GENERALIZED HYERS-ULAM-RASSIAS TYPE STABILITY OF ADDITIVE TYPE FUNCTIONAL EQUATIONS WITH 2k-VARIABLE IN (l,β)-NORMED SPACES

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Abstract. In this paper, we study to solve the Hyers-Ulam-Rassias stability type of the Cauchy functional equation and then Jensen functional equation in non-Archimdean (l,β)-normed space, and that of the pexiderized Cauchy functional equation in (l,β)-normed space. Then I will show that the solutions of equation are additive mapping. These are the main results of this paper.

Keywords: Hyers-Ulam-Rassias stability, (l,β)-normed space, non-Archimdean (l,β)-normed space, complete non-Archimdean (l,β)-normed space, Cauchy functional equation with 2k-variable, Jensen functional equation with 2k-variable, pexiderized Cauchy functional equation.

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1. Introduction

Let X and Y be a normed spaces on the same field K, and f : X → Y be a mapping. We use the notation ||·||β (||·||β) for corresponding the norms on X and Y. In this paper, we investigate the stability of the Cauchy functional equation and then Jensen functional equation in non-Archimdean (l,β)-normed space. In fact, when X is a non-Archimedean (n,β)-normed space with norm ||·||β, and that Y is a Banach non-Archimedean (n,β)-normed space with norm with norm ||·||β.

We solve and prove the Hyers-Ulam-Rassias type stability of the functional equation in non-Archimdean (l,β)-normed space, associated to the Cauchy type additive functional equation and Jensen type additive functional equation with 2k variable:

\[
    f\left(\sum_{j=1}^{k} x_j + \frac{1}{k} \sum_{j=1}^{k} x_{k+j}\right) = \sum_{j=1}^{k} f(x_j) + \sum_{j=1}^{k} f\left(\frac{x_{k+j}}{k}\right)
\]

(1.1)

and

\[
    2kf\left(\frac{1}{2k} \sum_{j=1}^{k} x_j + \frac{1}{2k} \sum_{j=1}^{k} x_{k+j}\right) = \sum_{j=1}^{k} f(x_j) + \sum_{j=1}^{k} f\left(\frac{x_{k+j}}{k}\right)
\]

(1.2)

The study of the functional equation stability originated from a question of S.M. Ulam [24], concerning the stability of group homomorphisms. Let \((G, \ast)\) be a group and let \((G', \circ, d)\) be a metric group with metric \(d(\cdot, \cdot)\). Given \(\epsilon > 0\), does there exist a \(\delta > 0\) such that if \(f : G \rightarrow G'\) satisfies

\[
    d(f(x \ast y), f(x) \circ f(y)) < \delta
\]

for all \(x, y \in G\) then there is a homomorphism \(h : G \rightarrow G'\) with

\[
    d(f(x), h(x)) < \epsilon
\]

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for all $x \in G$, if the answer, is affirmative, we would say that equation of homomophism $h(x * y) = h(y) \circ h(y)$ is stable. The concept of stability for a functional equation arises when we replace functional equation by an inequality which acts as a perturbation of the equation. Thus the stability question of functional equations is that how do the solutions of the inequality differ from those of the given function equation? The Hyers [8] gave first affirmative partial answer to the equation of homomorphism $h(x * y) = h(y) \circ h(y)$ is stable. The concept of stability for a functional equation arises when we replace functional equation by an inequality which acts as a perturbation of the equation. Thus the stability question of functional equations is that how do the solutions of the inequality differ from those of the given function equation?

Hyers’ Theorem was generalized by Aoki [1] for additive mappings and by Rassias [17] for linear mappings by considering an unbounded Cauchy difference. A generalization of the Rassias theorem was obtained by Gavruta [7] by replacing the unbounded Cauchy difference by a general control function in the spirit of Rassias approach. The functional equation
\[ f(x + y) = f(x) + f(y) \]
is called the Cauchy equation.

The functional equation
\[ f\left(\frac{x + y}{2}\right) = \frac{1}{2}f(x) + \frac{1}{2}f(y) \]
is called the Jensen additive functional equation.


concerning to the following Cauchy functional equation and Jensen functional equation
\[ f(x + y) = f(x) + f(y) \]
\[ 2f\left(\frac{x + y}{2}\right) = f(x) + f(y) \]

Recently, in [9, 10, 11, 25] the authors studied the on Hyers-Ulam-Rassias type stability the stability of the functional equation in non-Archimedean $(l, \beta)$-normed space, associated to the Cauchy type following additive functional equation and Jensen type additive functional equation.

\[ f\left(\sum_{j=1}^{k} x_j + \frac{1}{k} \sum_{j=1}^{k} x_{k+j}\right) = \sum_{j=1}^{k} f(x_j) + \sum_{j=1}^{k} f\left(\frac{x_{k+j}}{k}\right) \]

and
\[ 2kf\left(\frac{1}{2k} \sum_{j=1}^{k} x_j + \frac{1}{2k} \sum_{j=1}^{k} \frac{x_{k+j}}{k}\right) = \sum_{j=1}^{k} f(x_j) + \sum_{j=1}^{k} f\left(\frac{x_{k+j}}{k}\right) \]

ie the functional equation with $2k$-variables. Under suitable assumptions on spaces $X$ and $Y$, we will prove that the mappings satisfying the functional (1.1) and (1.2). Thus, the results in this paper are generalization of those in [10, 11, 25] for functional equation with $2k$-variables.

The paper is organized as follows:

In section preliminarie we remind some basic notations in [10,11,25] such as Banach space, Banach non-Archimedean space, non-Archimedean $(l, \beta)$-normed space, Banach non-Archimedean and solutions of the Cauchy function equation and Jensen function equation.
Section 3: is devoted to prove the Hyers-Ulam-Rassias type stability of the Cauchy type additive functional equations in non-Archimedean \((l, \beta)\)-normed space when \(X\) is a non-Archimedean \((l, \beta)\)-normed space with norm \(\| \cdot \|_\beta\) and \(Y\) is a complete non-Archimedean \((l, \beta)\)-normed space with norm \(\| \cdot \|_\beta\).

Section 4: is devoted to prove the Hyers-Ulam-Rassias type stability of the Jensen type additive functional equations in non-Archimedean \((l, \beta)\)-normed space when \(X\) is a vector and \(Y\) is a complete non-Archimedean \((l, \beta)\)-normed space with norm \(\| \cdot \|_\beta\).

Section 5: is devoted to prove the Hyers-Ulam-Rassias type stability of the pexiderized Cauchy type functional equations in non-Archimedean \((l, \beta)\)-normed space when \(X\) is a vector and \(Y\) is a complete non-Archimedean \((l, \beta)\)-normed space with norm \(\| \cdot \|_\beta\).

2. PRELIMINARIES

2.1. \((n, \beta)\)-normed spaces.

**Definition 2.1.**

Let \(\{x_n\}\) be a sequence in a normed space \(X\).

1. A sequence \(\{x_n\}_{n=1}^\infty\) in a space \(X\) is a Cauchy sequence iff the sequence \(\{x_{n+1} - x_n\}_{n=1}^\infty\) converges to zero.

2. The sequence \(\{x_n\}_{n=1}^\infty\) is said to be convergent if, for any \(\epsilon > 0\), there are a positive integer \(N\) and \(x \in X\) such that \(\|x_n - x\| \leq \epsilon. \forall n \geq N\), for all \(n,m \geq N\). Then the point \(x \in X\) is called the limit of sequence \(x_n\) and denote \(\lim_{n \to \infty} x_n = x\).

3. If every sequence Cauchy in \(X\) converges, then the normed space \(X\) is called a Banach space.

**Definition 2.2.**

Let \(X\) be a linear space over \(\mathbb{R}\) with \(\dim X \geq n, n \in \mathbb{N}\) and \(0 < \beta \leq 1\) let \(\|\cdot, \ldots, \cdot\| : X^n \to \mathbb{R}\) be a function satisfying the following properties:

1. \(\|x_1, \ldots, x_n\|_\beta = 0\) if and only if \(x_1, \ldots, x_n\) are linearly dependent,

2. \(\|x_1, \ldots, x_n\|_\beta\) is invariant under permutations of \(x_1, \ldots, x_n\)

3. \(\|\alpha x_1, \ldots, x_n\|_\beta = |\alpha|\|x_1, \ldots, x_n\|_\beta\)

4. \(\|x_1, \ldots, x_n, y + z\|_\beta \leq \|x_1, \ldots, x_n, y\|_\beta + \|x_1, \ldots, x_n, z\|_\beta, \forall x_1, \ldots, x_n, y, z \in X\) and \(\alpha \in \mathbb{R}\). Then the function \(\|\cdot, \ldots, \cdot\|\) is called an \((n, \beta)\)-norm on \(X\) and the pair \((X, \|\cdot, \ldots, \cdot\|)\) is called a linear \((n, \beta)\)-normed space or an \((n, \beta)\)-normed space.

* Note that the concept of a linear \((n, \beta)\)-normed space is a generalization of a linear \(n\)-normed space \((\beta = 1)\) and of a linear \(n\)-normed space \((n = 1)\).

**Definition 2.3.**

A sequence \(\{x_n\}\) in a linear \((n, \beta)\)-normed space \(X\) is called a convergent sequence if there is \(x \in X\) such that \(\lim_{n \to \infty} \|x_n - x, z_1, z_2, \ldots, z_{n-1}\|_\beta = 0\) for all \(z_1, z_2, \ldots, z_{n-1} \in X\).
*Note we call that \( \{ x_n \} \) convergent to or that \( x \) is the limit of \( \{ x_n \} \), when \( x_n \to x \) as \( n \to \infty \) or \( \lim_{n \to \infty} x_n = x \).

**Definition 2.4.**

A sequence \( \{ x_n \} \) in a linear \((n, \beta)\)-normed space \( X \) is called a Cauchy sequence if \( \lim_{n,m \to \infty} \| x_n - x_m, z_1, z_2, \ldots, z_{n-1} \|_\beta = 0 \) for all \( z_1, z_2, \ldots, z_{n-1} \in X \).

**Definition 2.5.**

A linear \((n, \beta)\)-normed space in which every Cauchy sequence is convergent is called a complete \((n, \beta)\)-normed space.

2.2. The properties of \((n, \beta)\)-normed spaces.

**Lemma 2.6.**

Let \(( X, \| \cdot \|, \ldots, \| \cdot \|_\beta )\) be a linear \((n, \beta)\)-normed space, \( k \geq 1, 0 < \beta \leq 1 \). If \( x_1 \in X \) and \( \| x_1, z_1, z_2, \ldots, z_{n-1} \| = 0 \) for all \( z_1, z_2, \ldots, z_{n-1} \in X \), then \( x_1 = 0 \).

