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Original Article

Fixed Point Theorems on Complete Cone Metric and Cone Rectangular Metric Spaces

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Abstract *- The study of common fixed point theory with different types of mappings and contractive conditions takes a significant role in present research activity. At first, the great mathematician Brouwer introduced the famous fixed-point theory in 1912. Accordingto Brouwer, "Every continuous function from the closed and bounded subset of R^n into itself has a fixed point". After that, the great mathematician S. Banach in 1922 invented the well-known fixed point theory, namely Banach Fixed Point Theory. "Huang & Zhang [3] have introduced the concept of cone metric space, where the set of real numbers is replaced by an ordered Banach space [3]". The present research article is about some common fixed-point theorems for "self-mappings and commuting mappings" on Cone and Cone Rectangular Metric Space.*

***Keywords* -** *Cone & Cone Rectangular Metric, Completeness, Cauchy Sequence, Fixed point.*

***AMS Subject Calcifications* -** *47H10; 54H25.*

1. Introduction

In Mathematical analysis, the metric space and Banach fixed-point theory are essential in nonlinear analysis. C.D. Birkhoff and Kellogg first gave the idea of infinite-dimensional fixed-point theory in 1922. They also prove some theorems on metric linear spaces, and about convexity and compactness. In 1930, Schauder proved that " Let S be a non-empty closed convex subset of a normed space, then every continuous function from S into a compact subset of S has a fixed point".

S. Gahler established the concept of a 2-metric space in 1962, which considers the distance between two triplets rather than pairs in a metric space. This study has generalized and extended the slandered metric spaces. S. Gahler proved that the well-known Banach fixed-point theory is satisfied. Besides this, in 1984, B. C. Dhage introduced the well-known D-Metrics Spaces, which also satisfied the condition of Banach fixed-point theory. After that, many mathematicians worked in this line. After that, many metric spaces, such as K-metric, B-Metric, Partially Ordered Metric, E-Metric, have been invented by different mathematicians over the last two decades of twenty century. All these metric spaces satisfy the Banach Contraction Principle and Banach Fixed Point Theory. After the invention of the continuous time scale, the Banach contraction has taken an important role in Mathematical Modelling in the real and complex world. A modeler can form a unique model using this technique.

The cone metric space was invented by "Huang Long Guang" and "Zang [3]" in 2007. After that, many researchers and mathematicians work in this line with the condition of a cone and a Banach contraction.

Definition: "([3]) Let S be a real Bench space and $\tau \subseteq S$. Let τ be a cone if and only if it holds the following properties,

(i) τ is a closed set, $\tau \neq \phi$ and $\tau \neq \{0\}$

(ii) $\zeta_1, \zeta_2 \in E$ & $\zeta_1, \zeta_2 \geq 0$, $\rho_1, \rho_2 \in \tau \Rightarrow \zeta_1 \rho_1 + \zeta_2 \rho_2 \in \tau$

(iii) ρ_1 and $-\rho_2 \in \tau \Rightarrow \rho_1 \cap (-\rho_2) = 0$".

Definition: "([3]) Let W be a non-empty set, suppose that the mapping d is a mapping from $W \times W \rightarrow E$ and satisfies the following properties,

(i) $d(\mu_1, \mu_2) > 0, \forall \mu_1, \mu_2 \in W$

(ii) $d(\mu_1, \mu_2) = 0$ iff $\mu_1 = \mu_2$

(iii) $d(\mu_1, \mu_2) = d(\mu_2, \mu_1), \forall \mu_1, \mu_2 \in W$

(iv) d (μ_1, μ_2) $\leq$ d (μ_1, μ_3) + d (μ_3, μ_2), $\forall$ $\mu_1, \mu_2, \mu_3 \in$ W, then the couple (W, d) is called a cone metric."

