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k-Rooted Product of Two Graphs

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Abstract:

Extending the idea of rooted product of two graphs, a new operation on two graphs is introduced and some of its properties are studied. We call it k-rooted product and denote it as $G \circ^k H$, where k-copies of a rooted graph H are connected at the root to every vertex of the graph G in the resulting product graph.

Keywords: *Rooted product graph, k-rooted product graphs.*

I. Introduction

Throughout this paper, we consider a finite simple connected graph G that has no loops or multiple edges. The vertex and the edge sets of a graph G are denoted by V (G) and E(G), respectively. The rooted product graphs $G\ o\ H$ was introduced by Godsil and Mckay in 1978 [4]. The rooted product of a graph *G* and a rooted graph *H* is defined as, take $|V(G)|$ copies of *H*, and for every vertex v_i of *G*, identify v_i with the root node of the *i*-th copy of *H*. The rooted product graph is denoted as $G\ o\ H$.

More formally, assuming that $V(G) = \{g_1, g_2 ..., g_m\}$, $V(H) = \{h_1, h_2\ ..., h_n\}$. Let the root node of *H* be h_1, then the vertex and edge set of G o H are as follows,

$V(G\ o\ H) = \{(g_i, h_j) \,/\, 1 \le i \le m,\ 1 \le j \le n\}$ and

$E(G\ o\ H) = \{((g_i, h_1),(g_k, h_1)) \text{ if } (g_i, g_k) \epsilon E(G)\} \cup \bigcup_{i=1}^{n}\{((g_i, h_j),(g_i, h_k)) \text{ if } (h_j, h_k) \in E(H)\}$,

Note that the rooted product $G\ o\ H$ is a subgraph of the cartesian product G x H. The number of vertices and edges in rooted product graph are $|V(G\ o\ H)| = |V(G)| + |V(G)|*(|V(H)| - 1) = |V(G)| * |V(H)|$ and $|E(G\ o\ H)| = |E(G)| + |V(G)| * |E(H)|$.

We extend this concept of rooted product, by attaching root of each of k-copies of rooted graph H at every vertex of G.

II. k-rooted product graph $G \circ^k H$

The k-rooted product of a graph G and a rooted graph H is defined as follows: Consider k-copies of rooted graph H and map it with every vertex v_i of G by identifying v_i with root node of each of the k-copies of H. This graph is denoted as $G \circ^k H$.

More formally, let G be a graph on m vertices and H be a graph on n vertices with vertex sets given as $V(G)$ = $\{g_1,g_2...,g_m\}$, $V(H)$ = $\{h_1,h_2 ..., h_n\}$. Let the root node of H be h_1, then we define the vertex set of k-rooted product graph $G \circ^k H$ as, $V(G \circ^k H) = V_R \cup V_{NR}$, with root vertices set V_R = $\{(g_i, h_1^1, h_1^2, h_1^3 \dots\dots h_1^k); i = 1,2,\dots\dots m\}$ and non-root vertices set V_{NR}=$\{(g_i, h_j^r) / g_i \in G , h_j^r \in H^r$;i=1,2,…..m , r=1,2,…k & j= 1,2,……n} where (g_i, h_j^r) is the jth vertex of rth copy of H attached at $g_i \in G$.

Illustration :

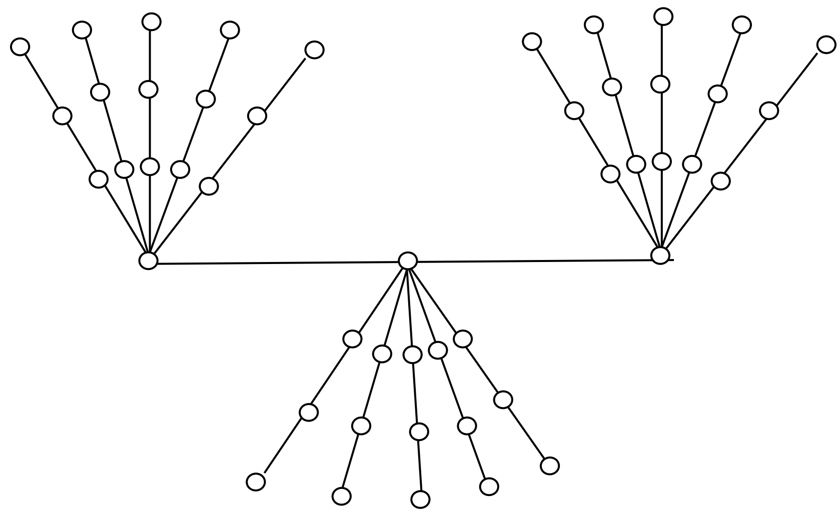

Fig. 1 : k-rooted Product Graph $P_3 \circ^5 P_4$

Next, the edge set of rooted product $G \circ^k H$ contains edges contributed from the graph G and edges contributed by each of the k-copies of H attached at each node of G. So edge set can be written as disjoint union of two sets, $E(G \circ^k H) = E_1 \cup E_2$ where,