**Lemma 2.7.**

For a convergent sequence \( \{ x_n \} \) in a linear \((n, \beta)\)-normed space \( X \),

\[
\lim_{n \to \infty} \| x_n, z_1, z_2, \ldots, z_{n-1} \|_\beta = \| \lim_{m \to \infty} x_m, z_1, z_2, \ldots, z_{n-1} \|_\beta = 0
\]

for all \( z_1, z_2, \ldots, z_{n-1} \in X \).

2.3. non-Archimedean \((n, \beta)\)-normed spaces. In this subsection we recall some basic notations from such as non-Archimedean fields, non-Archimedean normed spaces and non-Archimedean Banach spaces.

A valuation is a function \( | \cdot | \) from a field \( K \) into \([0, \infty)\) such that 0 is the unique element having the 0 valuation,

\[
|r| = 0 \iff r = 0
\]

\[
|r \cdot s| := |r||s|, \forall r, s \in K
\]

and the triangle inequality holds, i.e.,

\[
|r + s| \leq |r| + |s|, \forall r, s \in K.
\]

A field \( K \) is called a valued field if \( K \) carries a valuation. The usual absolute values of \( \mathbb{R} \) and \( \mathbb{C} \) are examples of valuation. Let us consider a valuation which satisfies a stronger condition than the triangle inequality. If the strong triangle inequality is replaced by

\[
|r + s| \leq \max\{|r|, |s|\}, \forall r, s \in K,
\]

then the function \( | \cdot | \) is called a non-Archimedean valuation. Clearly, \( |1| = |-1| = 1 \) and \( |n| \leq 1, \forall n \in \mathbb{N} \). A trivial example of a non-Archimedean valuation is the function \( | \cdot | \) talking everything except for 0 into 1 and \( |0| = 0 \). In this paper, we assume that the base field is a non-Archimedean field with \( |2| \neq 1 \), hence call it simply a field.

**Definition 2.8.** Let be a vector space over a filed \( K \) with a non-Archimedean \( | \cdot | \). A function \( \| \cdot \| : X \to [0, \infty) \) is said a non-Archimedean norm if it satisfies the following conditions:

1. \( \| x \| = 0 \) if and only if \( x = 0 \);
(2) $\|rx\| = |r| \|x\| (r \in \mathbb{K}, x \in X)$;

(3) $\|x + y\| \leq \max \left\{ \|x\|, \|y\| \right\} x, y \in X$ hold.

Then $(X, \|\cdot\|)$ is called a norm -Archimedean norm space.

**Definition 2.9.**

A sequence $\{x_n\}$ in a norm -Archimedean $(n, \beta)$-normed space $X$ is a Cauchy sequence if and only if $\{x_n - x_m\} \to 0$.

**Definition 2.10.** Let $\{x_n\}$ be a sequence in a norm -Archimedean normed space $X$.

1. A sequence $\{x_n\}_{n=1}^{\infty}$ in a non -Archimedean space is a Cauchy sequence iff the sequence $\{x_{n+1} - x_n\}_{n=1}^{\infty}$ converges to zero.

2. The sequence $\{x_n\}$ is said to be convergent if, for any $\epsilon > 0$, there are a positive integer $N$ and $x \in X$ such that $\|x_n - x\| \leq \epsilon, \forall n \geq N$.

3. If every sequence Cauchy in $X$ converger, then the norm -Archimedean normed space $X$ is called a norm -Archimedean Banach space.

**Definition 2.11.**

Let $X$ be a real space with $\dim X \geq n$ over a scalar filed $\mathbb{K}$ with a non -Archimedean nontrivial valuation $\|\cdot\|$, where $n$ is a positive integer and $\beta$ is a constant with $0 < \beta \leq 1$. A real-valued function let $\|\cdot, \ldots, \cdot\| : X^n \to \mathbb{R}$ is called an $(n, \beta)$-norm on $X$ satisfying the following properties:

1. $\|x_1, \ldots, x_n\|_\beta = 0$ if and only if $x_1, \ldots, x_n$ are linearly dependent,

2. $\|x_1, \ldots, x_n\|_\beta$ is invariant under permutations of $x_1, \ldots, x_n$.

3. $\|\alpha x_1, \ldots, x_n\|_\beta = |\alpha|_\beta \|x_1, \ldots, x_n\|$

4. $\|x_0 + x_1, \ldots, x_n\|_\beta \leq \max \left\{ \|x_0, \ldots, x_n\|_\beta, \\|x_1, \ldots, x_n\|_\beta \right\}$, $\forall x_0, x_1, \ldots, x_n \in X$ and $\alpha \in \mathbb{K}$. Then the function $\|\cdot, \ldots, \cdot\|$ is called an $(n, \beta)$-norm on $X$ and the pair $(X, \|\cdot, \ldots, \cdot\|)$ is called a non -Archimedean $(n, \beta)$-normed space or an $(n, \beta)$-normed space.

* Note that the concept of a non -Archimedean $(n, \beta)$-normed space is a non -Archimedean n-normed space if $(\beta = 1)$ and a a non -Archimedean $\beta$-normed space if n=1 respectively.

2.4. **Solutions of the equation.** The functional equation

\[ f(x + y) = f(x) + f(y) \]
is called the Cauchy equation. In particular, every solution of the Cauchy equation is said to be an additive mapping.

The functional equation \( f\left(\frac{x+y}{2}\right) = \frac{1}{2}f(x) + \frac{1}{2}f(y) \) called the Jensen equation. In particular, every solution of the Jensen equation is said to be a Jensen additive mapping.

Note: n is positive integer and \( l \geq 2 \).

3. Stability of the Cauchy type functional equation in non-Archimdean 
\((l, \beta)-\)normed space

In section, we assume that \(|2k| \neq 1\). Under this condition we prove the Hyers-Ulam-Rassias type stability of the Cauchy type additive functional equations in non-Archimdean \((l, \beta)-\)normed space when \( X \) is a non-Archimdean \((l, \beta)-\)normed space with norm \( \| \cdot \|_{\beta_1} \) and \( Y \) is a complete non-Archimdean \((l, \beta)-\)normed space with norm \( \| \cdot \|_{\beta} \). or \( X \) is a vector space and \( Y \) is a complete non-Archimdean \((l, \beta)-\)normed space with norm \( \| \cdot \|_{\beta} \).

Under this setting, we can show that the mapping satisfying (1.1) is additive. These results are give in the following.

**Theorem 3.1.**

Suppose that \( X \) is a non-Archimeidan \( \beta_1\)-normed space and that \( Y \) is a complete non-Archimeadan \((l, \beta)-\)normed space, where \( l \geq 2 \), \( 0 < \beta, \beta_1 \leq 1 \). Let \( \epsilon \in [0, \infty) \), \( p, q \in (0, \infty) \) with \( l\beta_1(p + q) > \beta \) and let \( \varphi : Y^{l-1} \to [0, \infty) \)

be a function. Suppose that a mapping

\[ f : X^{2k} \to Y \]

satisfying the inequality

\[
\left\| f\left(\sum_{j=1}^{k} x_j + \sum_{j=1}^{k} \frac{x_{k+j}}{k}\right) - \sum_{j=1}^{k} f(x_j) - \sum_{j=1}^{k} f\left(\frac{x_{k+j}}{k}\right), z_1, z_2, \ldots, z_{l-1}\right\|_{\beta} \\
\leq \epsilon \left(\prod_{j=1}^{k} \|x_j\|_{\beta_1}^p \cdot \prod_{j=1}^{k} \|x_{k+j}\|_{\beta_1}^q\right) \varphi\left(z_1, z_2, \ldots, z_{l-1}\right) 
\]

(3.1)

for all \( x_j, x_{k+j} \in X \) for all \( j = 1 \to k \) and \( z_1, z_2, \ldots, z_{l-1} \in Y \). Then there exists a unique additive mapping

\[ H : X \to Y \]

satisfying

\[
\left\| f(x) - H(x), z_1, z_2, \ldots, z_{l-1}\right\|_{\beta} \leq \epsilon (2k)^{-\beta} \|x\|_{\beta_1}^{k(p+q)} 
\]

(3.2)

for all \( x \in X \) and \( z_1, z_2, \ldots, z_{l-1} \in Y \).
Proof. Put \( x_j = x, x_{k+j} = kx \) for all \( j = 1 \rightarrow k \) in (3.1) and dividing both sides by \( |(2k)^{-\beta}| \), we get

\[
\left\| \frac{f(2kx)}{2k} - f(x), z_1, z_2, ..., z_{l-1} \right\|_\beta \leq \epsilon \cdot k^{q\beta_1} \left\| x \right\|_{\beta_1}^k (p+q) \varphi(z_1, z_2, ..., z_{l-1}) \tag{3.3}
\]

for all \( x \in X \) and \( z_1, z_2, ..., z_{l-1} \in Y \). Replacing \( x \) by \( (2k)^n x \) in (3.1) and dividing both sides by \( |(2k)^{n\beta}| \), we get

\[
\left\| \frac{f((2k)^{n+1}x)}{(2k)^{n+1}} - \frac{f((2k)^nx)}{(2k)^n}, z_1, z_2, ..., z_{l-1} \right\|_\beta \leq \epsilon \cdot k^{q\beta_1} \left\| \frac{1}{(2k)^n} \left\| (2k)^{nk\beta_1(p+q)} \right\| x \right\|_{\beta_1}^k (p+q) \varphi(z_1, z_2, ..., z_{l-1}) \tag{3.4}
\]

for all \( x \in X \) and \( z_1, z_2, ..., z_{2k-1} \in Y \). Since \( k(p+q)\beta_1 > \beta \) and \( |2k| \neq 1 \), we get

\[
\lim_{n \rightarrow \infty} \left\| \frac{f((2k)^{n+1}x)}{(2k)^{n+1}} - \frac{f((2k)^nx)}{(2k)^n}, z_1, z_2, ..., z_{l-1} \right\|_\beta = 0 \tag{3.5}
\]

for all \( x \in X \) and \( z_1, z_2, ..., z_{l-1} \in Y \).