Definition: "([3]) Let W be a non-empty set, suppose the mapping d: W× W to E satisfies the condition,

(i) o $\leq$ d (μ_1, μ_2); $\forall$ $\mu_1, \mu_2 \in$ W, d (μ_1, μ_2) = 0 iff $\mu_1 = \mu_2$

(ii) d ((μ_1, μ_2) = d ((μ_1, μ_2), $\forall$ $\mu_1, \mu_2 \in$ W.

(iii) d (μ_1, μ_2) $\leq$ d (μ_1, μ_3) + d (μ_4, μ_3) + d (μ_4, μ_2), $\forall$ $\mu_i \in$ W (rectangular property)

Then (W, d) is called a cone rectangular metric space".

"([3]) Definition: A sequence (ς_n) is said to be Cauchy in W if for c $\in$ E with o << c there is $n_o \in$ N such that for all n, m > n_o, d (ς_n, ς_m) << c.

"([3]) Definition: - Let {ς_n} be a sequence in (W, d) and $\varsigma \in$ (W, d). If for every c $\in$ E, with 0 << c there is $n_o \in$ N such that for all n > n_o, d (ς_n, ς) << c, then { ς_n } is said to be convergent and converges to ς_n. "

i.e. $\lim \varsigma_n = \varsigma$ or $\varsigma_n \to \varsigma$, for $n \to \infty$

"([3]) Definition**:** A cone metric space (W, d) is said to be a complete cone metric space if every Cauchy sequence on it is convergent on W."

([3]) Definition: "A Cone rectangular metric space will be a complete cone rectangular metric space if every Cauchy sequence converges on W".

2. Main Results

Theorem (1) Suppose (W, d) is a complete cone metric and τ is a 'normal cone' with normality κ. Assume f is a mapping from W into W and holds the Condition:

$$d(f(\xi), f(\zeta)) \leq \frac{\gamma_1\{(d(\xi,\zeta)+d(\xi,f(\xi))\}}{2} + \gamma_2 d\big(\zeta, f(\zeta)\big) + \gamma_3\, d(f(\xi), \zeta) \tag{1.1}$$

For all $\xi, \zeta \in$ W and $0 < \gamma_i < 1$, $0< (1-\gamma_1) > 1$, $0 < \gamma 1/(1-\gamma_2) < 1$, $0 < \gamma 1/(1-\gamma_{1-}\gamma_2) < 1$,

$(1 - \gamma_3 - \gamma_1/2) < 1$. Then f has a unique common fixed point.

Proof: Let us take an arbitrary element $\xi_0 \in$ W. Now, define the function f as follows:

$f(\xi_0) = \xi_1$, ξ_0 and $\xi_1 \in X$

In general,

$f(\xi_n) = \xi_{n+1}$.

From the above inequality, substituting, $\xi = \xi_0$ and $\zeta = \xi_1$, we get,

$$d(f(\xi_0),f(\xi_1)) \leq \frac{\gamma_1\{(d(\xi,\xi_1)+d(\xi_0,f(\xi_0)\}}{2} + \gamma_2 d(\xi_1,f(\xi_1)) + \gamma_3\, d(f(\xi_0), \xi_1)$$

$$= \frac{\gamma_1\{(d(\xi_0,\xi_1)+d(\xi_0,\xi_1)\}}{2} + \gamma_2 d(\xi_1,f(\xi_1)) + \gamma_3\, d(f(\xi_0), \xi_1)$$

This implies, $d(\xi_1, \xi_2) \leq \frac{2\gamma_1(d(\xi,\xi_1)\}}{2} + \gamma_2 d(\xi_1, \xi_2) + \gamma\, d(\xi_1, \xi_1) = \gamma_1 d(\xi_0, \xi_1) + \gamma_2 d(\xi_1, \xi_2)$

i.e, $(1-\gamma_2)\, d(\xi_1, \xi_2) \leq \gamma_1 d(\xi_0, \xi_1)$

$$\leq \frac{\gamma_1}{1-\gamma_2} d(\xi_0, \xi_1) \text{ ; let } \frac{\gamma_1}{1-\gamma_2} = \delta \text{ , i. e, } d(\xi_1, \xi_2) \leq \delta\, d(\xi_0, \xi_1) \tag{1.2}$$

