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Original Article

M-Polynomial of Windmill Graphs

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Abstract - *In this paper, we compute M-polynomial of certain windmill graphs such as French windmill graph $F_n^{(m)}$, Dutch windmill graph $D_n^{(m)}$, Kulli cycle windmill graph $C_{n+1}^{(m)}$, Kulli path windmill graph $P_{n+1}^{(m)}$. Furthermore, we derive some degree-based topological indices from the obtained M-polynomials.*

Mathematics Subject Classification 2020 - *05C07, 05C09, 05C31.*

Keywords - *M-polynomial, Windmill graphs, Edge partions, Degree-based topological indices.*

1. Introduction

Let $G = (V, E)$ be a simple, undirected graph, $V(G)$ be the vertex set and $E(G)$ be the edge set of the graph G. The degree $d_G(v)$ of a vertex $v \in V(G)$ is the number of edges incident to it in G. The graphs G_1 and G_2 have disjoint vertex sets V_1 and V_2 and edge sets X_1 and X_2 respectively. Their *union* [8] $G = G_1 \cup G_2$ has $V = V_1 \cup V_2$ and $X = X_1 \cup X_2$. Their *join* [8] denoted by $G_1 + G_2$ and it consists of $G_1 \cup G_2$ and all edges joining V_1 with V_2.

Definition 1. *[6] Let G be a graph. Then M-polynomial of G is defined as*

$$M(G; x, y) = \sum_{i \le j} m_{ij}(G) x^i y^j,$$

where $m_{ij}, i, j \ge 1$, is the number of edges uv of G such that $\{d_G(u)), d_G(v)\} = \{i, j\}$.

Recently, the study of M-polynomial is reported in [4, 15-17]. The topological indices play an important role in determining physico-chemical properties of chemical graphs and are used to predict the bioactivity of chemical compounds, among them the degree-based topological indices can be easily driven from an algebraic expression corresponding to the chemical graphs called M-polynomial. The study of topological indices are reported in [9-14]. The Table 1 shows the some degree based topological indices from the M-polynomial.

Table 1. Operators to derive of some degree-based topological indices from M-polynomial.

Notation	Topological index	f(x,y)	Derivation from $M(G; x, y)$
$M_1(G)$	First Zagreb	$x + y$	$(D_x + D_y)(M(G; x, y))\|_{x=y=1}$[6]
$M_2(G)$	Second Zagreb	xy	$(D_x D_y)(M(G; x, y))\|_{x=y=1}$[6]
$M_2^m(G)$	Second modified Zagreb	$\frac{1}{xy}$	$(S_x S_y)(M(G; x, y))\|_{x=y=1}$ [6]
$S_D(G)$	Symmetric division index	$\frac{x^2 + y^2}{xy}$	$(D_x S_y + D_y S_x)(M(G; x, y))\|_{x=y=1}$[6]
$H(G)$	Harmonic	$\frac{2}{x + y}$	$2S_x J(M(G; x, y))\|_{x=1}$ [6]

$I_n(G)$	Invesre sum index	$\frac{xy}{x+y}$	$S_x J D_x D_y(M(G;x,y))\|_{x=1}$ [6]
$R_\alpha(G)$	General Randic index	$(xy)^\alpha$	$D_x^\alpha D_y^\alpha(M(G;x,y))\|_{x=y=1}$[6]
$\chi_\alpha(G)$	General sum connectivity	$(x+y)^\alpha$	$D_x^\alpha(J(M(G;x,y)))\|_{x=1}$ [2]
$M_1^\alpha G$	First general Zagreb	$x^{\alpha-1}+y^{\alpha-1}$	$(D_x^{\alpha-1}+D_y^{\alpha-1})(M(G;x,y))\|_{x=y=1}$ [2]
$M_{(a,b)}(G)$	General Zagreb index	$x^a y^b + x^b y^a$	$(D_x^a D_y^b + D_x^b D_y^a)(M(G;x,y))\|_{x=y=1}$ [1]
$GA(G)$	Geometric-Arithmetic index	$\frac{2\sqrt{xy}}{x+y}$	$2S_x J D_x^{\frac{1}{2}} D_y^{\frac{1}{2}}(M(G;x,y))\|_{x=1}$ [3]

Where $D_x = x\frac{\partial f(x,y)}{\partial x}, D_y = y\frac{\partial f(x,y)}{\partial y}, S_x = \int_0^x \frac{f(t,y)}{t}dt S_y = \int_0^y \frac{f(x,t)}{t}dt$, $D_x^\alpha = D_x(D_x^{\alpha-1})\big(f(x,y)\big)$, $J(f(x,y)) = f(x,x)$ are the operators.