It follows from (3.5) that the sequence \( \left\{ \frac{f((2k)^nx)}{(2k)^n} \right\} \) is a Cauchy sequence for all \( x \in X \). Since \( Y \) is a complete space, the sequence \( \left\{ \frac{f((2k)^nx)}{(2k)^n} \right\} \) converges. So one can define the mapping \( H: X \rightarrow Y \) by

\[
H(x) := \lim_{n \rightarrow \infty} \frac{f((2k)^nx)}{(2k)^n} \tag{3.6}
\]

for all \( x \in X \).
It follows from (3.1) and (3.6) and lemma 2.7 that
\[
\left\| H\left(\sum_{j=1}^{k} x_j + \frac{1}{k} \sum_{j=1}^{k} x_{k+j}\right) - \sum_{j=1}^{k} H(x_j) - \sum_{j=1}^{k} H\left(\frac{x_{k+j}}{k}\right), z_1, z_2, \ldots, z_{l-1}\right\|_{\beta}
\]
\[
= \lim_{n \to \infty} \left(2k\right)^{-n\beta}\left| f\left(\sum_{j=1}^{n} x_j + \frac{1}{k} \sum_{j=1}^{n} x_{k+j}\right) - \sum_{j=1}^{k} f\left(\left(2k\right)^{n} x_j\right)\right|
\]
\[
- \sum_{j=1}^{k} f\left(\left(2k\right)^{n} \frac{x_{k+j}}{k}\right), z_1, z_2, \ldots, z_{l-1}\right\|_{\beta}
\]
\[
\leq \lim_{n \to \infty} \theta \left(2k\right)^{-n\beta} \left| \prod_{j=1}^{k} \left(2k\right)^{n} x_j \right|_{\beta_1}^{p} \cdot \sum_{j=1}^{k} \left(2k\right)^{n} x_{k+j} \right|_{\beta_1}^{q} \varphi(z_1, z_2, \ldots, z_{l-1})
\]
and so for all \(x_j, x_{k+j} \in X\) for all \(j = 1 \to k\) and \(z_1, z_2, \ldots, z_{2k-1} \in Y\). Since \(k\beta_1(p+q) > \beta\) and \(2k \neq 1\), we get
\[
\left\| H\left(\sum_{j=1}^{k} x_j + \frac{1}{k} \sum_{j=1}^{k} x_{k+j}\right) - \sum_{j=1}^{k} H(x_j) - \sum_{j=1}^{k} H\left(\frac{x_{k+j}}{k}\right), z_1, z_2, \ldots, z_{l-1}\right\| = 0
\]
for all \(x_j, x_{k+j} \in X\) for all \(j = 1 \to k\) and \(z_1, z_2, \ldots, z_{l-1} \in Y\). By lemma 2.6, we get
\[
H\left(\sum_{j=1}^{k} x_j + \frac{1}{k} \sum_{j=1}^{k} x_{k+j}\right) - \sum_{j=1}^{k} H(x_j) - \sum_{j=1}^{k} H\left(\frac{x_{k+j}}{k}\right) = 0
\]
for all \(x_j, x_{k+j} \in X\) for all \(j = 1 \to k\). So mapping \(H\) is additive. replace \(x\) by \(2kx\) in (3.3) and dividing both sides by \(\left(2k\right)^{\beta}\), we get
\[
\left\| \frac{f\left(2k^2 x\right)}{2k} - \frac{f\left(2kx\right)}{2k}, z_1, z_2, \ldots, z_{l-1}\right\|_{\beta}
\]
\[
\leq \epsilon \cdot k^{n+1}\left(2k\right)^{-3\beta} \left\| \left(2k\right)^{n} x \right\|_{\beta_1}^{k(p+q)} \varphi(z_1, z_2, \ldots, z_{l-1})
\]
and keep replacing \(x\) by \(2kx\) in (3.7)and dividing both sides by \(\left(2k\right)^{\beta}\), we get
\[
\left\| \frac{f\left(2k^3 x\right)}{2k} - \frac{f\left(2k^2 x\right)}{2k}, z_1, z_2, \ldots, z_{l-1}\right\|_{\beta}
\]
\[
\leq \epsilon \cdot k^{n+1}\left(2k\right)^{-3\beta} \left\| \left(2k\right)^{n} x \right\|_{\beta_1}^{k(p+q)} \varphi(z_1, z_2, \ldots, z_{l-1})
\]
and so on until
\[
\left\| \frac{f\left(2k^{n+1} x\right)}{2k^{n+1}}, z_1, z_2, \ldots, z_{l-1}\right\|_{\beta}
\]
\[
\leq \epsilon \cdot k^{n+1}\left(2k\right)^{-n\beta} \left\| \left(2k\right)^{n} x \right\|_{\beta_1}^{k(p+q)} \varphi(z_1, z_2, \ldots, z_{l-1})
\]
Thus by (3.7), (3.8) and (3.9) We get

\[
\left\| f(x) - \frac{f((2k)^{n+1}x)}{(2k)^{n+1}}, z_1, z_2, \ldots, z_{l-1} \right\|_\beta \\
\leq \max \left\{ \left\| \frac{f((2k)x)}{2k} - f(x), z_1, z_2, \ldots, z_{l-1} \right\|_\beta, \right. \\
\left. \left\| \frac{f((2k)^2 x)}{(2k)^2} - \frac{f(2kx)}{2k}, z_1, z_2, \ldots, z_{l-1} \right\|_\beta, \right. \\
\ldots, \left. \left\| \frac{f((2k)^{n+1}x)}{(2k)^{n+1}} - \frac{f((2k)^n x)}{(2k)^n}, z_1, z_2, \ldots, z_{l-1} \right\|_\beta \right\} \\
\leq \max \left\{ \epsilon \cdot k^{q\beta_1} \left\| (2k)^{-\beta} \right\| \beta_1^{k(p+q)} \varphi(z_1, z_2, \ldots, z_{l-1}), \right. \\
\epsilon \cdot k^{q\beta_1} \left\| \frac{1}{(2k)^{2\beta}} \right\| (2k)^2 x \beta_1^{k(p+q)} \varphi(z_1, z_2, \ldots, z_{2k-1}), \right. \\
\ldots, \left. \epsilon \cdot k^{q\beta_1} \left\| \frac{1}{(2k)^{n\beta}} \right\| (2k)^n x \beta_1^{k(p+q)} \varphi(z_1, z_2, \ldots, z_{l-1}) \right\} 
\]

for all \( x \in X \) and \( z_1, z_2, \ldots, z_{l-1} \in Y \). Since \( k(p+q)\beta_1 > \beta \) and \( |2k| \neq 1 \), we get

\[
\left\| f(x) - \frac{f(2kx)}{2k}, z_1, z_2, \ldots, z_{2k-1} \right\|_\beta \leq \epsilon \cdot k^{q\beta_1} \left\| (2k)^{-\beta} \right\| \beta_1^{k(p+q)} \varphi(z_1, z_2, \ldots, z_{l-1}) \tag{3.10}
\]

for all \( x \in X \) and \( z_1, z_2, \ldots, z_{l-1} \in Y \).

By induction on, \( n \) we can conclude that

\[
\left\| f(x) - \frac{f((2k)^n x)}{(2k)^n}, z_1, z_2, \ldots, z_{l-1} \right\|_\beta \leq \epsilon \cdot k^{q\beta_1} \left\| (2k)^{-\beta} \right\| \beta_1^{k(p+q)} \varphi(z_1, z_2, \ldots, z_{l-1}) \tag{3.11}
\]

for all \( x \in X \) and \( z_1, z_2, \ldots, z_{l-1} \in Y \)

for all \( n \in \mathbb{N} \), \( x \in X \) and \( z_1, z_2, \ldots, z_{l-1} \in Y \). Replacing \( x \) with \( 2kx \) in (d) and dividing both sides by \( (2k)^\beta \), we get

\[
\left\| \frac{f(2kx)}{2k} - \frac{f((2k)^{n+1}x)}{(2k)^{n+1}}, z_1, z_2, \ldots, z_{l-1} \right\|_\beta \leq \epsilon \cdot k^{q\beta_1} \left\| (2k)^{-\beta} \right\| \beta_1^{k(p+q)} \varphi(z_1, z_2, \ldots, z_{l-1}) \tag{3.12}
\]
for all \( x \in X \) and \( z_1, z_2, \ldots, z_{l-1} \in Y \) and \( n \in \mathbb{N} \). It follows from (3.3) and (3.12) that
\[
\left\| f(x) - f((2k)^{n+1}x), z_1, z_2, \ldots, z_{l-1} \right\|_\beta \\
\leq \epsilon \cdot k^{q \beta_1} (2k)^{-\beta} \left\| (2k)^n x \right\|_{\beta_1}^k (p+q) \varphi(z_1, z_2, \ldots, z_{l-1})
\]
(3.13)
for all \( x \in X \) and \( z_1, z_2, \ldots, z_{l-1} \in Y \) and \( n \in \mathbb{N} \). This is completes the proof of (3.13). Taking the limit as \( n \to \infty \) in (3.13) we can obtain (3.2) Now we prove the uniqueness of \( H \). Assume that \( H_1 : X \to Y \) is an additive mapping satisfying (3.2). Then we have
\[
\left\| H(x) - H_1(x), z_1, z_2, \ldots, z_{l-1} \right\|_\beta \\
= \left\| (2k)^{-n\beta} \right\| H((2k)^n x) - H_1((2k)^n x), z_1, z_2, \ldots, z_{l-1} \right\|_\beta \\
\leq \left\| (2k)^{-n\beta} \right\| \max \left\{ \left\| H((2k)^n x) - f((2k)^n x), z_1, z_2, \ldots, z_{l-1} \right\| , \right\}
\leq \epsilon \cdot k^{q \beta_1} (2k)^{-n\beta} \left\| (2k)^n x \right\|_{\beta_1}^k (p+q) \varphi(z_1, z_2, \ldots, z_{l-1})
\leq \epsilon \cdot k^{q \beta_1} (2k)^{k(p+q)\beta_1 - n\beta} \left\| (2k)^n x \right\|_{\beta_1}^k (p+q) \varphi(z_1, z_2, \ldots, z_{l-1})
\]
(3.14)
for all \( x \in X \) and \( z_1, z_2, \ldots, z_{l-1} \in Y \). Taking the limit as \( n \to \infty \), we have
\[
\left\| H(x) - H_1(x), z_1, z_2, \ldots, z_{l-1} \right\|_\beta = 0
\]
for all \( x \in X \) and \( z_1, z_2, \ldots, z_{l-1} \in Y \). By lemma 2.6, we get \( H(x) = H_1(x) \) for all \( x \in X \). So \( H \) is the unique additive mapping satisfying (3.2) .

