International Journal of Mathematics Trends and Technology
ISSN: 2231 – 5373 /doi:10.14445/22315373/IJMTT-V67I1P501
Volume 67 Issue 1, 1-8, January, 2021

Conditions For Geometrical Construction of Polygons

Rajasekhar Parameswara [#1], Parameswara Aiyer Nilakanta [*2]
[1]*General Manager, Technical, INKEL, Kakkanad, Cochin, India*

Abstract: *Construction of rectilinear figures using straight edge and compasses are dealt with prime importance in geometry. Three independent conditions (Data) are required to construct a scalene triangle. Similarly five independent conditions are required for constructing a quadrilateral. This paper deals with the conditions required to construct an irregular polygon having 'n' sides.*

Keywords: *Straight edge, Geometry, irregular polygons, compasses, Construction, conditions, Aryabhateeyam, Ganitha pada,*

I. INTRODUCTION

In geometry various constructions of rectilinear figures using a straight edge and compasses are dealt with prime importance .It is well known that three independent conditions are required to a construct a triangle. Similarly five conditions are required to draw a quadrilateral. Then a natural question would be how many number of conditions are thus required to draw an irregular polygon having "n" sides. This short paper deals with this topic.

II. CONSTRUCTION POLYGONS

Āryabhaṭīyam (CE 499) is an acclaimed Indian text on mathematics and astronomy written by *Aryabhata* which was translated worldwide including in Arabic (c.820 from Arabic to Latin in 12th Century. The text deals with the topic under discussion in a subtle manner. The second chapter in *Aryabhateeyam* named *"Ganitha Pada"* depicts pure mathematics. *Aryabhateeyam*, which is written in Sanskrit Language in ***'sutra'*** style. ***Sutra in*** Sanskrit language means condensed verses and thus for any ***sutra*** text brevity is its key feature. Thirteenth (13th) Sutra in Chapter II named ***'Ganitha Pada'*** states that:

"*Vrutham bhramena sadhyam Thribhujam cha chathurbhujam cha karnabhayam"*

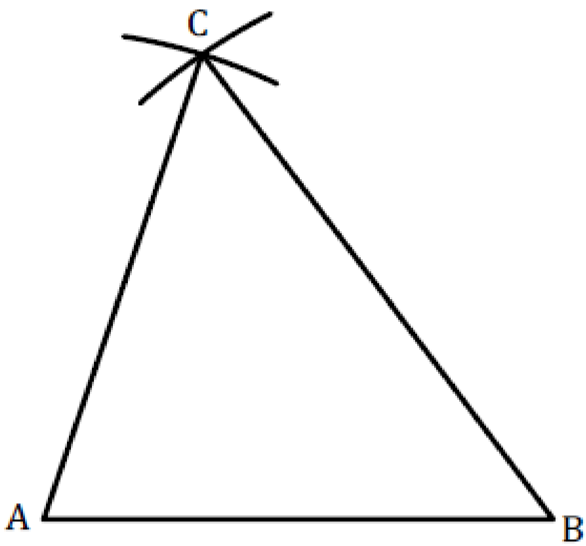
Literal meaning: *A circle is made possible (drawn) by rotation. A triangle, quadrilateral and so on from the diagonals'*

The implied meaning is that a circle or a part of a circle (arc) is drawn by using compasses. Then by knowing the sides a triangle can be drawn by cutting arcs from the base. Further by treating one of the arms as a diagonal two arcs can be cut and the quadrilateral can be drawn. The same procedure can be extended for any polygon. This has elaborately discussed the later mathematicians in India while writing commentaries on *Āryabhaṭīyam*, since it is written in condensed verse form as stated earlier many commentaries are available in Sanskrit as well as in other vernacular languages and these commentaries are called *'Vyakhyanas'*, in Indian scientific literature corpus. Many commentaries of *Āryabhaṭīyam,* are allowable of which '*Bhata Deepika'* by *Vatassery Paramasara* (c.1430) and Mahabhashya written *by Nilakanta Somayajin* (c. 1465) are splendid and very popular. The detailed explanations, proof, details of construction etc. are usually available in these commentaries. Thus *vyakhyanam* of any ancient text serves as good source of information especially for researchers and students.