2. M-polynomials of certain class of windmill graphs

In this section, we compute M-polynomials of certain windmill graphs such as, the French windmill graph $F_n^{(m)}$, the Dutch windmill graph $D_n^{(m)}$, the Kulli cycle windmill graph $C_{n+1}^{(m)}$ and the Kulli path windmill graph $P_{n+1}^{(m)}$. Furthermore, we derive some degree-based topological indices of these graphs from their respective M-polynomial.

Definition 2. [5] The French windmill graph $F_n^{(m)}$ is the graph obtained by taking $m \geq 2$ copies of the complete graph $K_n; n \geq 2$ with a vertex in common. This graph is shown in Figure 1 . The French windmill graph $F_2^{(m)}$ is called a star graph. The French windmill graph $F_3^{(m)}$ is called a friendship graph and the French windmill graph $F_3^{(2)}$ is called a butterfly graph.

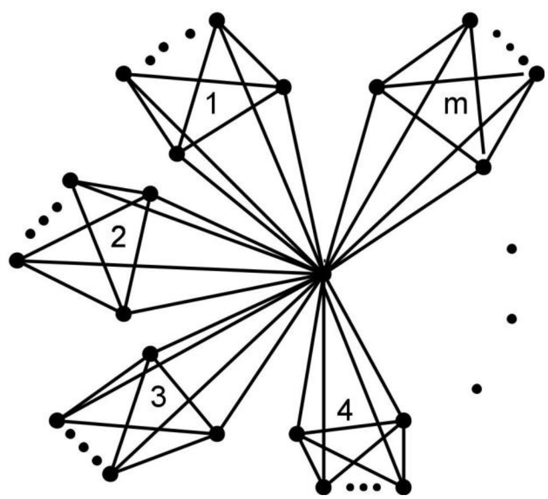

Fig. 1 French windmill graph $F_n^{(m)}$.

Theorem 2.1. *Let $F_n^{(m)}$ be a French windmill graph of order $(mn+1)$ and size $\frac{mn(n+1)}{2}$, then*

$$M(F_n^{(m)};x,y) = \left(\frac{mn(n-1)}{2}\right)x^n y^n + mnx^n y^{mn}.$$

Proof. The graph $F_n^{(m)}$ has $mn+1$ vertices and $\frac{mn(n+1)}{2}$ edges. The edge set of $F_n^{(m)}$ can be partitioned as

$$|E_{\{n,n\}}| = |uv \in E(F_n^{(m)}): d_u = n \ \ and \ \ d_v = n| = \frac{mn^2-mn}{2},$$

$$|E_{\{n,mn\}}| = |uv \in E(F_n^{(m)}): d_u = n \ \ and \ \ d_v = mn| = mn.$$

Using the above edge partition and definition of M-polynomial, we get the required result.

Corollary 2.2. *If $F_n^{(m)}$ is a French windmill graph, then*

1. $M_1(F_n^{(m)}) = mn(n^2 + mn)$,
2. $M_2(F_n^{(m)}) = \frac{mn(n^3-n^2+2mn^2)}{2}$,
3. $M_2^m(F_n^{(m)}) = \frac{(mn-m+2)}{2n}$,
4. $S_D(F_n^{(m)}) = mn^2 - mn + m^2n + n$,
5. $H(F_n^{(m)}) = \frac{m^2n^3-mn^3-m^2n^2+3mn^2}{2(mn^2+n^2)}$,
6. $I_n(F_n^{(m)}) = \frac{m^2n^5+mn^5-m^2n^4-mn^4+4mn^2}{4(mn^2+n^2)}$,
7. $R_\alpha(F_n^{(m)}) = \frac{mn(n-1)}{2}n^{2\alpha} + (mn)^{\alpha+1}n^\alpha$,
8. $\chi_\alpha(F_n^{(m)}) = mn(n+mn)^\alpha + (\frac{mn^2-mn}{2})(2n)^\alpha$,
9. $M_1^\alpha(F_n^{(m)}) = (mn^2-mn)n^{\alpha-1} + mn^\alpha + (mn)^\alpha$,
10. $M_{(a,b)}(F_n^{(m)}) = (mn^2-mn)n^{a+b} + mn^{a+b+1}(m^b + m^a)$,
11. $GA(F_n^{(m)}) = \frac{mn(n-1)}{2} + \frac{2\sqrt[3]{mn}\sqrt{n}}{n(1+m)}$.