**Theorem 3.2.**

Suppose That \( X \) be a vector space and that \( Y \) is a complete non-Archimedean \((l, \beta)\)-normed space, where \( l \geq 2 \), \( 0 < \beta \leq 1 \). Let
\[
\varphi : X^{2k} \to [0, \infty)
\]
be a function such that
\[
\lim_{n \to \infty} \left(2k\right)^{n\beta} \varphi\left(\frac{(2k)^n x_1}{(2k)^n}, (2k)^n x_2, \ldots, (2k)^n x_k, (2k)^n k x_{k+1}, (2k)^n k x_{k+2}, \ldots, (2k)^n k x_{2k}\right) = 0
\]
(3.15)
for all \( x_j, x_{k+j} \in X \) for all \( j = 1 \to k \), and suppose that a mapping
\[
\psi : Y^{1-1} \to [0, \infty)
\]
be a function. The limit
\[
\lim_{n \to \infty} \max \left\{ \frac{1}{(2k)^{i+j}} \left\| \varphi((2k)^{-1}x_1, (2k)^{-1}x_2, \ldots, (2k)^{-1}x_{2k}), 1 \leq i \leq n \right\| \right\} (3.16)
\]
exists for \( x \in X \), and it is denoted by \( \tilde{\varphi}(x) \). Suppose that a mapping
\[
f : X \to Y
\]
satisfying the inequality
\[
\left\| f \left( \sum_{j=1}^{k} x_j + \sum_{j=1}^{k} \frac{x_{k+j}}{k} \right) - \sum_{j=1}^{k} f(x_j) - \sum_{j=1}^{k} f \left( \frac{x_{k+j}}{k} \right), z_1, z_2, \ldots, z_{l-1} \right\|_\beta 
\leq \varphi(x_1, x_2, \ldots, x_k, x_{k+1}, x_{k+2}, \ldots, x_{2k}) \cdot \psi(z_1, z_2, \ldots, z_{l-1}) (3.17)
\]
for all \( x, x_{k+j} \in X \) for all \( j = 1 \to k \) and \( z_1, z_2, \ldots, z_{l-1} \in Y \). Then there exists a unique additive mapping
\[
H : X \to Y
\]
satisfying
\[
\left\| f(x) - H(x), z_1, z_2, \ldots, z_{l-1} \right\|_\beta \leq \tilde{\varphi}(x) \varphi(z_1, z_2, \ldots, z_{l-1}) (3.18)
\]
for all \( x \in X \) and \( z_1, z_2, \ldots, z_{l-1} \in Y \). Moreover, if
\[
\lim_{n \to \infty} \lim_{h \to \infty} \max \left\{ \frac{1}{(2k)^{i+j}} \varphi((2k)^{-1}x_1, (2k)^{-1}x_2, \ldots, (2k)^{-1}x_{k}, (2k)^{-1}kx_{k+1}, (2k)^{-1}kx_{k+2}, \ldots, (2k)^{-1}kx_{2k}), 1 + h \leq i \leq n + h \right\} = 0 (3.19)
\]
for all \( x \in X \), then \( H \) is a unique additive mapping satisfying (3.18).

\[\text{Proof.} \] Put \( x_j = x, x_{k+j} = kx \) for all \( j = 1 \to k \) in (3.17) and dividing both sides by \( (2k)^{i+j} \), we get
\[
\left\| \frac{f(2kx)}{2k} - f(x), z_1, z_2, \ldots, z_{l-1} \right\|_\beta 
\leq \left(2k\right)^{-\beta} \varphi(x, x, \ldots, x, kx, kx, \ldots, kx) \psi(z_1, z_2, \ldots, z_{l-1}) (3.20)
\]
for all \( x \in X \) and \( z_1, z_2, \ldots, z_{l-1} \in Y \). Replacing \( x \) by \( (2k)^i x \) in (3.20) and dividing both sides by \( (2k)^{i+j} \), we get
\[
\left\| \frac{f((2k)^{i+1}x)}{(2k)^{i+1}} - \frac{f((2k)^ix)}{(2k)^i}, z_1, z_2, \ldots, z_{2k-1} \right\|_\beta 
\leq \left(2k\right)^{-\beta} \left(2k\right)^{-i\beta} \varphi((2k)^i x, (2k)^i x, \ldots, (2k)^i x, (2k)^i kx, (2k)^i kx, \ldots, (2k)^i kx) \psi(z_1, z_2, \ldots, z_{2k-1}) (3.21)
\]
for all \( x \in X \) and \( z_1, z_2, ..., z_{l-1} \in Y \). Taking the limit as \( i \to \infty \) and considering (3.15)

\[
\lim_{i \to \infty} \left\| \frac{f((2k)^{i+1}x)}{(2k)^{i+1}} - \frac{f((2k)^ix)}{(2k)^i}, z_1, z_2, ..., z_{l-1} \right\|_\beta = 0 \tag{3.22}
\]

for all \( x \in X \) and \( z_1, z_2, ..., z_{l-1} \in Y \).

It follows from (3.22) that the sequence \( \left\{ \frac{f((2k)^nx)}{(2k)^n} \right\} \) is Cauchy sequence for all \( x \in X \). Since \( Y \) is completes space, the sequence \( \left\{ \frac{f((2k)^nx)}{(2k)^n} \right\} \) converges. So one can define the mapping \( H : X \to Y \) by

\[
H(x) := \lim_{n \to \infty} \frac{f((2k)^nx)}{(2k)^n} \tag{3.23}
\]

for all \( x \in X \).

It follows from (3.17), (3.23) and lemma 2.6 that

\[
\left\| H\left( \sum_{j=1}^k x_j + \frac{1}{k} \sum_{j=1}^k x_{k+j} \right) - \sum_{j=1}^k H(x_j) - \sum_{j=1}^k H\left( \frac{x_{k+j}}{k} \right), z_1, z_2, ..., z_{l-1} \right\|_\beta
\]

\[
= \lim_{n \to \infty} \left\| (2k)^{-n\beta} \left[ f\left( \sum_{j=1}^n x_j + \frac{1}{k} \sum_{j=1}^n x_{k+j} \right) - \sum_{j=1}^k f\left( \frac{(2k)^n x_j}{k} \right) \right] - \sum_{j=1}^k f\left( \frac{(2k)^n x_j}{k} \right), z_1, z_2, ..., z_{l-1} \right\|_\beta
\]

\[
\leq \lim_{n \to \infty} \left\| (2k)^{-n\beta} \varphi((2k)^nx, (2k)^nx, ..., (2k)^nx, (2k)^nx, (2k)^nx, (2k)^nx, (2k)^nx, (2k)^nx) \psi(z_1, z_2, ..., z_{2k-1}) \right\|_\beta
\]

and so for all \( x_j, x_{k+j} \in X \) for all \( j = 1 \to k \). and \( z_1, z_2, ..., z_{l-1} \in Y \). we get

\[
\left\| H\left( \sum_{j=1}^k x_j + \frac{1}{k} \sum_{j=1}^k x_{k+j} \right) - \sum_{j=1}^k H(x_j) - \sum_{j=1}^k H\left( \frac{x_{k+j}}{k} \right), z_1, z_2, ..., z_{l-1} \right\|_\beta = 0
\]

for all \( x_j, x_{k+j} \in X \) for all \( j = 1 \to k \). and \( z_1, z_2, ..., z_{l-1} \in Y \). By lemma 2.6, we get

\[
H\left( \sum_{j=1}^k x_j + \frac{1}{k} \sum_{j=1}^k x_{k+j} \right) - \sum_{j=1}^k H(x_j) - \sum_{j=1}^k H\left( \frac{x_{k+j}}{k} \right) = 0
\]

for all \( x_j, x_{k+j} \in X \) for all \( j = 1 \to k \). So mapping \( H \) is additive.

Replace \( x \) by \( 2kx \) in (3.20) and dividing both sides by \( (2k)^\beta \), we get

\[
\left\| \frac{f((2k)^2x)}{(2k)^2} - \frac{f(2kx)}{2k}, z_1, z_2, ..., z_{l-1} \right\|_\beta
\]

\[
\leq \left\| (2k)^{-2\beta} \varphi((2k)^nx, (2k)^nx, ..., (2k)^nx, (2k)^nx, (2k)^nx, ..., (2k)^nx) \psi(z_1, z_2, ..., z_{l-1}) \right\|_\beta
\]

\[
\leq \left\| (2k)^{-2\beta} \varphi((2k)^nx, (2k)^nx, ..., (2k)^nx, (2k)^nx, (2k)^nx, ..., (2k)^nx) \psi(z_1, z_2, ..., z_{l-1}) \right\|_\beta \tag{3.24}
\]
for all $x \in X, z_1, z_2, ..., z_{l-1} \in Y$. Considering (3.20), we get

$$
\left\| f(x) - \frac{f((2k)^{2}x)}{(2k)^{2}}, z_1, z_2, ..., z_{l-1} \right\| \beta \\
\leq \max \left\{ \left\| (2k)^{-\beta} \varphi(x, x, ..., x, kx, kx, ..., kx) \right\| (2k)^{-2\beta} \varphi((2k)x, (2k)x, ..., (2k)x, (2k)kx, (2k)kx, ..., (2k)kx) \right\} \psi(z_1, z_2, ..., z_{l-1})
$$

for all $x \in X, z_1, z_2, ..., z_{l-1} \in Y$. By induction on $n$, we get

$$
\left\| f(x) - \frac{f((2k)^{n}x)}{(2k)^{n}}, z_1, z_2, ..., z_{l-1} \right\| \beta \\
\leq \max \left\{ \varphi((2k)^{h-1}x, (2k)^{h-1}x, ..., (2k)^{h-1}x, (2k)^{h-1}kx, (2k)^{h-1}kx, ..., (2k)^{h-1}kx) \right\} \psi(z_1, z_2, ..., z_{l-1})
$$

replacing $x$ by $2kx$ in (3.26)and dividing both sides by $\left\| (2k)^{\beta} \right\|$, we get

$$
\left\| \frac{f((2k)x)}{(2k)} - \frac{f((2k)^{n+1}x)}{(2k)^{n+1}}, z_1, z_2, ..., z_{l-1} \right\| \beta \\
\leq \max \left\{ \varphi((2k)^{h}x, (2k)^{h}x, ..., (2k)^{h}x, (2k)^{h}kx, (2k)^{h}kx, ..., (2k)^{h}kx) \right\} \psi(z_1, z_2, ..., z_{l-1})
$$

for all $x \in X, z_1, z_2, ..., z_{l-1} \in Y$ and $n \in N$, which together with (3.20) implies .
\[ \| f((2k)x) - \frac{f((2k)^{n+1}x)}{(2k)^{n+1}}, z_1, z_2, \ldots, z_{l-1} \|_{\beta} \]
\[ \leq \max \{ \frac{\varphi(x, x, \ldots, x, kx, kx, \ldots, kx)}{(2k)^{\beta}} : \varphi((2k)^h x, (2k)^h x, \ldots, (2k)^h x, (2k)^h kx, (2k)^h kx, \ldots, (2k)^h kx) \quad , 1 \leq h \leq n \} \psi(z_1, z_2, \ldots, z_{l-1}) \]
\[ = \max \{ \frac{\varphi((2k)^h x, (2k)^h x, \ldots, (2k)^h x, (2k)^h kx, (2k)^h kx, \ldots, (2k)^h kx)}{(2k)^{(h+1)\beta}} : \varphi((2k)^h x, (2k)^h x, \ldots, (2k)^h x, (2k)^h kx, (2k)^h kx, \ldots, (2k)^h kx) \quad , 1 \leq h \leq n + 1 \} \psi(z_1, z_2, \ldots, z_{l-1}) \quad (3.28) \]

for all \( x \in X \), \( z_1, z_2, \ldots, z_{l-1} \in Y \) and \( n \in \mathbb{N} \). This is completes the proof of (3.26) Taking the limit as \( n \to \infty \) in (3.26). Now we need to prove the uniqueness of \( H \). Let \( H' \) be another additive mapping satisfying (3.18). Sence