III. REQUIRED CONDITIONS (DATA)

As stated earlier, construction of any triangle needs three independent conditions.

It is very evident that for drawing the base, the dimension of base is required and further the dimensions of two arms are required to cut the arcs and the point of intersection will be vertex. In lieu of the sides, two base angles can also serve the purpose. However three independent conditions are required for drawing the triangle. (Refer: Sketch: 1)

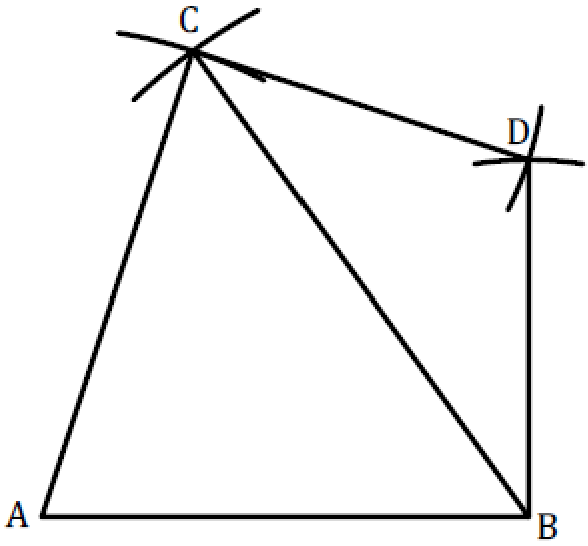


SKETCH -1 : Construction of Triangle

Further for drawing a quadrilateral, two more dimensions are required and one of the arms of the triangle will become a diagonal. Thus five conditions will be required for the quadrilateral (Refer: Sketch 2).

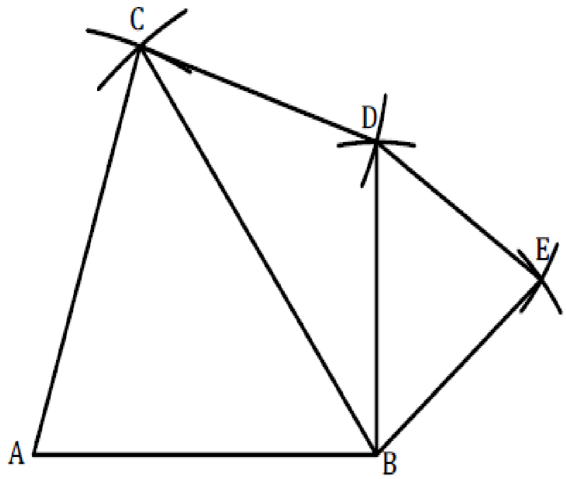
This is exactly what the verse quoted from Aryabhateeyam is also states. As quoted earlier, '*chathurbhujam cha karnabhayam'* meaning '*the qudarilaterals from the diagonals'.*

Treating one arm of a triangle as diagonal, if two arcs are cut, the next vertex of the quadrilateral can be fixed. This is depicted in the following sketch, wherein point 'D' is fixed by cutting arcs from point 'C 'and point 'B'.(Refer Sketch.2).



SKETCH -2: Construction of Quadrilateral

For a pentagon, another two (Refer Sketch: 3)



SKETCH -3: Construction of Pentagon

Thus further by induction it can be proved two more conditions are to be added for each additional side of the polygonal.

Summarizing the above, it can be concluded that for three sides (Triangle), three conditions required.

If from a triangle already drawn with given three conditions, 'm' further vertices are to be located $2m$ conditions are further required

Thus for any polygonal having $(3+m)$sides to be drawn, $(3+2m)$conditions are required

Let $(3+m)=n$

Then $m = (n-3)$

Then conditions required= $= 3 + 2(n-3)$

$= (2n-3)$

Thus it is established that for constructing an irregular polygon having 'n' sides $= (2n-3)$ conditions are required.