Again, putting the value of $\xi = \xi_1$, $\zeta = \xi_2$ (1.1) produces,

$$d(f(\xi_1),f(\xi_2)) \leq \frac{\gamma_1\{(d(\xi_1,\xi_2)+d(\xi_1,f(\xi_1)\}}{2} + \gamma_2 d(\xi_2,f(\xi_2)) + \gamma_3\, d(f(\xi_1), \xi_2)$$

$$= \frac{\gamma_1\{(d(\xi_1,\xi_2)+d(\xi_1,\xi_2)\}}{2} + \gamma_2 d(\xi_2, \xi_3) + \gamma_3\, d(\xi_2, \xi_2)$$

Which implies, $d(\xi_2, \xi_3) \leq \frac{2\gamma_1(d(\xi_1,\xi_2)\}}{2} + \gamma_2 d(\xi_2, \xi_3)$

$= \gamma_1 d(\xi_1, \xi_2) + \gamma_2 d(\xi_2, \xi_3)$,

i.e, $d(\xi_2, \xi_3) \leq \frac{\gamma_1}{1-\gamma_2} d(\xi_1, x_2)$

$$\Rightarrow d(\xi_2, \xi_3) \leq \delta^2 d(\xi_0, \xi_1) \text{ (from equation 1.2)} \tag{1.3}$$

Again, Introducing the value of $\xi = \xi_2$ and $\zeta = \xi_3$, (1.1) eyelids,

$d(f(\xi_2),f(\xi_3)) \leq \frac{\gamma_1\{(d(\xi_2,\xi_3)+d(\xi_2,f(\xi_2))\}}{2} + \gamma_2 d(\xi_3,f(\xi_3)) + \gamma_3 d(f(\xi_2), \xi_3)$

Which implies, $d(\xi_3, \xi_4) \leq \frac{\gamma_1\{(d(\xi_2,\xi_3)+d(\xi_2,\xi_3)\}}{2} + \gamma_2 d(\xi_3, \xi_4) + \gamma_3 d(\xi_3, \xi_3)$

$= \frac{2\gamma_1(d(\xi_2,\xi_3)\}}{2} + \gamma_2 d(\xi_2, \xi_3)$

i.e, $(1-\gamma_2) d(\xi_3, \xi_4) \leq \gamma_1 d(\xi_2, \xi_3)$, i.e, $d(\xi_3, \xi_4) \leq \frac{\gamma_1}{1-\gamma_2} d(\xi_2, \xi_3)$

$$\Rightarrow d(\xi_3, \xi_4) \leq \delta^3 d(\xi_0, \xi_1) \text{ (from equation 1.3)} \tag{1.4}$$

In general, the above gives,

$$d(\xi_n, \xi_{n+1}) \leq \delta^n d(\xi_0, \xi_1) \tag{1.5}$$

For n, m $\in$ N, (1.5) produces,

$d(\xi_n, \xi_m) \leq d(\xi_n, \xi_{n+1}) + d(\xi_{n+1}, \xi_m)$ (triangular property)

$\leq d(\xi_n, \xi_{n+1}) + d(\xi_{n+1}, \xi_{n+2}) + d(\xi_{n+2}, \xi_{n+1}) + d(\xi_{n+1}, \xi_{n+3}) + \ldots\ldots$

$\leq \delta^n d(\xi_0, \xi_1) + \delta^{n+1} d(\xi_0, \xi_1) + \delta^{n+3} d(\xi_0, \xi_1) + \delta^{n+4} d(\xi_0, \xi_1) + \ldots\ldots \ldots$

$\leq \delta^n [1+\delta + \delta^2 + \delta^3 + \ldots\ldots \ldots \ldots] d(\xi_0, \xi_1)$, $\Rightarrow d(\xi_n, \xi_m) \leq \frac{\delta^n}{1-\delta} d(\xi_0, \xi_1)$

Then, for n tending to infinity, $d(\xi_n, \xi_m) \rightarrow 0$, so, ξ_n is Cauchy.