Proof. The M-polynomial for French windmill graph $F_n^{(m)}$ is given by

$$M(F_n^{(m)}; x, y) = \sum_{i\le j} m_{ij}(F_n^{(m)})x^i y^j = \left(\frac{mn^2-mn}{2}\right)x^n y^n + mnx^n y^{mn}.$$

Then we have

$$D_x(f(x,y)) = \left(\frac{mn^3-mn^2}{2}\right)x^n y^n + mn^2 x^n y^{mn},$$

$$D_y(f(x,y)) = \left(\frac{mn^3-mn^2}{2}\right)x^n y^n + m^2n^2 x^n y^{mn},$$

$$S_x(f(x,y)) = \left(\frac{mn^2-mn}{2n}\right)x^n y^n + mx^n y^{mn},$$

$$S_y(f(x,y)) = \left(\frac{mn^2-mn}{2n}\right)x^n y^n + mx^n y^{mn},$$

$$D_xD_y(f(x,y)) = n\left(\frac{mn^3-mn^2}{2}\right)x^n y^n + m^2n^3 x^n y^{mn},$$

$$D_x^\alpha(f(x,y)) = \left(\frac{mn^2-mn}{2}\right)n^\alpha x^n y^n + mn^{\alpha+1}x^n y^{mn},$$

$$D_y^\alpha(f(x,y)) = \left(\frac{mn^2-mn}{2}\right)n^\alpha x^n y^n + (mn)^{\alpha+1}x^n y^{mn},$$

$$D_x^\alpha D_y^\alpha(f(x,y)) = \left(\frac{mn^2-mn}{2}\right)n^{2\alpha} x^n y^n + (mn)^{\alpha+1}n^\alpha x^n y^{mn},$$

$$D_xS_y(f(x,y)) = \left(\frac{mn^2-mn}{2}\right)x^n y^n + nmx^n y^{mn},$$

$$D_yS_x(f(x,y)) = \left(\frac{mn^2-mn}{2}\right)x^n y^n + m^2nx^n y^{mn},$$

$$S_xS_y(f(x,y)) = \left(\frac{mn-m}{2n}\right)x^n y^n + \frac{m}{n}x^n y^{mn}$$

$$D_x^a D_y^b(f(x,y)) = \left(\frac{mn^2-mn}{2}\right) n^{a+b}x^n y^n + (mn)^{b+1}n^a x^n y^{mn},$$

$$D_x^b D_y^a(f(x,y)) = \left(\frac{mn^2-mn}{2}\right) n^{a+b}x^n y^n + (mn)^{a+1}n^b x^n y^{mn},$$

$$D_y^{\frac{1}{2}}(f(x,y)) = n^{\frac{1}{2}}\left(\frac{mn^2-mn}{2}\right) x^n y^n + (mn)^{\frac{3}{2}}x^n y^{mn},$$

$$D_x^{\frac{1}{2}}D_y^{\frac{1}{2}}(f(x,y)) = n\left(\frac{mn^2-mn}{2}\right) x^n y^n + (mn)^{\frac{3}{2}}n^{\frac{1}{2}}x^n y^{mn},$$

$$2S_xJD_x^{\frac{1}{2}}D_y^{\frac{1}{2}}(f(x,y)) = \left(\frac{mn^2-mn}{2}\right) x^{2n} + \frac{2\sqrt[3]{mn}\sqrt{n}}{n(1+m)}.$$

Using the Theorem 2.1, and column 4 of Table 1, we get the desired results.

Definition 3. [5] The Dutch windmill graph $D_n^{(m)}$, $m \geq 2$, $n \geq 5$, is the graph obtained by taking m copies of the cycle C_n with a vertex in common. This graph is shown in Figure 2 . The Dutch windmill graph $D_3^{(m)} = F_3^{(m)}$ is called a friendship graph.

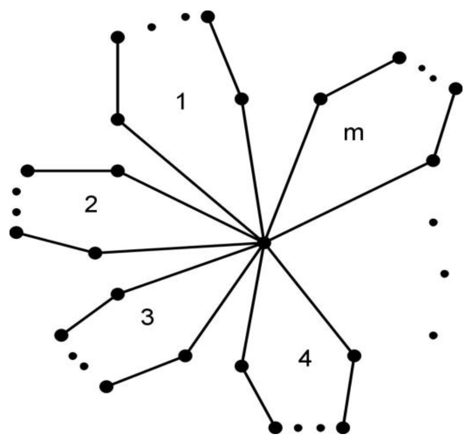

Fig. 2 The Dutch windmill graph $D_n^{(m)}$.