From the above, it can be seen that for triangles Number of conditions or data required

Since

$n = 3$

Conditions required

$= (2n-3)$

$= 2 \times 3 - 3 = 3.$

Similarly for a quadrilateral since $n = 4$

Number of conditions or data required

$= (2n-3)$

$= 2 \times 4 - 3 = 5$

If we proceed in similar lines, the number of conditions required for any number of sides can be evaluated. This is tabulated below. (Refer Table: I).

I.TABLE

Number of Sides	Conditions required
3	3
4	5
5	7
6	9
7	11
8	13
n	$(2n-3)$

IV. CYCLIC QUADRILATERALS

In the case of cyclic quadrilateral, the conditions needed are only four. It is because all quadrilaterals are not 'cyclic' meaning that all quadrilaterals cannot be inscribed in a circle. Only specific quadrilaterals are 'cyclic'. Thus the suffix 'cyclic' by itself defines a condition, which implies that the diagonals are computable from the given values of the sides using the formula given below. [Refer Equation (1), Equation (2) and Equation (3)].

Where a, b, c and d are the sides a cyclic quadrilateral and d_1, d_2, d_3 are the diagonals.

. In this context it may also be noted that in a cyclic quadrilateral imaginary third diagonal is also possible which is often designated as imaginary diagonal. The diagonals can be calculated from the sides using the formulae given below.

$$d_1 = \sqrt{\frac{(ab+cd)(ac+bd)}{(ad+bc)}} \quad \ldots\ldots\ldots\ldots\ldots \text{(1)}$$

$$d_2 = \sqrt{\frac{(ab+cd)(ad+bc)}{(ac+bd)}} \quad \ldots\ldots\ldots\ldots \text{(2)}$$

$$d_3 = \sqrt{\frac{(ac+bd)(ad+bc)}{(ab+cd)}} \quad \ldots\ldots\ldots\ldots\ldots\ldots \text{(3)}$$

The radius of the cyclic quadrilateral is given by the formula

$$R = \frac{1}{4}\sqrt{\frac{(ab+cd)(ac+bd)(ad+bc)}{(s-a)(s-b)(s-c)(s-d)}}$$

Where R is circum-radius, a, b ,c ,d are sides and

$$s = \frac{a+b+c+d}{2}$$

This formula was discovered by an Indian Mathematician belonging to medieval *'Kerala School of Mathematics'* named Vatasseri Parameswaran. (C.1365 CE -1355CE). Vatasseri Parameswaran is well known for new theory known as the 'Drikgganitha" system in south Indian Astronomy expounded in the CE 1343.

In the west, the above formula was rediscovered by Simon A Jean Lhuilier (1759 CE-1840CE) in the year CE 1782 and now it is widely known after him. The derivations of the above formulae can be found in the Kerala medieval text, '*Yukthi Bhasha' (c.1530CE)* authored by Jyeṣṭhadeva.

V. CYCLIC QUADRILATERAL-AN ILLUSTRATION

In a cyclic quadrilateral the given sides are

a=7.5 cm

b=4cm

c=5.1 cm

d= 6.8 cm

Diagonal BD can be calculated using the formula cited in the previous section.

$$d_1 = \sqrt{\frac{(cd+ab)(ac+bd)}{(ad+bc)}}$$

The diagonal BD

$$\sqrt{\frac{(5.1\times 6.8+4\times 7.5)(4\times 6.8+7.5\times 5.1)}{(7.5\times 6.8+4\times 5.1)}} = 7.7cm$$

Thus the quadrilateral can be draw as follows.

Step: 1

Construct the triangle Δ ABD as shown in sketch. (Refer Sketch: 4). initially construct a line with dimension equal to the one of the sides of the quadrilateral. Then cut an arc with one of the adjacent sides and the diagonal length as calculated earlier. .

Step: 2

Complete the quadrilateral as shown in sketch -5 by cutting arcs from B and D locating the vertex C.