For completeness of (W, d), it says that ξ_n must tend to ξ in W.
Now, we will verify the fixed point of f on W.

Inequality (1.1) produces,

$d(\xi_n,f(\xi_n)) \leq \frac{\gamma_1\{(d(\xi_{n-1},\xi_n)+d(\xi_{n-1},f(\xi_{n-1})\}}{2} + \gamma_2 d(\xi_n, f(\xi_n)) + \gamma_3 d(f(\xi_{n-1}), \xi_n)$

$= \frac{\gamma_1\{(d(\xi_{n-1},\xi_n)+d(\xi_{n-1},\xi_n)\}}{2} + \gamma_2 d(\xi_n, \xi_{n+1}) + \gamma_3 d(\xi_n, \xi_n)$

Simplifying the above yields,

i.e, $d(\xi_n, f(\xi_n)) \leq \gamma_1 d(\xi_{n-1}, \xi_n) + \gamma_2 d(\xi_n, \xi_{n+1})$

When n tends to ∞, x_n tends to x. Hence, we have from the above (1.1),

$d(\xi, f(\xi)) \leq \gamma_1 d(\xi, \xi) + \gamma_2 d(\xi, \xi)$. From the normality condition it produces,

$\| d(f(\xi), \xi) \| k \leq 0$, i.e, $d(f(x), x) \leq 0$, as d is positive, the only possibility is,

$$d(f(\xi), \xi) = 0, \text{ i.e, } f(\xi) = \xi. \tag{1.6}$$

Lastly, for the uniqueness of ξ, let ω is another point.

i.e., $f(\xi) = \xi$and $f(\omega) = \omega$

Now, we have from the given inequality of our theorem:

$d(\xi, \omega) \leq \frac{\gamma_1\{(d(\xi,\omega)+d(\xi,f(\xi))\}}{2} + \gamma_2 d(\omega, f(\omega)) + \gamma_3\, d(\xi, \omega)$

$= \frac{\gamma_1\{(d(\xi,x^{`})+d(\xi,\xi)\}}{2} + \gamma_2 d(\omega, \omega) + \gamma_3\, d(\xi, \omega)$

That is, $(1-\gamma_3)\, d(\xi, \omega) \leq \frac{\gamma_1\{(d(\xi,\omega)\}}{2}$. Applying the normality of Cone, we have:

$\|(1-\gamma_3-\frac{\alpha}{2})\, d(\xi, \omega)\|\, k \leq 0,$

As, $0< \gamma_1, \gamma_3<1$ & d is positive, above is not possible for the condition of Cone Metric.

The only possibility is, $d(\xi, \omega)=0$. Hence $\xi = \omega$

Therefore, ξ is a unique common fixed point of f in W.

Theorem (2): Consider (W, d) a complete cone rectangular metric with κ a normality constant of the cone τ. Let f, g, h, and I be four continuous mappings from W into W and hold the following property,

(i) f commutes with h and g commutes with I.

(ii) $f(W) \subset h(W)$ and $g(W) \subset I(W)$

(iii) $d(f(\xi), g(\zeta)) \leq \gamma_1\, d(h(\xi), I(\zeta)) + \gamma_2\, d(h(\xi),f(\xi)) + \gamma_3 d(I(\zeta),g(\zeta))$

$+ \gamma_4\, d(h(\xi),h(\zeta)) + \gamma_5\, d(k(\xi),k(\zeta))$ (2.1)

for all $\xi, \zeta \in W, 0< \gamma_i< 1, 0<\frac{\gamma_1+\gamma_2+\gamma_4+\gamma_5}{1-\gamma_3}<1$. Then, f, g, h, I have a unique common fixed point on W.

