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Exponential Generalization of a Ternary Symmetric Inequality

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Abstract - *In this paper, we establish an exponential generalization of a ternary symmetric inequality by using the theory of majorization and Schur convex.*

Keywords - *Ternary symmetric inequality, Inequality conjecture, Majorization, Schur convex function.*

I. INTRODUCTION

It has a wide application background to discuss the extremum of the following formula or study the related inequality problems.

$$F(x_1,x_2,\cdots,x_n)=\sum_{i=1}^{n} f(x_i), \tag{1.1}$$

where $f(x)$ is usually a derivable function in interval $I\subset R$. Suppose that $f(x)$ is a second derivatiable function with at least one inflection point in interval I. Then it is not suitable to directly use Jensen's inequality, but should use Schur's convex function to obtain its extreme value of (1.1) for I^n on hyperplane $x_1+x_1+\cdots+x_n=1$. We gave a counterexample to point out that the proofs of the two opposite conclusions in [2] and [3] are wrong, and proved the following inequality (see [4]).

Suppose that a,b,c are positive numbers. Then

$$\frac{2}{\sqrt{5}}\le\frac{a}{\sqrt{a^2+4(b^2+c^2)}}+\frac{b}{\sqrt{b^2+4(c^2+a^2)}}+\frac{c}{\sqrt{c^2+4(a^2+b^2)}}\le\frac{1}{3\sqrt{3}}\sqrt{3+8\sqrt{6}}+\frac{2}{3\sqrt{15}}\sqrt{4\sqrt{6}-9}. \tag{1.2}$$

In this paper, in order to further show the application skills of majorization and Schur convex function method, we will give the exponential generalization of inequality (1.2) and obtain the following inequality.

Theorem 1 Suppose that *a, b, c* are positive numbers and $n=3,4,5,\cdots$. Then

$$1\le\frac{a}{\sqrt[n]{a^n+4(b^n+c^n)}}+\frac{b}{\sqrt[n]{b^n+4(c^n+a^n)}}+\frac{c}{\sqrt[n]{c^n+4(a^n+b^n)}}\le\frac{3}{\sqrt[n]{9}}. \tag{1.3}$$

II. THE PROOF OF THEOREM

Proof of Theorem 1. Let $x=\frac{a^n}{a^n+b^n+c^n}$, $y=\frac{b^n}{a^n+b^n+c^n}$, $z=\frac{c^n}{a^n+b^n+c^n}$, then inequality (1.3) is equivalent to the following inequality for any $(x,y,z)\in\Omega=\{(x,y,z)\,|\,0<x,y,z<1,\ and\ x+y+z=1\}$.

$$1\le\sqrt[n]{\frac{x}{4-3x}}+\sqrt[n]{\frac{y}{4-3y}}+\sqrt[n]{\frac{z}{4-3z}}\le\frac{3}{\sqrt[n]{9}}. \tag{2.1}$$

Let $f(x)=\sqrt[n]{\frac{x}{4-3x}}$, $F:=F(x,y,z)=f(x)+f(y)+f(z)$, then $F(x,y,z)$ is a ternary symmetric function for Ω. Noticing that

$$f'(x)=\frac{4}{n}x^{1/n-1}(4-3x)^{-1/n-1},\ f''(x)=\frac{24}{n}x^{1/n-2}(4-3x)^{-1/n-2}\left(x-\frac{2(1-1/n)}{3}\right), \tag{2.2}$$

thus $f(x)$ has a unique inflection point $\beta:=\frac{2(1-1/n)}{3}$ in the interval $[0,\ 1]$.

By symmetric with inequality (2.1), we may assume that $x\ge y\ge z$.

For convenience, we wirte $a_n(x)=\sqrt[n]{x^{n-1}(4-3x)^{n+1}}$, then $f'(x)=\frac{4}{na_n(x)}$.

Case 1. If $n=3$, then $\beta=\frac{4}{9}$. This case can be further divided into the following three cases.