Theorem 2.3. *Let $D_n^{(m)}$ be a Dutch windmill graph of order $mn+1$ and size $m(n+1)$, then*

$$M(D_n^{(m)};x,y) = m(n-3)x^2y^2 + 2mx^2y^{2m}.$$

Proof. The graph $D_n^{(m)}$ has $(m(n-1)+1)$ vertices and $(m(n+1))$ edges. The edge partition of $D_n^{(m)}$ is as follows

$$|E_{\{2,2\}}| = |uv \in E(D_n^{(m)}): d_u = 2 \ \ and \ \ d_v = 2| = m(n-3),$$

$$|E_{\{2,2m\}}| = |uv \in E(D_n^{(m)}): d_u = 2 \ \ and \ \ d_v = 2m| = 2m.$$

Using the above edge partition and definition of M-polynomial, we get the required result.

Corollary 2.4. *If $D_n^{(m)}$ is a Dutch windmill graph, then.*

1. $M_1(D_n^{(m)}) = m(4n+4m-8)$,
2. $M_2(D_n^{(m)}) = 8m(m+n-3)$,
3. $M_2^m(D_n^{(m)}) = \frac{mn-3m+2}{4}$,
4. $S_D(D_n^{(m)}) = 2(mn-3m+1+m^2)$,

5. $H(D_n^{(m)}) = \frac{m(n+mn+1-3m)}{2(1+m)}$,
6. $I_n(D_n^{(m)}) = \frac{m(n+mn-3+m)}{1+m}$,
7. $R_\alpha(D_n^{(m)}) = m(n-3)2^{2\alpha} + (2m)^{\alpha+1}2^\alpha$,
8. $\chi_\alpha(D_n^{(m)}) = m(n-3)4^\alpha + 2m(2m+2)^\alpha$,
9. $M_1^\alpha(D_n^{(m)}) = m(n-3)2^\alpha + m2^\alpha + (2m)^\alpha$,
10. $M_{(a,b)}(D_n^{(m)}) = 2m(n-3)2^{a+b} + 2^{a+b+1}(m^{b+1} + m^{a+1})$.
11. $GA(D_n^{(m)}) = m(n-3) + \frac{\sqrt[3]{2m}\sqrt{2}}{1+m}$.

Proof. The M-polynomial for Dutch windmill graph $D_n^{(m)}$ is given by

$$M(D_n^{(m)}; x, y) = \sum_{i\le j} m_{ij}(D_n^{(m)})x^i y^j = m(n-3)x^2y^2 + 2mx^2y^{2m}.$$

Then we have

$$D_x(f(x,y)) = 2m(n-3)x^2y^2 + 4mx^2y^{2m},$$
$$D_y(f(x,y)) = 2m(n-3)x^2y^2 + 4m^2x^2y^{2m},$$
$$D_xD_y(f(x,y)) = 4m(n-3)x^2y^2 + 8m^2x^2y^{2m},$$
$$S_x(f(x,y)) = \left(\frac{m(n-3)}{2}\right)x^2y^2 + mx^2y^{2m},$$
$$S_y(f(x,y)) = \left(\frac{m(n-3)}{2}\right)x^2y^2 + x^2y^{2m},$$
$$S_xS_y(f(x,y)) = \left(\frac{m(n-3)}{4}\right)x^2y^2 + \frac{1}{2}x^2y^{2m},$$
$$D_xS_y(f(x,y)) = m(n-3)x^2y^2 + 2x^2y^{2m},$$
$$D_yS_x(f(x,y)) = m(n-3)x^2y^2 + 2m^2x^2y^{2m},$$
$$D_x^\alpha(f(x,y)) = m(n-3)2^\alpha x^2y^2 + 2^{\alpha+1}mx^2y^{2m},$$
$$D_y^\alpha(f(x,y)) = m(n-3)2^\alpha x^2y^2 + (2m)^{\alpha+1}x^2y^{2m},$$
$$D_x^\alpha D_y^\alpha(f(x,y)) = m(n-3)2^{2\alpha}x^2y^2 + 2^\alpha(2m)^{\alpha+1}x^2y^{2m},$$
$$D_x^a D_y^b(f(x,y)) = m(n-3)2^{a+b}x^2y^2 + 2^a(2m)^{b+1}x^2y^{2m},$$
$$D_x^b D_y^a(f(x,y)) = m(n-3)2^{a+b}x^2y^2 + 2^b(2m)^{a+1}x^2y^{2m},$$
$$D_y^{\frac{1}{2}}(f(x,y)) = 2^{\frac{1}{2}}m(n-3)x^2y^2 + 2m^{\frac{3}{2}}x^2y^{2m},$$
$$D_x^{\frac{1}{2}}D_y^{\frac{1}{2}}(f(x,y)) = 2m(n-3)x^2y^2 + (2m)2^{\frac{1}{2}}x^2y^{2m},$$
$$2S_xJD_x^{\frac{1}{2}}D_y^{\frac{1}{2}}(f(x,y)) = m(n-3)x^4 + \frac{\sqrt[3]{2m}\sqrt{2}}{1+m}x^{2(1+m)}.$$