Proof: Suppose $\xi_0, \zeta_0 \in W$, let us take $\{\zeta_n\} \in W$, as $f(\xi) \subset h(\xi)$ and $g(\xi) \subset I(\xi)$ define the mapping as follows,

$\zeta_{n+1} = f(\xi_n) = h(\xi_{n+1}),$

$\zeta_n = g(\xi_{n-1}) = I(\xi_n),\ n = 1,2,3\ldots.$

Now substituting $\xi = \xi_i$ and $\zeta = \xi_{i+1}$

$d(f(\xi_i),g(\xi_{i+1})) \leq \alpha\, d(h(\xi_i), I(\xi_{i+1})) + \beta\, d(h(\xi_i),f(\xi_i)) + \gamma\, d(I(\xi_{i+1}),g(\xi_{i+1}))$

$+ \delta\, d(h(\xi_i),h(\xi_{i+1})) + \eta\, d(k(\xi_i),k(\xi_{i+1}))$

Simplifying it produces,

$(1-\gamma) d(\zeta_{i+1}, \zeta_{i+2}) \leq (\alpha + \beta + \delta + \eta)\, d(\zeta_i, \zeta_{i+1})$

$d(\zeta_{i+1}, \zeta_{i+2}) \leq \frac{\alpha+\beta+\delta+\eta}{1-\gamma}\, d(\zeta_i, \zeta_{i+1})$

Let, $\sigma = \frac{\alpha+\beta+\delta+\eta}{1-\gamma}$ and $0 < \sigma < 1$

$d(\zeta_{i+1}, \zeta_{i+2}) \leq \sigma\, d(\zeta_i, \zeta_{i+1})$ (2.2)

Again, for $\xi = \xi_{i+1}$ and $y = \xi_{i+2}$, (2.1) produces,

$d(f(\xi_{i+1}),g(\xi_{i+2})) \leq \alpha\, d(h(\xi_{i+1}), I(\xi_{i+2})) + \beta\, d(h(\xi_{i+1}),f(\xi_{i+1})) + \gamma\, d(I(\xi_{i+2}),g(\xi_{i+3}))$

$+ \delta\, d(h(\xi_{i+1}),h(\xi_{i+2})) + \eta d(k(\xi_i),k(\xi_{i+1}))$

On simplifying the above yields,

$d(\zeta_{i+2}, \zeta_{i+3}) \leq \frac{\alpha+\beta+\delta+\eta}{1-\gamma} d(\zeta_{i+1}, \zeta_{i+2})$

i.e, $d(\zeta_{i+2}, \zeta_{i+3}) \leq \sigma^2 d(\zeta_i, \zeta_{i+1})$, (by using inequality 2.2) (2.3)

Again, for $\xi = \xi_{i+2}$ and $y = \xi_{i+3}$, we have

$d(f(\xi_{i+2}),g(\xi_{i+3})) \leq \alpha\, d(h(\xi_{i+2}), I(\xi_{i+3})) + \beta\, d(h(\xi_{i+2}),f(\xi_{i+2})) + \gamma\, d(I(\xi_{i+3}),g(\xi_{i+4}))$

$+ \delta\, d(h(\xi_{i+2}),h(\xi_{i+3})) + \eta\, d(k(\xi_{i+1}),k(\xi_{i+2}))$

i.e, $(1-\gamma)\, d(\zeta_{i+3}, \zeta_{i+4}) \leq (\alpha + \beta + \delta + \eta)\, d(\zeta_{i+2}, \zeta_{i+3})$

That implies $d(\zeta_{i+3}, \zeta_{i+4}) \leq \frac{\alpha+\beta+\delta+\eta}{1-\gamma} d(\zeta_{i+2}, \zeta_{i+3})$

