On a Mixed Arithmetic-Mean, Geometric-Mean, Harmonic-Mean Inequality

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Abstract - In 1992, F. Holland conjectured a mixed arithmetic-mean, geometric-mean inequality, and it was proved by K. Kedlaya in 1994. In this short communication, we provide more extended inequality: a mixed arithmetic-mean, geometric-mean, and harmonic-mean inequality.

Keywords - Arithmetic mean, Geometric mean, Harmonic mean, Mixed mean, Mixed mean inequality.

1. Introduction

In 1992, F. Holland [1] conjectured the following mixed arithmetic-mean, geometric-mean inequality:

\[
\left( \frac{x_1 \cdot \frac{x_1+x_2}{2} \cdot \frac{x_1+x_2+x_3}{3} \cdot \ldots \cdot \frac{x_1+x_2+\ldots+x_n}{n} }{n} \right)^{1/n} \geq \frac{1}{n} \left( x_1 + \sqrt{x_1x_2} + \sqrt[3]{x_1x_2x_3} + \ldots + \sqrt[n]{x_1x_2\ldots x_n} \right)
\]

(1)

Where \( x_1, x_2, \ldots, x_n \) are positive real numbers with equality if and only if \( x_1 = x_2 = \ldots = x_n \). And K. Kedlaya [2] proved it in 1994. We can extend this inequality, and the purpose of this paper is to address it.

2. Main Results

Before introducing the proof in [2], first, we'll state the following lemma from [2] without proof.

Lemma 1. The vectors \( a(i,j) = (a_1(i,j), a_2(i,j), \ldots, a_n(i,j)) \) given by

\[
a_k(i,j) = \frac{c(n-k)(n-k-1)}{c(n-k+1)} = \frac{(n-k)(n-k+1)!}{(n-k+1)(n-k+2)(n-k+3)!} \quad (2)
\]

\((i, j = 1, 2, \ldots, n)\) satisfy

i. \( a_i(i,j) \geq 0 \) for all \( i, j, k \),

ii. \( a_i(i,j) = 0 \) for \( k \geq \min(i, j) \),

iii. \( a_i(i,j) = a_k(i, j) \) for all \( i, j, k \),

iv. \( a_1(i,j) + a_2(i,j) + \ldots + a_n(i,j) = 1 \) for all \( i, j \),

v. \( a_1(i,j) + a_2(i,j) + \ldots + a_n(i,j) = n/j \) for \( k \leq j \), \( a_1(i,j) + a_2(i,j) + \ldots + a_n(i,j) = 0 \) for \( k > j \).

And the proof of (1) in [2] followed as:

Proposition 2. Let \( x_1, x_2, \ldots, x_n \) be positive real numbers. Then,

\[
\left( \prod_{j=1}^{n} \frac{x_1+x_2+\ldots+x_j}{j} \right)^{1/n} \geq \frac{1}{n} \sum_{i=1}^{n} \sqrt[n]{x_1x_2\ldots x_i} . \quad (3)
\]

There is equality if and only if \( x_1 = x_2 = \ldots = x_n \).

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Proof. Let us define the weighted arithmetic mean and geometric mean of tuple $x$ as $A(x, a) = a_1x_1 + a_2x_2 + \ldots + a_nx_n$ and $G(x, a) = x_1^{a_1}x_2^{a_2}\ldots x_n^{a_n}$ where $a = (a_1, a_2, \ldots, a_n)$ is an $n$-tuple of nonnegative real numbers such that $a_1 + a_2 + \ldots + a_n = 1$. By the AM-GM inequality [6, 7, 8, 9, 10, 11, 12, 13, 14, 15], $A(x, a) \geq G(x, a)$ with equality if and only if $x_i$ is constant over all $k$ for which $a_k > 0$. Let $A(i, j)$ and $G(i, j)$ be the means obtained by setting $a = a(i, j)$ in $A(x, a)$ and $G(x, a)$. Using Lemma 1,

$$
\frac{x_1 + x_2 + \cdots + x_j}{j} = \frac{1}{n} \sum_{k=1}^{n} x_k \sum_{i=1}^{n} a_k(i, j) = \frac{1}{n} \sum_{i=1}^{n} A(i, j) \geq \frac{1}{n} \sum_{i=1}^{n} G(i, j)
$$

(4)

Taking the geometric mean of both sides over $j$, we get

$$
\left(\Pi_{j=1}^{n} \frac{x_1 + x_2 + \cdots + x_j}{j}\right)^{1/n} \geq \frac{1}{n} \Pi_{j=1}^{n} (\sum_{i=1}^{n} G(i, j))^{1/n}
$$

(5)

By Hölder's inequality [3],

$$
\frac{1}{n} \Pi_{j=1}^{n} (\sum_{i=1}^{n} G(i, j))^{1/n} \geq \frac{1}{n} \sum_{i=1}^{n} \Pi_{j=1}^{n} G(i, j)^{1/n}
$$

(6)

Equality holds only if every two $g_1, g_2, \ldots, g_n$ are proportional where $g_i = (G(i, 1), G(i, 2), \ldots, G(i, n)) (i = 1, 2, \ldots, n)$. Since $G(i, 1) = x_1$ and $G(i, n) = x_n$ for all $i$, this would imply that $g_1 = g_2 = \ldots = g_n$ and that would imply $x_1 = x_2 = \ldots = x_n$. Also, by Lemma 1,

$$
\Pi_{j=1}^{n} G(i, j)^{1/n} = \Pi_{k=1}^{n} x_k^{a_k(i, j)/n} = \Pi_{k=1}^{n} x_k^{1/j} = \sqrt[j]{x_1x_2\cdots x_l}
$$

(7)

Combining (5), (6), and (7) completes the proof.