\[ \lim_{h \to \infty} \frac{1}{(2k)^{h\beta}} \overline{\varphi}((2k)^h x) = \lim_{h \to \infty} \lim_{n \to \infty} \max \{ \frac{1}{(2k)^{i\beta}} \varphi((2k)^{i+h-1} x_1, (2k)^{i+h-1} x_2, \ldots, (2k)^{h+i-1} x_k, (2k)^{h+i-1} k x_{k+1}, (2k)^{h+i-1} k x_{k+2}, \ldots, (2k)^{h+i-1} k x_{2k}) : 1 \leq i \leq n \} \]
\[ = \lim_{h \to \infty} \lim_{n \to \infty} \max \{ \frac{1}{(2k)^{i\beta}} \varphi((2k)^{i-1} x_1, (2k)^{i-1} x_2, \ldots, (2k)^{i-1} x_k, (2k)^{i-1} k x_{k+1}, (2k)^{i-1} k x_{k+2}, \ldots, (2k)^{i-1} k x_{2k}) : 1 + h \leq i \leq n + h \} \quad (3.29) \]

for all \( x \in X \), \( z_1, z_2, \ldots, z_{l-1} \in Y \), it follows from then \( H \) is a unique additive mapping satisfying (3.19) that.
\[
\begin{align*}
\left\| H(x) - H'(x), z_1, z_2, ..., z_{l-1} \right\|_\beta \\
&= \lim_{h \to \infty} \left| \frac{1}{(2k)^{h\beta}} \left\| H((2k)^h x) - H'((2k)^h x), z_1, z_2, ..., z_{l-1} \right\|_\beta \right| \\
&\leq \lim_{h \to \infty} \left| \frac{1}{(2k)^{h\beta}} \max \left\{ \left\| H((2k)^h x) - f((2k)^h x), z_1, z_2, ..., z_{l-1} \right\|_\beta \right\} \right| \\
&\leq \lim_{h \to \infty} \left| \frac{1}{(2k)^{h\beta}} \varphi((2k)^{-h\beta} x) \psi(z_1, z_2, ..., z_{l-1}) = 0 \right| 
\end{align*}
\]

for all \( x \in X \) and \( z_1, z_2, ..., z_{l-1} \in Y \). Considering lemma 2.6 we prove that \( H \) is unique. \( \square \)

4. Stability of the Jensen type functional equation in non-Archimedean \((l, \beta)\)-normed space

In section, we assume that \(|2| \neq 1\). Under this condition we prove the Hyers-Ulam-Rassias type stability of the Jensen type additive functional equations in non-Archimedean \((l, \beta)\)-normed space when \( X \) is a vector and \( Y \) is a complete non-Archimedean \((l, \beta)\)-normed space with norm \( \left\| \cdot \right\|_\beta \).

Under this setting, we can show that the mapping satisfying (1.2) is Jensen additive. These results are given in the following.

**Theorem 4.1.**

Suppose that \( X \) be a vector space and that \( Y \) is a complete non-Archimedean \((l, \beta)\)-normed space, where \( l \geq 2 \), \( 0 < \beta \leq 1 \). Let

\[ \varphi : X^{2k} \to [0, \infty) \]

be a function such that

\[ \lim_{n \to \infty} \left| 2^{n\beta} \varphi\left( x_1, x_2, ..., x_k, \frac{x_{k+1}}{2^n}, \frac{x_{k+2}}{2^n}, ..., \frac{x_{2k}}{2^n} \right) \right| = 0 \] (4.1)

for all \( x_j, x_{k+j} \in X \) for all \( j = 1 \to k \), and suppose that a mapping

\[ \psi : Y^{l-1} \to [0, \infty) \]

be a function. The limit

\[ \lim_{n \to \infty} \max \left\{ \left| \psi^{i\beta} \varphi\left( x_1, x_2, ..., x_k, 0, 0, ..., 0 \right) \right|, 1 \leq i \leq n - 1 \right\} \] (4.2)

exists for \( x \in X \), which is denoted by \( \tilde{\varphi}(x) \). Suppose that a mapping

\[ f : X \to Y \]
and \( f(0) = 0 \) satisfying the inequality
\[
\left\| 2k f \left( \frac{1}{2k} \sum_{j=1}^{k} x_j + \frac{1}{2k} \sum_{j=1}^{k} x_{k+j} \right) - \sum_{j=1}^{k} f \left( \frac{x_j}{k} \right) \right\|_\beta \\
\leq \varphi(x_1, x_2, \ldots, x_k, x_{k+1}, x_{k+2}, \ldots, x_{2k}) \cdot \psi(z_1, z_2, \ldots, z_{l-1})
\]
(4.3)
for all \( x, x_{k+j} \in X \) for all \( j = 1 \to k \) and \( z_1, z_2, \ldots, z_{l-1} \in Y \). Then there exists a unique additive mapping
\[ H : X \to Y \]
satisfying
\[
\left\| f(x) - H(x) , z_1, z_2, \ldots, z_{l-1} \right\|_\beta \leq \tilde{\varphi}(x) \varphi(z_1, z_2, \ldots, z_{2k-1})
\]
(4.4)
for all \( x \in X \) and \( z_1, z_2, \ldots, z_{l-1} \in Y \). Moreover, if
\[
\lim_{h \to \infty} \lim_{n \to \infty} \max \left\{ 2^{i\beta} \left| \varphi \left( \frac{x_1}{2^i}, \frac{x_2}{2^i}, \ldots, \frac{x_k}{2^i}, 0, 0, \ldots, 0 \right) \right| , h \leq i \leq n + h - 1 \right\} = 0
\]
(4.5)
for all \( x \in X \), then \( H \) is a unique additive mapping satisfying (4.4).

Proof. Put \( x_j = x, x_{k+j} = 0 \) for all \( j = 1 \to k \) in (4.3) we get
\[
\left\| 2f \left( \frac{x}{2} \right) - f(x) , z_1, z_2, \ldots, z_{l-1} \right\|_\beta \\
\leq \left| k^{-\beta} \right| \varphi(x, x, \ldots, x, 0, 0, \ldots, 0) \psi(z_1, z_2, \ldots, z_{l-1})
\]
(4.6)
for all \( x \in X \) and \( z_1, z_2, \ldots, z_{l-1} \in Y \). Replacing \( x \) by \( \frac{x}{2^n} \) in (4.6)
and multiplying both sides by \( \left| 2^{n\beta} \right| \), we get
\[
\left\| 2^{n+1} f \left( \frac{x}{2^{n+1}} \right) - 2^n f \left( \frac{x}{2^n} \right) , z_1, z_2, \ldots, z_{l-1} \right\|_\beta \\
\leq \left| k^{-\beta} \right| \left| 2^{n\beta} \right| \varphi \left( \frac{x_1}{2^n}, \frac{x_2}{2^n}, \ldots, \frac{x_k}{2^n}, 0, 0, \ldots, 0 \right) \psi(z_1, z_2, \ldots, z_{l-1})
\]
(4.7)
for all \( x \in X \) and \( z_1, z_2, \ldots, z_{l-1} \in Y \). Taking the limit as \( i \to \infty \) and considering (4.1)
\[
\lim_{n \to \infty} \left\| 2^{n+1} f \left( \frac{x}{2^{n+1}} \right) - 2^n f \left( \frac{x}{2^n} \right) , z_1, z_2, \ldots, z_{l-1} \right\|_\beta = 0
\]
(4.8)
for all \( x \in X \) and \( z_1, z_2, \ldots, z_{l-1} \in Y \).

It follows from (3.22) that the sequence \( \left\{ 2^n f \left( \frac{x}{2^n} \right) \right\} \) is Cauchy sequence for all \( x \in X \). Since \( Y \) is complete space, the sequence \( \left\{ 2^n f \left( \frac{x}{2^n} \right) \right\} \) converges. So one can define the mapping \( H : X \to Y \) by
\[
H(x) := \lim_{n \to \infty} 2^n f \left( \frac{x}{2^n} \right)
\]
(4.9)
for all \( x \in X \).
By induction on \( n \), we have
\[
\left\| 2^n f \left( \frac{x}{2^n} \right) - f(x), z_1, z_2, \ldots, z_{l-1} \right\|_{\beta} \\
\leq \max \left\{ 2^{n+1} \beta \left| \varphi \left( \frac{x_1}{2^n}, \frac{x_2}{2^n}, \ldots, \frac{x_k}{2^n}, 0, 0, \ldots, 0 \right), 1 \leq h \leq n-1 \right\} \psi (z_1, z_2, \ldots, z_{l-1})
\]
for all \( x \in X \) and \( z_1, z_2, \ldots, z_{l-1} \in Y \) and \( n \in \mathbb{N} \). Replacing \( x \) by \( \frac{x}{2^n} \) in (4.10)
and multiplying both sides by \( 2^\beta \), we get
\[
\left\| 2^{n+1} f \left( \frac{x}{2^{n+1}} \right) - 2f \left( \frac{x}{2} \right), z_1, z_2, \ldots, z_{l-1} \right\|_{\beta} \\
\leq \max \left\{ 2^{(h+1)\beta} \left| \varphi \left( \frac{x_1}{2^{h+1}}, \frac{x_2}{2^{h+1}}, \ldots, \frac{x_k}{2^{h+1}}, 0, 0, \ldots, 0 \right), 1 \leq h \leq n-1 \right\} \psi (z_1, z_2, \ldots, z_{l-1})
\]
for all \( x \in X \) and \( z_1, z_2, \ldots, z_{l-1} \in Y \) and \( n \in \mathbb{N} \). Considering the above inequality and (4.6) we have
\[
\left\| 2^{n+1} f \left( \frac{x}{2^{n+1}} \right) - 2f \left( \frac{x}{2} \right), z_1, z_2, \ldots, z_{l-1} \right\|_{\beta} \\
\leq \max \left\{ \varphi (x_1, x_2, \ldots, x_k, 0, 0, \ldots, 0), \right. \\
\left. 2^{(h+1)\beta} \left| \varphi \left( \frac{x_1}{2^{h+1}}, \frac{x_2}{2^{h+1}}, \ldots, \frac{x_k}{2^{h+1}}, 0, 0, \ldots, 0 \right), 1 \leq h \leq n-1 \right\} \psi (z_1, z_2, \ldots, z_{l-1})
\]
\[
= \max \left\{ 2^\beta \left| \varphi \left( \frac{x_1}{2^n}, \frac{x_2}{2^n}, \ldots, \frac{x_k}{2^n}, 0, 0, \ldots, 0 \right), 1 \leq h \leq n-1 \right\} \psi (z_1, z_2, \ldots, z_{l-1})
\]
for all \( x \in X \) and \( z_1, z_2, \ldots, z_{l-1} \in Y \) and \( n \in \mathbb{N} \). This completes the proof of (4.6) Taking
the limit as \( n \to \infty \) in (4.10), we obtain (4.4).