$\Rightarrow d(\zeta_{i+3}, \zeta_{i+4}) \leq \sigma^3 d(\zeta_i, \zeta_{i+1})$, (by using inequality (2.3) (2.4)

In general, for $n \in N$, it produces,

$d(\zeta_{i+n}, \zeta_{i+n+1}) \leq \sigma^n d(\zeta_i, \zeta_{i+1})$ (2.5)

Now, for $m > n > i$, (2.1) yields,

$d(\zeta_{i+n}, \zeta_{i+n+m}) \leq d(\zeta_{i+n}, \zeta_{i+n+1}) + d(\zeta_{i+n+1}, \zeta_{i+n+m})$

$d(\zeta_{i+n}, \zeta_{i+n+m}) \leq d(\zeta_{i+n}, \zeta_{i+n+1}) + d(\zeta_{i+n+1}, \zeta_{i+n+2}) + d(\zeta_{i+n+1}, \zeta_{i+n+2}) + \ldots\ldots\ldots\ldots + d(\zeta_{i+n+m-1}, \zeta_{i+n+m})$

Which implies,

$d(\zeta_{i+n}, \zeta_{i+n+m}) \leq [\sigma^n + \sigma^{n+1} + \sigma^{n+2} + \ldots\ldots\ldots\ldots]\, d(\zeta_i, \zeta_{i+1})$

$= \sigma^n[1 + \sigma + \sigma^2 + \cdots]\, d(\zeta_i, \zeta_{i+1})$

$d(\zeta_{i+n}, \zeta_{i+n+m}) \leq \frac{\sigma^n}{1-\sigma} d(\zeta_i, \zeta_{i+1})$

Applying the normality k of the cone above produces,

$\| d(\zeta_{i+n}, \zeta_{i+n+m}) \| \leq \| \frac{\sigma^n}{1-\sigma} \| k\, d(\zeta_i, \zeta_{i+1}) \|$

Now tending (i+n) and $n \to \infty$ above inequality gives,

$\| d(\zeta_{i+n}, \zeta_{i+n+m}) \| \to 0$

It shows ζ_n is a Cauchy. For completeness of W, ζ_n must converge to a point, say υ.

i.e., $\zeta_n \to \upsilon$ as n tends ∞,

As $\{\zeta_n\} \to \upsilon$ then $\{\xi_n\}$ It also converges to υ.

i.e. ζ_n tends to υ, ξ_n tends to υ as n tends to infinity,

i.e., $\upsilon = f(\upsilon) = h(\upsilon)$ & $\upsilon = g(\upsilon) = I(\upsilon)$, as $f(W) \subset h(W)$ and $g(W) \subset I(W)$

Now setting $\xi = \xi_n$ & $\zeta = \zeta_n$, (2.1) yields,

$d(f(\xi_n), \zeta_n) = d(f(\xi_n),g(\xi_{n-1}))$

$\leq \alpha\, d(h(\xi_n), I(\xi_{n-1})) + \beta\, d(h(\xi_n),f(\xi_{n-1})) + \gamma\, d(I(\xi_{n-1}),g(\xi_{n-1}))$

$+ \delta\, d(h(\xi_n),h(\xi_{n-1})) + \eta d(k(\xi_n),k(\xi_{n-1}))$

$= \alpha\, d(\zeta_n, \zeta_{n-1}) + \beta\, d(\zeta_n,f(\zeta_{n-1})) + \gamma\, d(\zeta_{n-1}, \zeta_n)$

$+ \delta\, d(\zeta_n, \zeta_{n-1}) + \eta\, d(\zeta_n, \zeta_{n-1})$

When 'n' tends to infinity, the above produces,

$d(f(\upsilon), \upsilon) \leq \alpha\, d(\upsilon,\upsilon) + \beta\, d(\upsilon,f(\upsilon)) + \gamma\, d(\upsilon,\upsilon) + \delta\, d(\upsilon,\upsilon) + \eta\, d(\upsilon,\upsilon)$. Applying the normality of the cone we have,