Also, in [4], the authors proved the mixed arithmetic-mean, harmonic-mean inequality for matrices. Here we provide proof for scalars.

Proposition 3. Let $x_1, x_2, \ldots, x_n$ be positive real numbers. Then,

$$
\left[\frac{1}{n} \sum_{i=1}^{n} \left(\frac{x_1 + x_2 + \cdots + x_j}{j}\right)^{-1}\right]^{-1} \geq \frac{1}{n} \sum_{i=1}^{n} \left(\frac{1}{n} \sum_{k=1}^{n} x_k^{-1}\right)^{-1}
$$

(8)

There is equality if and only if $x_1 = x_2 = \ldots = x_n$.

Proof. Let us define the weighted arithmetic mean and the geometric mean of tuple $x$ as $A(x, a) = a_1x_1 + a_2x_2 + \ldots + a_nx_n$ and $H(x, a) = (a_1/x_1 + a_2/x_2 + \ldots + a_n/x_n)^{-1}$ where $a = (a_1, a_2, \ldots, a_n)$ is an $n$-tuple of nonnegative real numbers such that $a_1 + a_2 + \ldots + a_n = 1$. By the AM-HM inequality [6, 7, 8, 9, 10, 11, 12, 13, 14, 15], $A(x, a) \geq H(x, a)$ with equality if and only if $x_i$ is constant over all $k$ for which $a_k > 0$. Let $A(i, j)$ and $H(i, j)$ be the means obtained by setting $a = a(i, j)$ in $A(x, a)$ and $G(x, a)$. Using Lemma 1,

$$
\frac{x_1 + x_2 + \cdots + x_j}{j} = \frac{1}{n} \sum_{k=1}^{n} x_k \sum_{i=1}^{n} a_k(i, j) = \frac{1}{n} \sum_{i=1}^{n} A(i, j) \geq \frac{1}{n} \sum_{i=1}^{n} H(i, j)
$$

(9)

Taking the harmonic mean of both sides over $j$, we get

$$
\left[\frac{1}{n} \sum_{j=1}^{n} \left(\frac{x_1 + x_2 + \cdots + x_j}{j}\right)^{-1}\right]^{-1} \geq \frac{1}{n} \sum_{j=1}^{n} \left(\frac{1}{n} \sum_{i=1}^{n} H(i, j)\right)^{-1}
$$

(10)

From [3, 5],

$$
\left[\frac{1}{n} \sum_{j=1}^{n} \left(\frac{1}{n} \sum_{i=1}^{n} H(i, j)\right)^{-1}\right]^{-1} \geq \frac{1}{n} \sum_{i=1}^{n} \left(\frac{1}{n} \sum_{j=1}^{n} H(i, j)^{-1}\right)^{-1}
$$

(11)
Equality holds only if \( h_1, h_2, \ldots, h_n \) are proportional where \( h_i = (H(i, 1), H(i, 2), \ldots, H(i, n)) \) \((i = 1, 2, \ldots, n)\). Since \( H(i, 1) = x_i \) and \( H(i, n) = x_i \) for all \( i \), this would imply that \( h_i = h_2 = \ldots = h_n \) and that would imply \( x_1 = x_2 = \ldots = x_n \). Also, by Lemma 1,

\[
\frac{1}{n} \sum_{j=1}^{n} H(i, j)^{-1} = \frac{1}{n} \sum_{k=1}^{n} \sum_{j=1}^{n} a_k(i, j)x_k^{-1} = \frac{1}{n} \sum_{k=1}^{n} x_k^{-1} \sum_{j=1}^{n} a_k(i, j) = \frac{1}{n} \sum_{k=1}^{n} x_k^{-1}
\]

Combining (10), (11), and (12) completes the proof.

Here, we can extend these inequalities to a mixed arithmetic-mean, geometric-mean, and harmonic-mean inequality. First, we define mixed arithmetic-mean, geometric-mean, and harmonic-mean of the first, second, and third kinds as follows:

\[
AGH(x_1, x_2, \ldots, x_n) = \left[ \frac{1}{n} \sum_{i=1}^{n} \left( \prod_{j=1}^{n} \frac{x_1 + x_2 + \cdots + x_j}{j} \right)^{1/n} \right]^{-1}
\]

Equality holds only if \( h_1 = h_2 = \ldots = h_n \) and that would imply \( x_1 = x_2 = \ldots = x_n \). Also, by Lemma 1,

\[
\frac{1}{n} \sum_{i=1}^{n} \left( \prod_{j=1}^{n} \frac{x_1 + x_2 + \cdots + x_j}{j} \right)^{1/n} \geq \frac{1}{n} \sum_{i=1}^{n} \left( \frac{1}{n} \sum_{j=1}^{n} \sqrt[1/n]{x_1 x_2 \cdots x_j} \right)^{-1}
\]

Equality holds only if \( x_1 = x_2 = \ldots = x_n \). And by Proposition 3,

\[
\frac{1}{n} \sum_{i=1}^{n} \left( \frac{1}{n} \sum_{j=1}^{n} \sqrt[1/n]{x_1 x_2 \cdots x_j} \right)^{-1} \geq \frac{1}{n} \sum_{i=1}^{n} \left( \sqrt[1/n]{x_1 x_2 \cdots x_j} \right)^{-1}
\]

Equality holds only if \( x_1 = x_2 = \ldots = x_n \). This completes the proof.

3. Conclusion

There are many versions of mixed mean and its inequalities. We just proved a mixed arithmetic-mean, harmonic-mean inequality and a mixed arithmetic-mean, geometric-mean, harmonic-mean inequality, which is just one of the mixed mean inequalities. Undoubtedly, more mixed mean inequality will be studied and discovered.

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References


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