Using the Theorem 2.3, and column 4 of Table 1, we get the desired results.

Definition 4 . [5] *The Kulli cycle windmill graph* $C_{n+1}^{(m)}$ *is the graph obtained by taking* m *copies of the graph* $K_1 + C_n$ *for* $n \ge 3$ *with a vertex* K_1 *in common. This graph shown in Figure 3. The Kulli cycle windmill graph* $C_4^{(m)}$ *is a French windmill graph and it is denoted by* $F_3^{(m)}$.

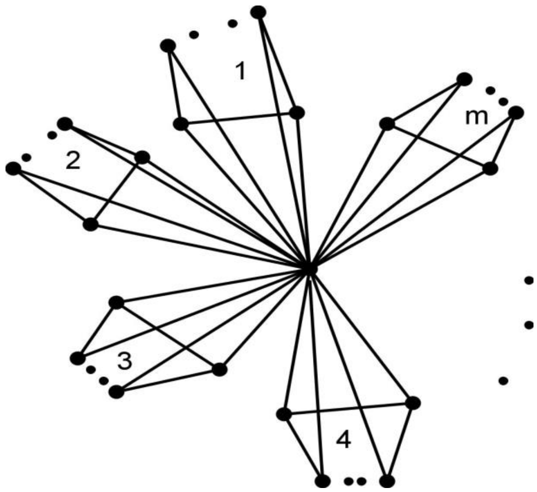

Fig. 3 Kulli cycle windmill graph $C_{n+1}^{(m)}$.

Theorem 2.5. *Let $C_{n+1}^{(m)}$ be a Kulli cycle windmill graphs of order($mn+1$) and size $2mn$, then*

$$M(C_{n+1}^{(m)};x,y) = mnx^3y^3 + mnx^3y^{mn}.$$

Proof. Let $C_{n+1}^{(m)}$ be a Kulli cycle windmill graph having $mn+1$ vertices and $2mn$ edges. The edge partition of $C_{n+1}^{(m)}$ is given by

$$|E_{\{3,3\}}| = |uv \in E(C_{n+1}^{(m)}): d_u = 3 \ \ and \ \ d_v = 3| = mn,$$

$$|E_{\{3,mn\}}| = |uv \in E(C_{n+1}^{(m)}): d_u = 3 \ \ and \ \ d_v = mn| = mn.$$

Using the above edge partition and definition of M-polynomial, we get the required result.

Corollary 2.6. *If $C_{n+1}^{(m)}$ is Kulli cycle windmill graph, then.*

1. $M_1(C_{n+1}^{(m)}) = mn(9+mn)$,
2. $M_2(C_{n+1}^{(m)}) = 3mn(3+mn)$,
3. $M_2^m(C_{n+1}^{(m)}) = \frac{mn+3}{9}$,
4. $S_D(C_{n+1}^{(m)}) = \frac{m^2n^2+6mn+9}{3}$,
5. $H(C_{n+1}^{(m)}) = \frac{m^2n^2+9mn}{3mn+9}$,
6. $I_n(C_{n+1}^{(m)}) = \frac{9(mn+m^2n^2)}{2mn+6}$,
7. $R_\alpha(C_{n+1}^{(m)}) = (mn)3^{2\alpha} + (mn)^{\alpha+1}3^\alpha$,
8. $\chi_\alpha(C_{n+1}^{(m)}) = mn(6^\alpha + (mn+3)^\alpha)$,
9. $M_1^\alpha(C_{n+1}^{(m)}) = (mn)3^\alpha + (mn)^\alpha$,
10. $M_{(a,b)}(C_{n+1}^{(m)}) = 2mn3^{a+b} + m^{b+1}n^{b+1}3^a + m^{a+1}n^{a+1}3^b$.
11. $GA(C_{n+1}^{(m)}) = mn + \frac{2\sqrt{3}\sqrt[3]{mn}}{3+mn}$.