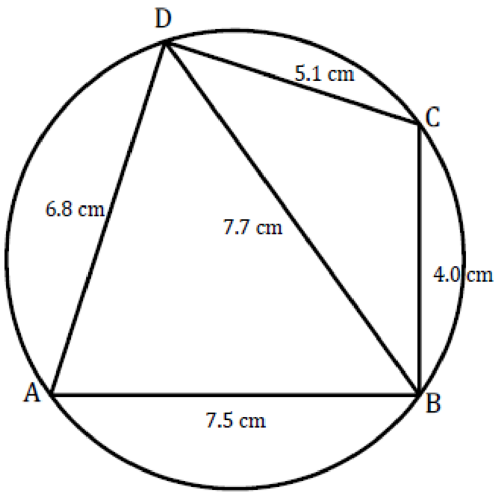
Thus the 4th vertex can be fixed. (Refer Sketch: 4)

Step: 3

The radius of the circle can be found using the

Formula cited earlier Refer quadrilateral circumscribed by the circle. (Refer Sketch: 4).

Draw the circum-circle of the Δ ABD. This will pass through the point D since ABCD is a cyclic quadrilateral of the circle can be found using the formula cited earlier. Refer the quadrilateral circumscribed by the circle. (Refer Sketch: 4).



SKETCH.4: Cyclic Quadrilateral

From the above it can be seen that cyclic quadrilateral can be constructed if four conditions are given since the lengths of diagonals are dependent sides and hence it can be computed. Thus with the given values of four sides cyclic quadrilaterals can be constructed. However one of the diagonals need be calculated using the formula cited elsewhere. There are some methods are available to construct it using straight edge and compasses, however these methods are lengthy. Thus there is a further scope for finding a simple method for constructing a cyclic quadrilateral with given four sides.

VI. CONCLUSION

This short paper deals with the number of conditions that are required to construct polygons and an explicit formula is presented. The formula arrived can be sated as follows:

For the geometrical construction using straight edge and compasses of any polygon (applicable to a triangle also) having ''n sides, (2n-3) independent conditions are essentially required.

The above formula is arrived and a deductive proof is also given. Some historical notes especially with reference to traditional Indians texts are also cited.

An interesting problem is arising from the discussions on cyclic quadrilateral; Is there any method to construct a cyclic quadrilateral if all the four sides are given using a straight edge and compasses only without calculating the diagonal. ? .

The new simple method(s) for construction is/are worth exploring and the problem is left to the readers. A few methods are available also but are not simple.

VII. ACKNOWLEDGEMENT

The formula cited in this paper is suggested Sri (Late) Parameswara Aiyer Nilakanta Aiyer, Kunnemadom, Vaikom the co-author of this paper.

VIII. REFERENCE

[1] Aiyer, Parameswara Nilakanta Vaikom, The Class Book of Geometry, MS (unpublished), c.1930
[2] Barnard. S and Child J.M, Elements of Geometry, Macmillan and Co Lmimited, London, 1948
[3] Joseph, Gheverghese, The Crest of the Peacock, non-European Rules of Mathematics, Princeton and Oxford: Princeton University Press, London, 2000.
[4] Joseph, Gheverghese. Kerala Mathematics History and its Possible Transmission to Europe. B.R Publishing Corporation, Delhi, 2009.
[5] Panicker, V.B., Līlāvati, Kurukshetra Prakashasan, Ernakulam, 2006.
[6] Pillai, Kunjan Surand, The Āryabhaṭīyam, The Bhaskara Press, Trivandrum, 1957
[7] Parameswaran, S., The golden age of Indian Mathematics, Swadeshi Science Movement, Trivandrum, 1990.
[8] Rajasekhar, P. (2018) New Extension for Pythagoras theorem and its applications, International Journal Mathematics Trends and Technology, 54(II), 236-239.
[9] Rajasekhar, P. (2011) Derivation of remainder term for the series expansion of π as depicted in Yuktibhāṣā and its modern interpretation, Bulletin of Kerala Mathematical Association,8(1), 17-39