Next, we prove that \( H \) is additive. Considering (4.1), (4.3) and (4.9)
\[
\left\| 2kH \left( \frac{1}{2k} \sum_{j=1}^k x_j + \frac{1}{2k} \sum_{j=1}^k \frac{x_{k+j}}{k} \right) - \sum_{j=1}^k H \left( \frac{x_j}{k} \right), z_1, z_2, \ldots, z_{l-1} \right\|_{\beta} \\
= \lim_{n \to \infty} \left\| 2^n f \left( \frac{1}{k} \sum_{j=1}^n \frac{x_j}{2^n} + \frac{1}{k^2} \sum_{j=1}^n \frac{x_{k+j}}{2^n} \right) - \sum_{j=1}^k f \left( \frac{x_j}{2^n} \right), z_1, z_2, \ldots, z_{l-1} \right\|_{\beta} \\
\leq \lim_{n \to \infty} \left\| 2^n f \left( \frac{x_1}{2^n}, \frac{x_2}{2^n}, \ldots, \frac{x_k}{2^n}, 0, 0, \ldots, 0 \right) \psi (z_1, z_2, \ldots, z_{l-1})
\right\|_{\beta}
and so for all $x_j, x_{k+j} \in X$ for all $j = 1 \to k$. and $z_1, z_2, ..., z_{l-1} \in Y$. we get

$$\left\| 2kH \left( \frac{1}{2} \sum_{j=1}^{k} x_j + \frac{1}{2k^2} \sum_{j=1}^{k} x_{k+j} \right) - \sum_{j=1}^{k} H(x_j) - \sum_{j=1}^{k} \left( \frac{x_{k+j}}{k} \right), z_1, z_2, ..., z_{l-1} \right\| = 0$$

for all $x_j, x_{k+j} \in X$ for all $j = 1 \to k$. and $z_1, z_2, ..., z_{l-1} \in Y$. By lemma 2.6, we get

$$H\left( \sum_{j=1}^{k} x_j + \frac{1}{k} \sum_{j=1}^{k} x_{k+j} \right) - \sum_{j=1}^{k} H(x_j) - \sum_{j=1}^{k} \left( \frac{x_{k+j}}{k} \right) = 0$$

for all $x_j, x_{k+j} \in X$ for all $j = 1 \to k$. So mapping $H$ is additive.

Now we need to prove the uniqueness of $H$. Let $H'$ be another additive mapping satisfying (4.4). Since

$$\lim_{h \to \infty} \left\| \frac{1}{(2k)^{h\beta}} \tilde{\varphi} ((2k)^h x) \right\| = \lim_{h \to \infty} \lim_{n \to \infty} \max \left\{ \left\| \frac{1}{(2k)^{i\beta}} \varphi ((2k)^{i+h-1}x_1, (2k)^{i+h-1}x_2, ..., (2k)^{h+i-1}x_k, (2k)^{h+i-1}k x_{k+1}, (2k)^{h+i-1}k x_{k+2}, ..., (2k)^{h+i-1}k x_{2k}), 1 \leq i \leq n \right\| \right\}$$

$$= \lim_{h \to \infty} \lim_{n \to \infty} \max \left\{ \left\| \frac{1}{(2k)^{i\beta}} \varphi ((2k)^{i-1}x_1, (2k)^{i-1}x_2, ..., (2k)^{i-1}x_k, (2k)^{i-1}k x_{k+1}, (2k)^{i-1}k x_{k+2}, ..., (2k)^{i-1}k x_{2k}), 1 \leq i \leq n+h \right\| \right\}$$

(4.13)

for all $x \in X$, $z_1, z_2, ..., z_{l-1} \in Y$, it follows from then $H$ is a unique additive mapping satisfying (3.19) that.

$$\left\| H(x) - H'(x), z_1, z_2, ..., z_{l-1} \right\|_{\beta}$$

$$= \lim_{h \to \infty} \left\| \frac{1}{(2k)^{h\beta}} \right\| \left\| H'((2k)^h x) - H'((2k)^h x), z_1, z_2, ..., z_{2k-1} \right\|_{\beta}$$

$$\leq \lim_{h \to \infty} \left\| \frac{1}{(2k)^{h\beta}} \right\| \max \left\{ \left\| H((2k)^h x) - f((2k)^h x), z_1, z_2, ..., z_{l-1} \right\|_{\beta} \right\}$$

$$\left\| f((2k)^h x) - H'((2k)^h x), z_1, z_2, ..., z_{l-1} \right\|_{\beta}$$

$$\leq \lim_{h \to \infty} \left\| \frac{1}{(2k)^{h\beta}} \right\| \tilde{\varphi} ((2k)^{-h\beta}x) \psi (z_1, z_2, ..., z_{l-1}) = 0$$

(4.14)

for all $x \in X$ and $z_1, z_2, ..., z_{l-1} \in Y$. Considering lemma 2.6 we prove that $H$ is unique \□
5. Stability of the Pexiderized Cauchy Functional Equation

is devoted to prove the Hyers-Ulam-Rassias type stability of the pexiderized Cauchy type functional equations in non-Archimedean \((l, \beta)\)-normed space when \(X\) is a vector and \(Y\) is a complete non-Archimedean \((l, \beta)\)-normed space with norm \(\| \cdot \|_\beta\).

Theorem 5.1.

Suppose that \(X\) be a vector space and that \(Y\) is a complete non-Archimedean \((l, \beta)\)-normed space, where \(l \geq 2\), \(0 < \beta \leq 1\). Let

\[
\varphi : X^{2k} \to [0, \infty)
\]

be a function such that

\[
\Gamma(x_1, x_2, \ldots, x_k, x_{k+1}, x_{k+2}, \ldots, x_{2k}) = \sum_{i=1}^{\infty} k^{i\beta} \left( \varphi \left( \frac{x_1}{2k^{i-1}}, \frac{x_2}{2k^{i-1}}, \ldots, \frac{x_k}{2k^{i-1}}, 0, 0, \ldots \right) + \varphi \left( 0, 0, \ldots, 0, \frac{x_{k+1}}{2k^{i-1}}, \frac{x_{k+2}}{2k^{i-1}}, \ldots, \frac{x_{2k}}{2k^{i-1}} \right) + \varphi \left( \frac{x_1}{2k^{i-1}}, \frac{x_2}{2k^{i-1}}, \ldots, \frac{x_k}{2k^{i-1}}, \frac{x_{k+1}}{2k^{i-1}}, \frac{x_{k+2}}{2k^{i-1}}, \ldots, \frac{x_{2k}}{2k^{i-1}} \right) \right) < \infty
\]

and

\[
\lim_{n \to \infty} \left| k^{n\beta} \varphi \left( \frac{x_1}{2k^n}, \frac{x_2}{2k^n}, \ldots, \frac{x_k}{2k^n}, \frac{x_{k+1}}{2k^n}, \frac{x_{k+2}}{2k^n}, \ldots, \frac{x_{2k}}{2k^n} \right) \right| = 0
\]

for all \(x_j, x_{k+j} \in X\) for all \(j = 1 \to k\), and suppose that a mapping

\[
\psi : Y^{l-1} \to [0, \infty)
\]

be a function.

If mapping

\[
f, g, p : X \to Y
\]

satisfying the inequality

\[
\left\| f \left( \sum_{j=1}^{k} x_j + \sum_{j=1}^{k} \frac{x_{k+j}}{k} \right) - \sum_{j=1}^{k} g(x_j) - \sum_{j=1}^{k} p \left( \frac{x_{k+j}}{k} \right) , z_1, z_2, \ldots, z_{l-1} \right\|_\beta \\
\leq \varphi(x_1, x_2, \ldots, x_k, x_{k+1}, x_{k+2}, \ldots, x_{2k}) \cdot \psi(z_1, z_2, \ldots, z_{l-1})
\]

for all \(x_j, x_{k+j} \in X\) for all \(j = 1 \to k\) and \(z_1, z_2, \ldots, z_{l-1} \in Y\), then there exists a unique additive mapping

\[
H : X \to Y
\]

satisfying

\[
\left\| f(x) - H(x), z_1, z_2, \ldots, z_{l-1} \right\|_\beta \\
\leq \Gamma(x) \psi(z_1, z_2, \ldots, z_{l-1}) + \left\| p(0), z_1, z_2, \ldots, z_{l-1} \right\|_\beta + \left\| g(0), z_1, z_2, \ldots, z_{l-1} \right\|_\beta
\]

(5.4)
\[
\left\| g(x) - H(x), z_1, z_2, \ldots, z_{l-1} \right\|_\beta \\
\leq \Gamma(x) \psi(z_1, z_2, \ldots, z_{l-1}) + \left\| g(0), z_1, z_2, \ldots, z_{l-1} \right\|_\beta + 2 \left\| p(0), z_1, z_2, \ldots, z_{l-1} \right\|_\beta \\
+ \varphi((2k)^{i-1}x_1, (2k)^{i-1}x_2, \ldots, (2k)^{i-1}x_k, 0, 0, \ldots, 0) \psi(z_1, z_2, \ldots, z_{l-1}) \\
\leq \Gamma(x) \psi(z_1, z_2, \ldots, z_{l-1}) + 2 \left\| p(0), z_1, z_2, \ldots, z_{l-1} \right\|_\beta \\
+ \varphi(0, 0, \ldots, 0, (2k)^{i-1}x_{k+1}, (2k)^{i-1}x_{k+2}, \ldots, (2k)^{i-1}x_{2k}) \psi(z_1, z_2, \ldots, z_{l-1}) \\
\leq \varphi(\frac{x}{2k}, \frac{z_1}{2k}, \frac{z_2}{2k}, \ldots, \frac{x}{2k}, \frac{z_1}{2k}, \frac{z_2}{2k}, \ldots, \frac{x}{2k}, \frac{z_1}{2k}, \frac{z_2}{2k}, \ldots, \frac{x}{2k}, \frac{z_1}{2k}, \frac{z_2}{2k}, \ldots, \frac{x}{2k}) \psi(z_1, z_2, \ldots, z_{l-1}) \\
\leq \varphi(\frac{x}{2k}, \frac{z_1}{2k}, \frac{z_2}{2k}, \ldots, \frac{x}{2k}, 0, 0, \ldots, 0) \psi(z_1, z_2, \ldots, z_{l-1}) \\
\leq \varphi(\frac{x}{2k}, \frac{z_1}{2k}, \frac{z_2}{2k}, \ldots, \frac{x}{2k}, 0, 0, \ldots, 0) \psi(z_1, z_2, \ldots, z_{l-1}) + \left\| kp(0), z_1, z_2, \ldots, z_{l-1} \right\|_\beta \\
\leq \varphi(0, 0, \ldots, 0, \frac{x}{2k}, \frac{x}{2k}, \ldots, \frac{x}{2k}) \psi(z_1, z_2, \ldots, z_{l-1}) \\
\leq \varphi(0, 0, \ldots, 0, \frac{x}{2k}, \frac{x}{2k}, \ldots, \frac{x}{2k}) \psi(z_1, z_2, \ldots, z_{l-1}) \\
\leq \varphi(0, 0, \ldots, 0, \frac{x}{2k}, \frac{x}{2k}, \ldots, \frac{x}{2k}) \psi(z_1, z_2, \ldots, z_{l-1}) \right\|_\beta \\
\right.
\]
Using (5.7), (5.9) and (5.11), we have
\[\left\| f\left(\frac{x}{2}\right) - kp\left(\frac{x}{2k}\right), z_1, z_2, ..., z_{l-1}\right\|_\beta \leq \varphi(0, 0, ..., 0, \frac{x}{2}, \frac{x}{2}, ... \frac{x}{2})\psi(z_1, z_2, ..., z_{l-1}) + \left\| kg(0), z_1, z_2, ..., z_{l-1}\right\|_\beta \] (5.11)
for all \(x \in X\) and \(z_1, z_2, ..., z_{l-1} \in Y\).