(i) If $\frac{4}{9}\ge x\ge y\ge z>0$, then $F(x,y,z)$ is a Schur concave function on region $(0,\ 4/9)^3$. Since

$$\left(\frac{1}{3},\frac{1}{3},\frac{1}{3}\right)\prec(x,y,z)\prec\left(\frac{4}{9},\frac{4}{9},\frac{1}{9}\right), \tag{2.3}$$

We have

$$2f\left(\frac{4}{9}\right)+f\left(\frac{1}{9}\right)\le F\le 3f\left(\frac{1}{3}\right). \tag{2.4}$$

(ii) If $x\ge\frac{4}{9}\ge y\ge z>0$, then $F_1(y,z):=F(1-y-z,y,z)$ is a Schur concave function on region $(0,\ 4/9)^2$. Since

$$\left(\frac{y+z}{2},\frac{y+z}{2}\right)\prec(y,z)\prec\begin{cases}\left(\frac{4}{9},y+z-\frac{4}{9}\right), & \frac{4}{9}\le x\le\frac{5}{9}\\ (y+z,0), & \frac{5}{9}\le x<1\end{cases}, \tag{2.5}$$

We have

$$\left.\begin{matrix} f\left(\frac{4}{9}\right)+f\left(\frac{5}{9}-x\right)+f(x), & \frac{4}{9}\le x\le\frac{5}{9}\\ f(x)+f(1-x), & \frac{5}{9}<x<1\end{matrix}\right\}\le F\le f(x)+2f\left(\frac{1-x}{2}\right),\ \frac{4}{9}\le x<1. \tag{2.6}$$

(iii) If $x\ge y\ge\frac{4}{9}\ge z>0$, then $F_2(x,y):=F(x,y,1-x-y)$ is a Schur convex function on region $(4/9,\ 5/9)^2$.

Since

$$\left(\frac{x+y}{2},\frac{x+y}{2}\right)\prec(x,y)\prec\left(x+y-\frac{4}{9},\frac{4}{9}\right), \tag{2.7}$$

we have

$$f(z)+2f\left(\frac{1-z}{2}\right)\le F\le f\left(\frac{4}{9}\right)+f\left(\frac{5}{9}-z\right)+f(z),\ 0<z\le\frac{1}{9}. \tag{2.8}$$

Wirte

$$h_1(x):=f(x)+2f\left(\frac{1-x}{2}\right),\ h_2(x):=f(x)+f\left(1-x\right),\ h_3(x):=f\left(\frac{4}{9}\right)+f\left(\frac{5}{9}-x\right)+f(x), \tag{2.9}$$

then

$$h_1'(x)=f'(x)-f'\left(\frac{1-x}{2}\right),\ h_2'(x)=f'(x)-f'\left(1-x\right),\ h_3'(x)=f'(x)-f'\left(\frac{5}{9}-x\right). \tag{2.10}$$

Furthemore,

$$h_1'(x)=\frac{-(3x-1)(63x^3-213x^2+133x+25)(27x^2-48x+25)}{48a_3(x)a_3\left(\frac{1-x}{2}\right)\left\{a_3^2(x)+a_3(x)a_3\left(\frac{1-x}{2}\right)+a_3^2\left(\frac{1-x}{2}\right)\right\}}=\begin{cases}>0,\ 0<x<1/3\\<0,\ 1/3<x<1\end{cases},$$