Proof. The M-polynomial for Kulli cycle windmill graph $C_{n+1}^{(m)}$ is given by

$$M(C_{n+1}^{(m)};x,y)=\sum_{i\le j} m_{ij}(C_{n+1}^{(m)})x^iy^j=mnx^3y^3+mnx^3y^{mn}.$$

Then we have

$$D_x(f(x,y))=3mnx^3y^3+3mnx^3y^{mn},$$
$$D_y(f(x,y))=3mnx^3y^3+m^2n^2x^3y^{mn},$$
$$D_xD_y(f(x,y))=9mnx^3y^3+3m^2n^2x^3y^{mn},$$
$$S_x(f(x,y))=\left(\frac{mn}{3}\right)x^3y^3+\left(\frac{mn}{3}\right)x^3y^{mn},$$
$$S_y(f(x,y))=\left(\frac{mn}{3}\right)x^3y^3+x^3y^{mn},$$
$$S_xS_y(f(x,y))=\left(\frac{mn}{9}\right)x^3y^3+\frac{1}{3}x^3y^{mn},$$
$$D_xS_y(f(x,y))=mnx^3y^3+3x^3y^{mn},$$
$$D_yS_x(f(x,y))=mnx^3y^3+\left(\frac{m^2n^2}{3}\right)x^3y^{mn},$$
$$D_x^{\alpha}(f(x,y))=3^{\alpha}mnx^3y^3+3^{\alpha}mnx^3y^{mn},$$
$$D_y^{\alpha}(f(x,y))=3^{\alpha}mnx^3y^3+(mn)^{\alpha+1}x^3y^{mn},$$
$$D_x^{\alpha}D_y^{\alpha}(f(x,y))=3^{2\alpha}mnx^3y^3+3^{\alpha}(mn)^{\alpha+1}x^3y^{mn},$$
$$D_x^{a}D_y^{b}(f(x,y))=3^{a+b}mnx^3y^3+3^{a}(mn)^{b+1}x^3y^{mn},$$
$$D_x^{b}D_y^{a}(f(x,y))=3^{a+b}mnx^3y^3+3^{b}(mn)^{a+1}x^3y^{mn},$$
$$D_y^{\frac{1}{2}}(f(x,y))=mn3^{\frac{1}{2}}x^3y^3+mn^{\frac{3}{2}}x^3y^{mn},$$
$$D_x^{\frac{1}{2}}D_y^{\frac{1}{2}}(f(x,y))=3mnx^3y^3+3^{\frac{1}{2}}mn^{\frac{3}{2}}x^3y^{mn},$$
$$2S_xJD_x^{\frac{1}{2}}D_y^{\frac{1}{2}}(f(x,y))=mnx^6+\frac{2\sqrt{3}\sqrt[3]{mn}}{3+mn}x^{3+mn}.$$

Using the Theorem 2.5, and column 4 of Table 1, we get the desired results.

Definition 5. [5] The Kulli path windmill graph $P_{n+1}^{(m)}$, $m\ge 2, n\ge 5$, is the graph obtained by taking m copies of the graph K_1+P_n for $n\ge 2$ with a vertex K_1 in common. This graph is shown in Figure 4. The Kulli path windmill graph $P_3^{(m)}$ is friendship graph and it is denoted by $F_3^{(m)}$.

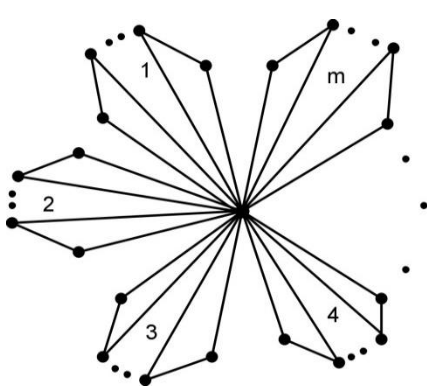

Fig. 4 Kulli path windmill graph $P_{n+1}^{(m)}$.