Using (5.7), (5.9) and (5.11), we have
\[\eta\left(\frac{x}{2}, z_1, z_2, ..., z_k, z_{k+1}, z_{k+2}, ..., z_{l-1}\right) = \left\| kg(0), z_1, z_2, ..., z_{2k-1}\right\|_\beta + \left\| kp(0), z_1, z_2, ..., z_{l-1}\right\|_\beta \]
\[+ \varphi\left(\frac{x}{2k}, \frac{x}{2k}, ..., \frac{x}{2}, \frac{x}{2}, ... \frac{x}{2}\right)\psi(z_1, z_2, ..., z_{l-1}) + \varphi\left(\frac{x}{2k}, \frac{x}{2k}, ..., \frac{x}{2k}, 0, 0, ..., 0\right)\psi(z_1, z_2, ..., z_{l-1}) + \varphi(0, 0, ..., 0, \frac{x}{2}, \frac{x}{2}, ... \frac{x}{2})\psi(z_1, z_2, ..., z_{l-1})\] (5.12)
for all \(x \in X\) and \(z_1, z_2, ..., z_{l-1} \in Y\).

So
\[\left\| f\left(\frac{x}{2}\right) - \frac{1}{2} f\left(\frac{x}{2}\right), z_1, z_2, ..., z_{l-1}\right\|_\beta \leq 2^{-\beta} \eta\left(\frac{x}{2}, z_1, z_2, ..., z_{l-1}\right)\] (5.14)
for all \(x \in X\) and \(z_1, z_2, ..., z_{l-1} \in Y\). Replacing \(x\) by \(\frac{x}{2}\) in (5.14), we get
\[\left\| f\left(\frac{x}{2}\right) - \frac{1}{2} f\left(\frac{x}{2}\right), z_1, z_2, ..., z_{l-1}\right\|_\beta \leq \eta\left(\frac{x}{2}, z_1, z_2, ..., z_{l-1}\right)\] (5.15)
for all \( x \in X \) and \( z_1, z_2, ..., z_{l-1} \in Y \). It then follows from (5.14) and (5.15)

\[
\left\| f\left( \frac{x}{2^2} \right) - \frac{1}{2^2} f\left( x \right), z_1, z_2, ..., z_{l-1} \right\| \leq \left\| f\left( \frac{x}{2^2} \right) - \frac{1}{2} f\left( \frac{x}{2} \right), z_1, z_2, ..., z_{l-1} \right\| \\
+ 2^{-\beta} \left\| f\left( \frac{x}{2} \right) - \frac{1}{2} f\left( x \right), z_1, z_2, ..., z_{l-1} \right\| \leq \eta\left( \frac{x}{2^2}, z_1, z_2, ..., z_{l-1} \right) + 2^{-\beta} \eta\left( \frac{x}{2}, z_1, z_2, ..., z_{2k-1} \right)
\]

(5.16)

for all \( x \in X \) and \( z_1, z_2, ..., z_{l-1} \in Y \). Applying an induction argument on \( n \), we will prove that

\[
\left\| f\left( \frac{x}{2^n} \right) - \frac{1}{2^n} f\left( x \right), z_1, z_2, ..., z_{l-1} \right\| \leq \sum_{i=1}^{n} \frac{1}{2^{(i-1)\beta}} \varphi\left( \frac{x}{2^{n-i}}, z_1, z_2, ..., z_{l-1} \right)
\]

(5.17)

for all \( x \in X, z_1, z_2, ..., z_{l-1} \in Y \) and \( n \in \mathbb{N} \). In view of (5.14) is true for \( n=1 \). Assume that (5.14) is true for \( n > 1 \). Substituting \( \frac{x}{2} \) for \( x \) in (5.14), we obtain

\[
\left\| f\left( \frac{x}{2^n+1} \right) - \frac{1}{2^{n+1}} f\left( x \right), z_1, z_2, ..., z_{l-1} \right\| \leq \sum_{i=1}^{n} \frac{1}{2^{(i-1)\beta}} \eta\left( \frac{x}{2^{n-i}}, z_1, z_2, ..., z_{l-1} \right)
\]

(5.18)

for all \( x \in X \) and \( z_1, z_2, ..., z_{l-1} \in Y \). Hence, it follows from (5.17) that

\[
\left\| f\left( \frac{x}{2^{n+1}} \right) - \frac{1}{2^{n+1}} f\left( x \right), z_1, z_2, ..., z_{l-1} \right\| \leq \left\| f\left( \frac{x}{2^n+1} \right) - \frac{1}{2^n} f\left( x \right), z_1, z_2, ..., z_{l-1} \right\| \\
+ 2^{-n\beta} \left\| f\left( \frac{x}{2^n} \right) - \frac{1}{2^n} f\left( x \right), z_1, z_2, ..., z_{l-1} \right\| \\
\leq \sum_{i=1}^{n} \frac{1}{2^{(i-1)\beta}} \eta\left( \frac{1}{2^{n+1-i}}, z_1, z_2, ..., z_{l-1} \right) + 2^{-n\beta} \eta\left( \frac{x}{2^n}, z_1, z_2, ..., z_{l-1} \right)
\]

\[
= \sum_{i=1}^{n} \frac{1}{2^{(i-1)\beta}} \eta\left( \frac{1}{2^{n+1-i}}, z_1, z_2, ..., z_{l-1} \right)
\]

(5.19)

for all \( x \in X \) and \( z_1, z_2, ..., z_{l-1} \in Y \), which proves inequality (5.17) by (5.17), we have

\[
\left\| 2^n f\left( \frac{x}{2^n} \right) - f\left( x \right), z_1, z_2, ..., z_{l-1} \right\| \leq \sum_{i=1}^{n} \frac{2^{n\beta}}{2^{(i-1)\beta}} \eta\left( \frac{1}{2^{n-i}}, z_1, z_2, ..., z_{l-1} \right)
\]

(5.20)

for all \( x \in X, z_1, z_2, ..., z_{l-1} \in Y \) and \( n \in \mathbb{N} \). Moreover, if \( n, q \in \mathbb{N} \) with \( n < q \), then it follows from (5.14) that
\[ \left\| 2^i f\left(\frac{x}{2^i}\right) - 2^n f\left(\frac{x}{2^n}\right), z_1, z_2, \ldots, z_{l-1} \right\|_\beta \]

\[ \leq \sum_{i=n}^{q-1} \left\| 2^i f\left(\frac{x}{2^i}\right) - 2^{i+1} f\left(\frac{x}{2^{i+1}}\right), z_1, z_2, \ldots, z_{l-1} \right\|_\beta \]

\[ \leq \sum_{i=n}^{q-1} 2^{(i+1)\beta} \left\| 2^{-1} f\left(\frac{x}{2^i}\right) - f\left(\frac{x}{2^{i+1}}\right), z_1, z_2, \ldots, z_{l-1} \right\|_\beta \]

\[ = \sum_{i=n}^{q-1} 2^{(i+1)\beta} \eta \left(\frac{1}{2^i}, x, z_1, z_2, \ldots, z_{l-1}\right) \]

\[ = \sum_{i=1}^{\infty} 2^{(i+1)\beta} \left[ \varphi \left(\frac{x}{2^i}, \frac{x_2}{2^i}, \ldots, \frac{x_k}{2^i}, \frac{x_{k+1}}{2^i}, \frac{x_{k+2}}{2^i}, \ldots, \psi \left(\frac{x_k}{2^i}, x, z_1, z_2, \ldots, z_{l-1}\right) \right] \]

\[ + \varphi \left(0, 0, \ldots, 0, \frac{x_{k+1}}{2^i}, \frac{x_{k+2}}{2^i}, \ldots, \frac{x_{2k}}{2^i}\right) \psi \left(\frac{x_{k+1}}{2^i}, x, z_1, z_2, \ldots, z_{l-1}\right) \]

\[ + \left\| p(0), z_1, z_2, \ldots, z_{l-1} \right\|_\beta \leq \left\| g(0), z_1, z_2, \ldots, z_{l-1} \right\|_\beta \] (5.21)

for all \( x \in X \), \( z_1, z_2, \ldots, z_{l-1} \in Y \). Hence, it follows from

Taking the limit as \( n, q \to \infty \) and considering (5.1)

\[ \lim_{n,q \to \infty} \left\| 2^i f\left(\frac{1}{2^i} x\right) - 2^n f\left(\frac{1}{2^n} x\right), z_1, z_2, \ldots, z_{l-1} \right\|_\beta = 0 \] (5.22)

for all \( x \in X \) and \( z_1, z_2, \ldots, z_{l-1} \in Y \). According to Definition 2.4, we know that

that the sequence \( \{2^q f\left(\frac{1}{2^q} x\right)\} \) is Cauchy sequence for all \( x \in X \). Since \( Y \) is completes (\( n, \beta \) space, the sequence \( \{2^q f\left(\frac{1}{2^q} x\right)\} \) converges.