(2.11)

$$h_2'(x)=\frac{4(2x-1)(9x^2-9x+1)(21x^2-21x-1)}{3a_3(x)a_3\left(1-x\right)\left\{a_3^2(x)+a_3(x)a_3\left(1-x\right)+a_3^2\left(1-x\right)\right\}}=\begin{cases}>0,\ \text{for } 0<x<\frac{1}{2}-\frac{\sqrt{5}}{2},\ \text{or } \frac{1}{2}<x<\frac{1}{2}+\frac{\sqrt{5}}{6},\\<0,\ \text{for } \frac{1}{2}-\frac{\sqrt{5}}{2}<x<\frac{1}{2},\ \text{or } \frac{1}{2}+\frac{\sqrt{5}}{6}<x<1,\end{cases} \tag{2.12}$$

$$h_3'(x)=\frac{4(18x-5)(81x^2-45x+49)(2673x^2-1485x-245)}{19683a_3(x)a_3\left(\frac{5}{9}-x\right)\left\{a_3^2(x)+a_3(x)a_3\left(\frac{5}{9}-x\right)+a_3^2\left(\frac{5}{9}-x\right)\right\}}=\begin{cases}>0,\ \text{for } 0<x<\frac{5}{18},\ \text{or } \frac{5}{18}+\frac{19\sqrt{165}}{594}<x<1,\\<0,\quad \text{for } \frac{5}{18}<x<\frac{5}{18}+\frac{19\sqrt{165}}{594},\end{cases}. \tag{2.13}$$

Therefore,

$$\max_{4/9\le x<1} h_1(x)=h_1\left(\frac{4}{9}\right)=f\left(\frac{4}{9}\right)+2f\left(\frac{5}{18}\right), \tag{2.14}$$

$$\min_{5/9\le x<1} h_2(x)=\min\left\{h_2\left(\frac{5}{9}\right),h_2(1)\right\}=1,\ \min_{4/9\le x\le 5/9} h_3(x)=h_3\left(\frac{5}{9}\right)=f\left(\frac{5}{9}\right)+f\left(\frac{4}{9}\right), \tag{2.15}$$

$$\min_{0<x<1/9} h_1(x)=h_1\left(0\right)=2f\left(\frac{1}{2}\right),\ \max_{0<x\le 1/9} h_3(x)=h_3\left(\frac{1}{9}\right)=2f\left(\frac{4}{9}\right)+f\left(\frac{1}{9}\right). \tag{2.16}$$

From (2.6) and (2.14) ,together with (2.15), we have

$$1 \le F \le f\left(\frac{4}{9}\right)+2f\left(\frac{5}{18}\right). \tag{2.17}$$

From (2.16), together with (2.14), we have

$$2f\left(\frac{1}{2}\right) \le F \le 2f\left(\frac{4}{9}\right)+f\left(\frac{1}{9}\right). \tag{2.18}$$

Combining (2.4), (2.17) and (2.18), we get

$$1 \le F \le \sqrt[3]{3}. \tag{2.19}$$

Case 1 has been proved.

Case 2. If $n \ge 4$, then $\frac{1}{2} \le \beta < \frac{2}{3}$. This case can be further divided into the following two cases.

(i) If $\beta \ge x \ge y \ge z > 0$, then $F(x,y,z)$ is a Schur concave function on region $(0,\ \beta)^3$. Since

$$\left(\frac{1}{3},\frac{1}{3},\frac{1}{3}\right) \prec (x,y,z) \prec (\beta, 1-\beta, 0), \tag{2.20}$$

We have

$$f(\beta)+f(1-\beta) \le F \le 3f\left(\frac{1}{3}\right). \tag{2.21}$$

(ii) If $x \ge \beta \ge y \ge z > 0$, then $F_1(y,z) := F(1-y-z,y,z)$ is a Schur concave function on region $(0,\ \beta)^2$. Noticing that $y+z = 1-x \le \beta$, therefore

$$\left(\frac{y+z}{2},\frac{y+z}{2}\right) \prec (y,z) \prec (y+z,0), \tag{2.22}$$

We have

$$f(x)+f(1-x) \le F \le f(x)+2f\left(\frac{1-x}{2}\right),\ \beta \le x < 1. \tag{2.23}$$