Theorem 2.7. *Let $P_{n+1}^{(m)}$ be a Kulli Path windmill graph of order $mn+1$ and size $m(2n-1)$, then*

$$M(P_{n+1}^{(m)};x,y) = 2mx^2y^3 + m(n-3)x^3y^3 + 2mx^2y^{mn} + m(n-2)x^3y^{mn}.$$

Proof. The $P_{n+1}^{(m)}$ is a graph having $mn+1$ vertices and $m(2n-1)$ edges. The edge partition of $P_{n+1}^{(m)}$ is given by

$$|E_{\{2,3\}}| = |uv \in E(P_{n+1}^{(m)}): d_u = 2 \ \ and \ \ d_v = 3| = 2m,$$

$$|E_{\{3,3\}}| = |uv \in E(P_{n+1}^{(m)}): d_u = 3 \ \ and \ \ d_v = 3| = m(n-3),$$

$$|E_{\{2,mn\}}| = |uv \in E(P_{n+1}^{(m)}): d_u = 2 \ \ and \ \ d_v = mn| = 2m,$$

$$|E_{\{3,mn\}}| = |uv \in E(P_{n+1}^{(m)}): d_u = 3 \ \ and \ \ d_v = mn| = m(n-2).$$

Using the above edge partition and definition of M-polynomial, we get the required result.

Corollary 2.8. *If $P_{n+1}^{(m)}$ is a Kulli path windmill graph, then*

1. $M_1(P_{n+1}^{(m)}) = 9mn + m^2n^2 - 10m$,
2. $M_2(P_{n+1}^{(m)}) = 9mn + 15m - 2m^2n + 3m^2n^2$,
3. $M_2^m(P_{n+1}^{(m)}) = \frac{mn^2+3n+3}{9n}$,
4. $S_D(P_{n+1}^{(m)}) = \frac{m^2n^2+6mn^2+m^2n^3-5mn-9n-6}{3n}$,
5. $H(P_{n+1}^{(m)}) = \frac{28m+4m^2n}{5(mn+2)} + \frac{m^2n^2-3m^2n+9mn-21m}{3(mn+3)}$,
6. $I_n(P_{n+1}^{(m)}) = \frac{9m^2n^2+9mn-21m^2n-27m}{2(mn+3)} + \frac{32m^2n+24m}{5(mn+2)}$,
7. $R_\alpha(P_{n+1}^{(m)}) = m(mn)^\alpha 2^{\alpha+1} + m3^\alpha 2^{\alpha+1} + m(n-3)3^{2\alpha} + m(n-2)(mn)^\alpha 3^\alpha$,
8. $\chi_\alpha(P_{n+1}^{(m)}) = 2m(mn+2)^\alpha + 2m5^\alpha + m(n-3)6^\alpha + m(n-2)(mn+3)^\alpha$,
9. $M_1^\alpha(P_{n+1}^{(m)}) = m2^{\alpha+1} + 2m(n-3)3^{\alpha-1} + m(n-2)3^{\alpha-1} + m(n-2)(mn)^{\alpha-1} + 2m(mn)^{\alpha-1} + 2m3^{\alpha-1}$,
10. $M_{(a,b)}(P_{n+1}^{(m)}) = 2m[2^a(mn)^b + 2^b(mn)^a + 2^a3^b + 2^b3^a] + 2m(n-3)3^{(a+b)} +$
    $m(n-2)[3^a(mn)^b + 3^b(mn)^a]$,
11. $GA(P_{n+1}^{(m)}) = \frac{4m\sqrt{6}}{5} + m(n-3) + \frac{m\sqrt{mn}\,\sqrt[5]{2}}{2+mn} + \frac{2m(n-2)\sqrt{3}\sqrt{mn}}{3+mn}$.

Proof. The M-polynomial for Kulli path windmill graph $P_{n+1}^{(m)}$ is given by

$$M\left(P_{n+1}^{(m)};x,y\right) = \sum_{i\le j} m_{ij}\left(P_{n+1}^{(m)}\right)x^iy^j = 2mx^2y^3 + m(n-3)x^3y^3 + 2mx^2y^{mn} + m(n-2)x^3y^{mn}.$$