So one can define the mapping \( H: X \to Y \) by

\[ H(x) := \lim_{n \to \infty} 2^n f\left(\frac{1}{2^n} x\right) \] (5.23)

for all \( x \in X \). in (5.3)

\[ \left\| f\left(\sum_{j=1}^{k} x_j + \frac{1}{k} \sum_{j=1}^{k} \frac{x_{k+j}}{k}\right) - \sum_{j=1}^{k} g(x_j) - \sum_{j=1}^{k} p\left(\frac{x_{k+j}}{k}\right), z_1, z_2, \ldots, z_{l-1} \right\|_\beta \]

\[ \leq \lim_{n \to \infty} 2^n \left\| \varphi \left(\frac{x_1}{2^n}, \frac{x_2}{2^n}, \ldots, \frac{x_k}{2^n}, \frac{x_{k+1}}{2^n}, \frac{x_{k+2}}{2^n}, \ldots, \frac{x_{2k}}{2^n}\right), z_1, z_2, \ldots, z_{l-1} \right\|_\beta \]
and so for all $x_j, x_{k+j} \in \mathbb{X}$ for all $j = 1 \to k$. and $z_1, z_2, ..., z_{l-1} \in \mathbb{Y}$. It follows from (5.9)

$$\left\| k^n f\left( \frac{x}{2k^n} \right) - k^n g\left( \frac{x}{2k^n} \right), z_1, z_2, ..., z_{l-1} \right\|_\beta$$

$$\leq k^{n\beta} \left[ \varphi\left( \frac{x}{k}, ..., \frac{x}{k}, 0, 0, ..., 0 \right) \psi(z_1, z_2, ..., z_{l-1}) + \left\| k^n f(0), \psi(z_1, z_2, ..., z_{l-1}) \right\|_\beta \right]$$

(5.24)

for all $x \in \mathbb{X}$ and $z_1, z_2, ..., z_{l-1} \in \mathbb{Y}$. Considering (5.1)

$$k^{n\beta} \varphi\left( \frac{x_1}{2k^n}, \frac{x_2}{2k^n}, ..., \frac{x_k}{2k^n}, 0, 0, ..., 0 \right) \psi(z_1, z_2, ..., z_{l-1})$$

$$\leq k^{\beta} \sum_{i=1}^{\infty} k^{(i+1)\beta} \left[ \varphi\left( \frac{x_1}{2k^n}, \frac{x_2}{2k^n}, ..., \frac{x_k}{2k^n}, 0, 0, ..., 0 \right) \psi(z_1, z_2, ..., z_{l-1}) \right.$$

$$+ \varphi(0, 0, ..., 0, \frac{x_{k+1}}{2k^n}, ..., \frac{x_{k+2}}{2k^n}) \psi(z_1, z_2, ..., z_{l-1})$$

$$+ \varphi\left( \frac{x_1}{2k^n}, \frac{x_2}{2k^n}, ..., \frac{x_k}{2k^n}, \frac{x_{k+1}}{2k^n}, ..., \frac{x_{k+2}}{2k^n} \right) \psi(z_1, z_2, ..., z_{l-1})$$

(5.25)

$\to 0$ as $n \to \infty$.

It follows from (5.25) that

$$H(x) := \lim_{n \to \infty} k^n f\left( \frac{1}{2k^n} x \right) = \lim_{n \to \infty} k^n g\left( \frac{1}{2k^n} x \right)$$

(5.26)

for all $x \in \mathbb{X}$. Also, by (5.11)

$$\left\| k^n f\left( \frac{x}{2k^n} \right) - k^n p\left( \frac{x}{2k^n} \right), z_1, z_2, ..., z_{l-1} \right\|_\beta$$

$$\leq k^{n\beta} \left[ \varphi(0, 0, ..., 0, \frac{x}{2k^n}, ..., \frac{x}{2k^n}) \psi(z_1, z_2, ..., z_{l-1}) \right. + \left\| k^n p(0), \psi(z_1, z_2, ..., z_{l-1}) \right\|_\beta$$

(5.27)

for all $x \in \mathbb{X}$ and $z_1, z_2, ..., z_{l-1} \in \mathbb{Y}$. Similarly, it follows from (5.27) that

$$H(x) := \lim_{n \to \infty} k^n f\left( \frac{1}{2k^n} x \right) = \lim_{n \to \infty} k^n p\left( \frac{1}{2k^n} x \right)$$

(5.28)

for all $x \in \mathbb{X}$ and $z_1, z_2, ..., z_{l-1} \in \mathbb{Y}$. Thus, by (5.2), (5.25), (5.28) and lemma 2.7, we get

$$\left\| H \left( \sum_{j=1}^{k} x_j + \frac{1}{k} \sum_{j=1}^{k} \frac{x_{k+j}}{k} \right) - \sum_{j=1}^{k} H(x_j) + \sum_{j=1}^{k} H \left( \frac{x_{k+j}}{k} \right), z_1, z_2, ..., z_{l-1} \right\|_\beta$$

$$= \lim_{n \to \infty} \left\| k^{n\beta} \left\| f \left( \sum_{j=1}^{n} \frac{1}{2k^n} x_j + \frac{1}{k} \sum_{j=1}^{n} \frac{1}{2k^n} x_{k+j} \right) \right\|_\beta \right. - \sum_{j=1}^{k} f\left( \frac{1}{2k^n} x_j \right) - \sum_{j=1}^{k} f\left( \frac{1}{2k^n} \frac{x_{k+j}}{k} \right), z_1, z_2, ...$$

, $z_{l-1}$

$$\leq \lim_{n \to \infty} \left\| k^{n\beta} \varphi\left( \frac{x_1}{2k^n}, \frac{x_2}{2k^n}, ..., \frac{x_k}{2k^n}, \frac{x_{k+1}}{2k^n}, ..., \frac{x_{k+2}}{2k^n} \right) \psi(z_1, z_2, ..., z_{l-1}) \right. - 0$$

and so for all $x_j, x_{k+j} \in \mathbb{X}$ for all $j = 1 \to k$. and $z_1, z_2, ..., z_{l-1} \in \mathbb{Y}$. 30
we get
\[
\left\| H\left(\sum_{j=1}^{k} x_j + \frac{1}{k} \sum_{j=1}^{k} x_{k+j}\right) - \sum_{j=1}^{k} H\left(x_j\right) - \sum_{j=1}^{k} H\left(\frac{x_{k+j}}{k}\right), z_1, z_2, \ldots, z_{l-1}\right\| = 0
\]
for all \(x_j, x_{k+j} \in X\) for all \(j = 1 \rightarrow k\). and \(z_1, z_2, \ldots, z_{l-1} \in Y\). By lemma 2.6, we get
\[
H\left(\sum_{j=1}^{k} x_j + \frac{1}{k} \sum_{j=1}^{k} x_{k+j}\right) - \sum_{j=1}^{k} H\left(x_j\right) - \sum_{j=1}^{k} H\left(\frac{x_{k+j}}{k}\right) = 0
\]
for all \(x_j, x_{k+j} \in X\) for all \(j = 1 \rightarrow k\). So mapping \(H\) is additive. Taking the limit as \(n \to \infty\) in (5.17)
\[
\left\| H(x) - f(x), z_1, z_2, \ldots, z_l \right\|_\beta
\]
\[
\leq \lim_{n \to \infty} \sum_{i=1}^{n} \frac{1}{2^{k(i-1)-n}} \varphi\left(\frac{x}{2^{k(i-1)-n}}, z_1, z_2, \ldots, z_{l-1}\right)
\]
\[
= \lim_{n \to \infty} \left(1 - 2^{k^n}\right) \left(\left\| k\left(0\right), z_1, z_2, \ldots, z_{l-1}\right\|_\beta + \left\| k\left(0\right), z_1, z_2, \ldots, z_{l-1}\right\|_\beta\right)
\]
\[
+ \lim_{n \to \infty} \sum_{i=1}^{n} \frac{1}{2^{k(i-1)-n}} \varphi\left(\frac{x_1}{2^{k(i-1)-n}}, \frac{x_2}{2^{k(i-1)-n}}, \ldots, \frac{x_k}{2^{k(i-1)-n}}, 0, 0, \ldots, 0\right) \psi\left(z_1, z_2, \ldots, z_{l-1}\right)
\]
\[
+ \varphi\left(0, 0, \ldots, 0, \frac{x_1}{2^{k(i-1)-n}}, \frac{x_2}{2^{k(i-1)-n}}, \ldots, \frac{x_k}{2^{k(i-1)-n}}\right) \psi\left(z_1, z_2, \ldots, z_{l-1}\right)
\]
\[
+ \varphi\left(\frac{x_1}{2^{k(i-1)-n}}, \frac{x_2}{2^{k(i-1)-n}}, \ldots, \frac{x_k}{2^{k(i-1)-n}}\right) \psi\left(z_1, z_2, \ldots, z_{l-1}\right)
\]
\[
= \left\| k\left(0\right), z_1, z_2, \ldots, z_{l-1}\right\|_\beta + \left\| k\left(0\right), z_1, z_2, \ldots, z_{l-1}\right\|_\beta + \Gamma(x) \psi\left(z_1, z_2, \ldots, z_{l-1}\right) \quad (5.29)
\]
for all \(x \in X, z_1, z_2, \ldots, z_{l-1} \in Y\), Which prover (5.4) Prover the uniqueness of \(H\). Assume That
\[
H' : X \to Y
\]
is another additive mapping which satisfying (5.4)
\[
\left\| H(x) - H'(x), z_1, z_2, \ldots, z_{l-1}\right\|_\beta
\]
\[
\leq k^{n\beta} \left\| H\left(\frac{x}{2^{kn}}\right) - f\left(\frac{x}{2^{kn}}\right), z_1, z_2, \ldots, z_{l-1}\right\|_\beta + k^{n\beta} \left\| f\left(\frac{x}{2^{kn}}\right) - H'\left(\frac{x}{2^{kn}}\right), z_1, z_2, \ldots, z_{l-1}\right\|_\beta
\]
\[
k^{n\beta+1} \left(\left\| k\left(0\right), z_1, z_2, \ldots, z_{l-1}\right\|_\beta + \left\| k\left(0\right), z_1, z_2, \ldots, z_{l-1}\right\|_\beta + \Gamma\left(\frac{x}{2^{kn}}\right) \psi\left(z_1, z_2, \ldots, z_{l-1}\right)\right)
\]
\[
= \left\| k\left(0\right), z_1, z_2, \ldots, z_{l-1}\right\|_\beta + \left\| k\left(0\right), z_1, z_2, \ldots, z_{l-1}\right\|_\beta + \Gamma\left(\frac{x}{2^{kn}}\right) \psi\left(z_1, z_2, \ldots, z_{l-1}\right)
\]
\[
+ \varphi\left(\frac{x_1}{2^{k(i-1)-n}}, \frac{x_2}{2^{k(i-1)-n}}, \ldots, \frac{x_k}{2^{k(i-1)-n}}, 0, 0, \ldots, 0\right) \psi\left(z_1, z_2, \ldots, z_{l-1}\right)
\]
\[
+ \varphi\left(0, 0, \ldots, 0, \frac{x_1}{2^{k(i-1)-n}}, \frac{x_2}{2^{k(i-1)-n}}, \ldots, \frac{x_k}{2^{k(i-1)-n}}\right) \psi\left(z_1, z_2, \ldots, z_{l-1}\right)
\]
\[
+ \varphi\left(\frac{x_1}{2^{k(i-1)-n}}, \frac{x_2}{2^{k(i-1)-n}}, \ldots, \frac{x_k}{2^{k(i-1)-n}}\right) \psi\left(z_1, z_2, \ldots, z_{l-1}\right)
\]
\[
\quad (5.30)
\]
n \to 0 as \(n \to \infty\) for all \(x \in X\) and \(z_1, z_2, \ldots, z_{l-1} \in Y\). Which together with lemma 2.6 implies that \(H(x) = H'(x)\) for all \(x \in X\).
6. Conclusion

In this paper, I have shown that the solutions of the (1.1) and (1.2) are additive mappings. The Hyers-Ulam-Rassia stability for these given from theorems. These are the main results of the paper, which are the generalization of the results [10, 11, 25].

References