$$E_1 = \{(g_i, h_1^1, h_1^2, h_1^3 \dots\dots h_1^k), (g_j, h_1^1, h_1^2, h_1^3 \dots\dots h_1^k); if\ (g_i, g_j) \in E(G)\}$$

$$E_2 = \cup_{r=1}^{k}\{((g_j, h_p^r)\ (g_j, h_q^r)\ if\ (h_p, h_q) \in H\}$$

Proposition 2.1: The number of vertices and edges in k-rooted product graph $G \circ^k H$ is given by

1. $|V(G \circ^k H)| = |V(G)| + |V(G)|.k(|V(H)| - 1)$
2. $|E(G \circ^k H)| = |E(G)| + |V(G)|.(k|E(H)|)$

Proof: From definition of k-rooted product graph vertex set we have, $V(G \circ^k H) = V_R \cup V_{NR}$ where root vertices set V_R= { $(g_i, h_1^1, h_1^2, h_1^3 \dots\dots h_1^k); i = 1,2,\dots\dots m\}$ contains |V(G)| = m vertices and non-root vertices set V_{NR}= {($g_i, h_j^r) / g_i \in G$,

$h_j \in H^r$; i=1,2,…..m & j= 1,2,……n} has |V(H)|-1 vertices from each copy of H attached at g_i as root vertex is identified with the vertex g_i, giving total $k|V(H)| - 1$ vertices at each vertex g_i hence total number of vertices in the k-rooted product $G \circ^k H$ is,

$$|V(G \circ^k H)| = |V(G)| + |V(G)| * k * [|V(H)| - 1] \text{ ------------- (1)}$$

Similarly, $E(G \circ^k H) = E_1 \cup E_2$ a disjoint union gives $|E(G \circ^k H)| = |E_1| + |E_2|$ Here we have $E_1 = \{(g_i, h_1^1, h_1^2, h_1^3 \dots\dots h_1^k), (g_j, h_1^1, h_1^2, h_1^3 \dots\dots h_1^k); if\ (g_i, g_j) \in E(G)\}$ giving equal number of edges as in G so $|E_1| = |E(G)|$. Next the edges contributed by k copies of H attached at each vertex of G included in $E_2 = \cup_{r=1}^{k}\{((g_j, h_p^r)\ (g_j, h_q^r)\ if\ (h_p, h_q) \in H\}$, gives $|E_2| = |V(G)| * k|E(H)|$.

Thus, total number of edges in the k- rooted product $G \circ^k H$ is,

$$|E(G \circ^k H)| = |E(G)| + |V(G)|[kE(H)] \text{ ---------------(2)}$$

Proposition 2.2: The difference between number of vertices in k-rooted product graph $G \circ^k H$, the number of vertices in G and the difference between number of vertices in the rooted product graph $G \circ H$, the number of vertices in G, maintain a fixed ratio k.

Proof : We know that the number of vertices in $G \circ H$ is

$$|V(G \circ H)| = |V(G)| + |V(G)|[|V(H)| - 1] \text{ ----------------- (3)}$$

Equation (1) and (3) gives,

$$|V(G \circ^k H)| - |V(G)| = k * [|V(G)|(|V(H)| - 1)]$$
$$|V(G \circ^k H)| - |V(G)| = k * [|V(G \circ H)| - |V(G)|]$$
$$\frac{|V(G \circ^k H)| - |V(G)|}{|V(G \circ H)| - |V(G)|} = k$$

Proposition 2.3: The difference between number of edges in k-rooted product graph $G \circ^k H$, the number of edges in G and the difference between number of edges in the rooted product graph $G \circ H$, the number of edges in G, maintain a fixed ratio k.

Proof : We know that the number of edges in $G \circ H$ is

$$|E(G \circ H)| = |E(G)| + |V(G)|[E(H)] \text{ -------------------------- (4)}$$

Equation (2) and (4) gives,

$$|E(G \circ^k H)| - |E(G)| = k * |V(G)||E(G)|$$
$$|E(G \circ^k H)| - |E(G)| = k * [|E(G \circ H)| - |E(G)|]$$
$$\frac{|E(G \circ^k H)| - |E(G)|}{|E(G \circ H)| - |E(G)|} = k$$

Proposition 2.4: If G and H are connected graphs then $G \circ^k H$ is a connected graph.

Proof: By the definition of the k-rooted graph $G \circ^k H$, root vertex h_1 of each of the k-copies of the connected graph H is attached to each vertex of G. As G is a connected graph, and each copy of H is connected with vertices of G through the root vertex. Thus as G and H are connected, it follows that $G \circ^k H$ is a connected graph.