We still wirte

$$h_1(x) := f(x)+2f\left(\frac{1-x}{2}\right),\ h_2(x) := f(x)+f(1-x), \tag{2.24}$$

where

$$h_1'(x) = \frac{4\left\{a_n^n\left(\frac{1-x}{2}\right)-a_n^n(x)\right\}}{na_n(x)a_n\left(\frac{1-x}{2}\right)\left\{\sum_{i=0}^{n-1} a_n^i(x)a_n^{n-i-1}\left(\frac{1-x}{2}\right)\right\}}, \tag{2.25}$$

$$h_2'(x)=\frac{4\left\{a_n^n\left(1-x\right)-a_n^n(x)\right\}}{na_n(x)a_n\left(1-x\right)\left\{\sum_{i=0}^{n-1}a_n^i(x)a_n^{n-i-1}\left(1-x\right)\right\}}\ . \tag{2.26}$$

We now prove the following results.

(a) If $n\geq 4$, then

$$a_n^n\left(1-x\right)-a_n^n(x)=\begin{cases}>0, & 0<x<\frac{1}{2}\\ <0, & \frac{1}{2}<x<1\end{cases}. \tag{2.27}$$

(b) If $n\geq 3$, then

$$a_n^n\left(\frac{1-x}{2}\right)-a_n^n(x)<0,\ \frac{1}{3}<x<1. \tag{2.28}$$

First, $g_1(x):=a_n^n\left(1-x\right)-a_n^n(x)$ is symmetric about x = 1 / 2 in interval $(0,\ 1)$. Thus, we only need to prove $g_1(x)>0$ when $0<x<1/2$ about (2.25). In fact, If $0<x<1/2$, then $\ln\frac{(1-x)(1+3x)}{x(4-3x)}>0$. Since

$$\begin{aligned} g_1(x)>0 &\Leftrightarrow (1-x)^{n-1}(1+3x)^{n+1}>x^{n-1}(4-3x)^{n+1}\\ &\Leftrightarrow n\ln\frac{(1-x)(1+3x)}{x(4-3x)}+\ln\frac{x(1+3x)}{(1-x)(4-3x)}>0, \end{aligned} \tag{2.29}$$

and

$$\begin{aligned} g_1(x)\big|_{n=4} &= (1-x)^3(1+3x)^5>x^3(4-3x)^5\\ &=-(2x-1)^3(162x^4-324x^3+144x^2+18x+1)\\ &>0. \end{aligned} \tag{2.30}$$

Therefore, (2.29) holds for $n\geq 4$.

Second, write $g_2(x):=a_n^n\left(\frac{1-x}{2}\right)-a_n^n(x)$. We note that if $\frac{1}{3}<x<1$, then $\ln\frac{4x(4-3x)}{(1-x)(5+3x)}>0$. Since

$$\begin{aligned} g_2(x)<0 &\Leftrightarrow x^{n-1}(4-3x)^{n+1}>\left(\frac{1-x}{2}\right)^{n-1}\left(4-3\cdot\frac{1-x}{2}\right)^{n+1}\\ &\Leftrightarrow n\ln\frac{4x(4-3x)}{(1-x)(5+3x)}-\ln\frac{4x(5+3x)}{(1-x)(4-3x)}>0. \end{aligned} \tag{2.31}$$

Thus (2.31) hold for $n\geq 3$ with (2.11).

Combining (2.25) with (2.28), and (2.26) with (2.27), we know that $h_1(x)$ is monotonically decreasing in the interval

$(1/3,\ 1)$, and $h_2(x)$ is monotonically decreasing in the interval $(\beta,\ 1)$ when $n \geq 4$. Thus we have

$$\max_{\beta \le x < 1} h_1(x) = h_1(\beta) = f(\beta) + 2f\left(\frac{1-\beta}{2}\right) \le f\left(\frac{1}{3}\right), \quad \min_{\beta \le x < 1} h_2(x) = h_2(1) = 1, \tag{2.32}$$

Therefore

$$1 = f(1) \le F \le 3f\left(\frac{1}{3}\right) = \frac{3}{\sqrt[n]{9}}. \tag{2.33}$$

This finishes the proof.

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