Then we have

$$D_x(f(x,y)) = 4mx^2y^{mn} + 4mx^2y^3 + 3m(n-3)x^3y^3 + 3m(n-2)x^3y^{mn},$$

$$D_y(f(x,y)) = 2m^2nx^2y^{mn} + 6mx^2y^3 + 3m(n-3)x^3y^3 + (m^2n^2 - 2m^2n)x^3y^{mn},$$

$$S_x(f(x,y)) = mx^2y^{mn} + mx^3y^3 + \left(\frac{m(n-3)}{3}\right)x^3y^3 + \left(\frac{m(n-2)}{3}\right)x^3y^{mn},$$

$$S_y(f(x,y)) = \left(\frac{2}{n}\right)x^2y^{mn} + \left(\frac{2m}{3}\right)x^2y^3 + \left(\frac{m(n-3)}{3}\right)x^3y^3 + \left(\frac{n-2}{n}\right)x^3y^{mn},$$

$$D_xD_y\left(f(x,y)\right) = 4m^2nx^2y^{mn} + 12mx^2y^3 + 9m(n-3)x^3y^3 + 3(m^2n^2 - 2m^2n)x^3y^{mn},$$

$$S_xS_y(f(x,y)) = \left(\frac{2}{2n}\right)x^2y^{mn} + \left(\frac{m}{3}\right)x^2y^3 + \left(\frac{m(n-3)}{9}\right)x^3y^3 + \left(\frac{n-2}{3n}\right)x^3y^{mn},$$

$$D_xS_y(f(x,y)) = \left(\frac{4}{n}\right)x^2y^{mn} + \left(\frac{4m}{3}\right)x^2y^3 + m(n-3)x^3y^3 + \left(\frac{3(n-2)}{n}\right)x^3y^{mn},$$

$$D_yS_x(f(x,y)) = m^2nx^2y^{mn} + 3mx^3y^3 + m(n-3)x^3y^3 + \left(\frac{m^2n(n-2)}{3}\right)x^3y^{mn},$$

$$D_x^\alpha(f(x,y)) = 2^{\alpha+1}mx^2y^3 + 3^\alpha m(n-3)x^3y^3 + 2^{\alpha+1}mx^2y^{mn} + 3^\alpha m(n-2)x^3y^{mn},$$

$$D_y^\alpha(f(x,y)) = 3^\alpha 2mx^2y^3 + 3^\alpha m(n-3)x^3y^3 + (mn)^\alpha 2mx^2y^{mn} + (mn)^\alpha m(n-2)x^3y^{mn},$$

$$D_x^\alpha D_y^\alpha(f(x,y)) = 2^\alpha 3^\alpha 2mx^2y^3 + 3^{2\alpha}m(n-3)x^3y^3 + 2^\alpha(mn)^\alpha 2mx^2y^{mn} + 3^\alpha(mn)^\alpha m(n-2)x^3y^{mn},$$

$$D_x^aD_y^b(f(x,y)) = 2^a3^b2mx^2y^3 + 3^{a+b}m(n-3)x^3y^3 + 2^a(mn)^b2mx^2y^{mn} + 3^a(mn)^bm(n-2)x^3y^{mn},$$

$$D_x^bD_y^a(f(x,y)) = 2^b3^a2mx^2y^3 + 3^{a+b}m(n-3)x^3y^3 + 2^b(mn)^a2mx^2y^{mn} + 3^b(mn)^am(n-2)x^3y^{mn},$$

$$D_y^{\frac{1}{2}}(f(x,y)) = 2^{\frac{3}{2}}mx^2y^3 + 3^{\frac{1}{2}}m(n-3)x^3y^3 + (mn)^{\frac{1}{2}}2mx^2y^{mn} + (mn)^{\frac{1}{2}}m(n-2)x^3y^{mn},$$

$$D_x^{\frac{1}{2}}D_y^{\frac{1}{2}}(f(x,y)) = 4mx^2y^3 + 3m(n-3)x^3y^3 + mn^{\frac{1}{2}}2^{\frac{3}{2}}mx^2y^{mn} + 3^{\frac{1}{2}}(mn)^{\frac{1}{2}}m(n-2)x^3y^{mn},$$

$$2S_xJD_x^{\frac{1}{2}}D_y^{\frac{1}{2}}\big(f(x,y)\big) = \frac{4m\sqrt{6}}{5}x^5 + m(n-3)x^6 + \frac{m\sqrt{mn}\sqrt[5]{2}}{2+mn}x^{2+mn} + \frac{2m(n-2)\sqrt{3}\sqrt{mn}}{3+mn}x^{3+mn}.$$

Using the Theorem 2.7, and column 4 of Table 1, we get the desired result.

3. Conclusion

In the present paper, we obtained M-polynomial of certain winmdmill graphs and derived their degree-based topological indices using the obtained polynomials.

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