the gravity there are two vectors formed one tangent and one normal to the particle on the curve. Let us call those vectors as T and N respectively.

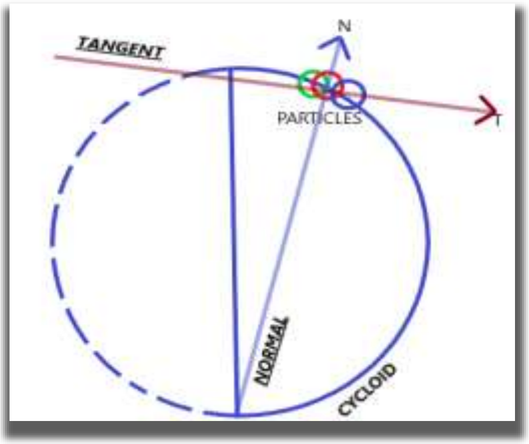

Figure 2. Depiction of both types of tangents in the curve.

The T and N vector is given as: $T = \frac{dx}{ds}\hat{x} + \frac{dy}{ds}\hat{y}$ & $N = -\frac{dy}{ds}\hat{x} + \frac{dx}{ds}\hat{y}$

Taking the force of gravity and friction into consideration along the curve.

$F_{gravity} = mg\hat{y}$ and $F_{Friction} = -\mu(F_{gravity)}).N.T = -\mu(mg\frac{dx}{dt}).T$

Where 'μ' is coefficient of friction.

$F_{gravity}\,T = mg\frac{my}{ds}$; $F_{friction}\,T = -\mu mg\frac{dx}{ds}$

According to Newton's law **F=m*a**

Thus, the total force F is the summation of force of gravity and the force of friction ∴

$$F = \overrightarrow{F_{gravity}} + \overrightarrow{F_{friction}} \quad = mg\frac{dy}{ds} + \left(-\mu mg\frac{dx}{ds}\right) \quad = mg\frac{dy}{ds} - \mu mg\frac{dx}{ds}$$

$\frac{dv}{dt} = v\frac{dv}{ds}$ so, it will be $\frac{1}{2}\frac{d}{ds}v^2$

$$\frac{v^2}{2} = g(y - \mu x)\ ;\ v^2 = 2g(y - \mu x) \quad \therefore v = \sqrt{2g(y - \mu x)}$$

From the initialization of the derivation, we know that $t_{AB} = \int \frac{dS}{v}$

So, we execute the final substitution of the equations we have solved till now into the integral ∴

$$t_{AB} = \int \frac{\sqrt{1 + y'^2}}{\sqrt{2g(y - \mu x)}} = \int \sqrt{\frac{1 + y'^2}{2g(y - \mu x)}}\,dx$$

Using Euler-Lagrange differential equation, we can solve this integral through easy simplifying.

$$[1 + y'^2](1 + \mu y') + 2(y - \mu x)y'' = 0 \therefore \text{ rewritting it as } \frac{1+y'^2}{(1+\mu y')} = \frac{c}{y-\mu x} \text{ where } y' = \cot(\frac{\theta}{2})$$

$\therefore \frac{1+\cot(\frac{\theta}{2})^2}{(1+\mu)\frac{\cos\theta^2}{2}} = \frac{c}{y-\mu x}$ So, finally the value of x and y variables is:

$x = \frac{k^2}{2}[(\theta - \sin\theta) + \mu(1 - \cos\theta)$ & $y = \frac{k^2}{2}[(1 - cos\theta) + \mu(\theta + \sin\theta)]$ -----Equation (2)

In conclusion the equation no.2 proves that the curvilinear path will be a shorter path for a particle with gravity and friction acting on it when it is travelling between 2 points.

III. THE EFFECT OF GRAVITY AND FRICTION ON THE PARTICLE IN THESE ISOCHRONE CURVES

As its apparent by the derivation above but it's still worth discoursing that both gravity and friction that is acting on the particles are the principal factors of the physical aces that are verified by the derivation of these curves without these natural physical phenomena's the result will just be an ordinary curve with regular physical constraints and nothing else.

The outcome of the significance of friction and gravity on the curves can be observed by altering the coefficient of friction and likewise by varying the value of gravitational constant and thus to articulate these facts properly graphs are presented to the readers with referenced values. We can perceive from the graphs that when the coefficient of friction is reduced the curve become as so circular and when the value of 'g' is lessened the cycloid gets isochronal so we should have maximum value of 'μ'i.e., coefficient of friction and minimized value of 'g'

And this reflects to the datum that without exact rate of friction and gravity these isochronal curves will just be regular curves.

Note: In this paragraph I have referred the brachistochrone and tautochrone curve together as 'isochrone' curve because isochrone curves are curves which follow fixed path in time and so does these.

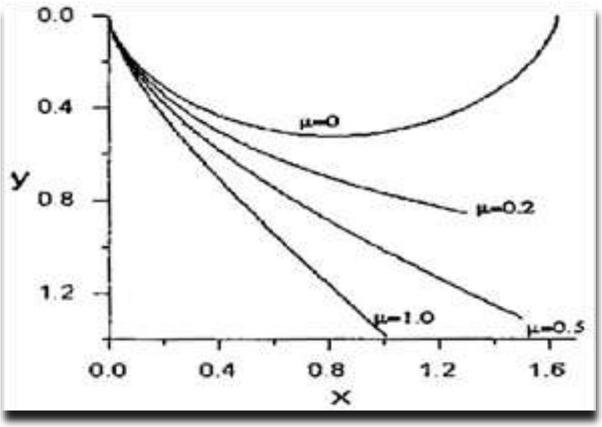

Figure 3. Representation of the flow of the cycloids with different value of coefficient of friction.

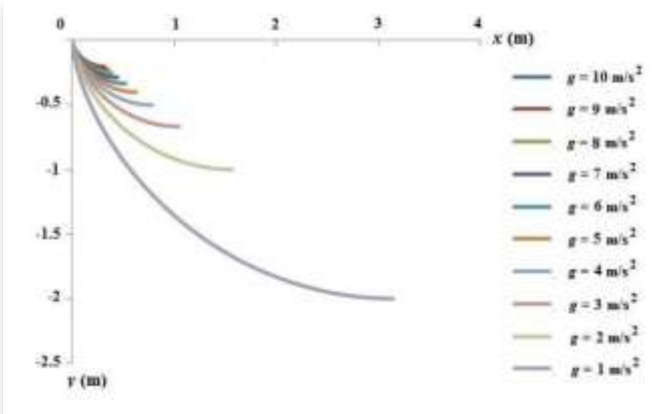

Figure 4. Representation of the curvature of the cycloids with different value of 'g'.

IV. RESULTS AND CONCLUSION

From the equations inscribed in the mathematical methodology we can draw the premise that these two curves have an analogous relationship among each other and this summative approach can be used for designing instruments such as curvilinear roads and highways for reducing the distances between places, sporting techniques for improving accuracy and various other apparatus. It can also be used to prove different phenomenon in the field of science and mathematics such as alternative parallax method, scheming parallel curvilinear geometry rather than the conventional rectilinear geometry for designs.

Consequently, I would like to conclude this precis by mentioning that the mathematical reckonings that I have figured out is tested and has led to efficacious results and as mentioned earlier this idea will be useful in the overhead discussed domains and also in other disciplines such as physics, space sciences etc. And it is also made evident that the force of gravity and friction is important for the outcome of the result of these equations and the methodology can contrast if the any reader varies the parameters used in the derivation such as complementary calculus techniques or the diverse laws of physics used.

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