Proposition 2.5: The degree sequence of connected graph $G \circ^k H$ is ,
$\{(d_1 + kr_i)^{m_1}, (d_2 + kr_i)^{m_2}, \ldots (d_t + kr_i)^{m_t}, {r_1}^{kn_1}, {r_2}^{kn_2}, \ldots {r_i}^{k(n_i-1)}, \ldots {r_s}^{kn_s}\}$

Proof: Let G=$\{g_1, g_2, g_3, \ldots g_m\}$ and H=$\{h_1, h_2, h_3, \ldots h_n\}$ be two connected graphs with degree sequences $\{{d_1}^{t_1}, {d_2}^{t_2}, {d_3}^{t_3}, \ldots {d_p}^{t_p}\}$ and $\{{r_1}^{s_1}, {r_2}^{s_2}, {r_3}^{s_3}, \ldots {r_q}^{s_q}\}$ respectively. Where $t_1 + t_2 + \cdots + t_p = m$ & $s_1 + s_2 + \cdots + s_q = n$.

By the definition of k-rooted product graph, with the root node of H being h_1, we have, $V(G \circ^k H) = V_R \cup V_{NR}$, with root vertices set V_R = $\{(g_i, h_1^1,\ h_1^2,\ h_1^3 \ldots h_1^k); i = 1,2, \ldots m\}$ and non-root vertices set V_{NR}=$\{(g_i, h_j^r)\ / g_i \in G\ ,\ h_j^r \in H^r$;i=1,2,…..m , r=1,2,…k & j= 1,2,……n} where (g_i, h_j^r) is the jth vertex of rth copy of H attached at $g_i \in G$. The degree of vertices in V_R becomes, d$(g_i, h_1^1,\ h_1^2,\ h_1^3 \ldots h_1^k)$ = $d_G(g_i) + k\ d_H(h_1)$ --------- (1)

And the degree of vertices from V_{NR} becomes d(g_i, h_j^r) = d(h_j) ------------(2)

(1) And (2) together gives the degree sequence of k-rooted product graph as ,
$d_G(G \circ^k H)$ =$\{(d_1 + kr_i)^{t_1}, (d_2 + kr_i)^{t_2}, \ldots (d_p + kr_i)^{t_p}, {r_1}^{ks_1}, {r_2}^{ks_2}, \ldots {r_i}^{k(s_i-1)}, \ldots {r_q}^{ks_q}\}$

Proposition 2.6: The girth of k-rooted product graph $G \circ^k H$ is
girth $(G \circ^k H)$ = girth $(G \circ H) = \min\{girth(G),\ girth(H)\}$

Proof: By definition girth of a graph is the length of a shortest cycle contained in the graph. If the graph does not contain any cycles, its girth is defined to be infinity. By definition of the rooted product graph and k-rooted product graph, it is very obvious that girth $(G \circ^k H)$ = girth $(G \circ H)$. Also, while constructing the rooted product graph, there are no edges added either to part of G or each of the k|V(G)| copies of H, length of shortest cycle depends on the girth of G and H, so we get following cases.

Case(i) : If G and H both do not contain a cycle (or G,H are cycle free graphs)

The rooted product of $G \circ^k H$ does not have a cycle.

Hence Girth $(G \circ^k H)$ = Girth (G) = Girth (H) = ∞

Case (ii): If girth(G) $<\infty$ and Girth (H) = ∞

The rooted product graph $G\circ^k H$ contains the cycle of smallest size same as in G, giving

$$\text{girth}\,(G\circ^k H)=\text{girth (G)}=\min\{girth(G),\ girth(H)\}$$

Case(iii) : **If girth(G)= ∞ and girth (H) $<\infty$**

The rooted product graph $G\circ^k H$ has $k|V(G)|$ number of cycles from each copy of H , but with same size as in H, therefore girth $(G\circ^k H)=$ girth (H) $=\min\{girth(G),\ girth(H)\}$

Case(iv) : If girth(G) $<\infty$ and girth (H) $<\infty$

Subcase(i) : girth(G) $\leq$ girth (H), then the smallest size cycle in $G\circ^k H$ is same as the smallest size cycle in G, so girth($G\circ^k H$) = girth(G) = $\min\{girth(G),\ girth(H)\}$

Subcase(ii) : girth(G) $>$ $g$irth (H) then as no edges are added to any copy of H, size of any cycle does not change so the smallest size cycle in $G\circ^k H$ is same as the size of k|V(G)| smallest size cycles in each copy of H giving,

$$\text{girth}(G\circ^k H)=\text{girth(H)}=\min\{girth(G),\ girth(H)\}$$

Hence combining all the above cases we get,

$$\text{girth}\,(G\circ^k H)=\text{girth}\,(G\circ H)=\min\{girth(G),\ girth(H)\}.$$

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