$\| k\,(1-\beta)d(f(\upsilon),\upsilon) \| \leq 0$, which is $\beta \in (0,1)$, then $(1-\beta) > 0$ and $d(f(\upsilon),\upsilon) > 0$, so only possibility is that $d(f(\upsilon),\upsilon) = 0$

$$\text{So, } f(\upsilon) = \upsilon \tag{2.6}$$

Again, putting $\xi = y_n$, $\zeta = \xi_{n-1}$, then (2.1) yields,

$d(\zeta_n,g(\xi_{n-1})) = d(f(\xi_n),g(\xi_{n-1}))$

$\leq \alpha\, d(h(\xi_n), I(\xi_{n-1})) + \beta\, d(h(\xi_n),f(\xi_{n-1}))$

$+\gamma d(I(\xi_{n-1}),g(\xi_{n-1}))+\delta d(h(\xi_n),h(\xi_{n-1})) +\eta d(k(\xi_n),k(\xi_{n-1}))$

$= \alpha\, d(\zeta_n, g(\xi_{n-1})) + \beta\, d(\zeta_n, \zeta_n) + \gamma\, d(\zeta_{n-1},g(\xi_{n-1})) + \delta\, d(\zeta_n, \zeta_{n-1}) + \eta\, d(\zeta_n, \zeta_{n-1})$

When 'n' $\rightarrow$ to ∞, above yields,
$d(\upsilon,g(\upsilon))$
$\leq \alpha\, d(\upsilon,\upsilon) + \beta\, d(\upsilon,\upsilon) + \gamma\, d(\upsilon,g(\upsilon)) + \delta\, d(\upsilon,\upsilon) + \eta\, d(\upsilon,\upsilon)$
$(1-\gamma)d(\upsilon, g(\upsilon)) \leq 0$, as $0 < 1-\gamma < 1$, $d(\upsilon,g(\upsilon))$ is positive, so only option is that,

$$d(\upsilon,g(\upsilon)) = 0, \text{ i.e, } g(\upsilon) = \upsilon \tag{2.7}$$

From equation (2.6) & (2.7), we have $f(\upsilon) = g(\upsilon) = \upsilon$.
As $f(w) \subset h(w)$ and $g(w) \subset I(w)$, this implies $f(\upsilon) \subset h(\upsilon)$ & $g(\upsilon) \subset I(\upsilon)$
Hence, $f(\upsilon) = h(\upsilon) = g(\upsilon) = I(\upsilon) = \upsilon$. Therefore, υ is a common fixed point.

Lastly, consider μ is one more fixed point then, we have:
$d(\upsilon, \mu) = d(f(\upsilon), g(\mu))$
$d(\upsilon, \mu) \leq \alpha\, d(h(\upsilon), I(\mu)) + \beta\, d(h(\upsilon),f(\mu)) + \gamma\, d(I(\mu),g(\mu)) + \delta\, d(h(\upsilon),h(\mu)) + \eta\, d(k(\upsilon),k(\mu))$
$= \alpha\, d(\upsilon, \mu) + \beta\, d(\upsilon, \mu) + \gamma\, d(\mu,\mu) + \delta\, d(\upsilon,\mu) + \eta\, d(\upsilon,\mu)$
$\{1-(\alpha+\beta+\eta)\}\, d(\upsilon, \mu) \leq 0$, as $0 < 1-(\alpha+\beta+\eta) < 1$ and d is positive, the only possibility, $d(\upsilon, \mu) = 0$. i.e, $\upsilon = \mu$.
Hence, our theorem is proved.

3. Novelty of The Research Work

This generalization of the fixed point theory on cone metric and cone rectangular metric spaces represents a more flexible and significant way to find the existence and uniqueness of fixed points on distance spaces, taking different mappings. The method used here is to establish the existence and uniqueness of a fixed point by using the Banach condition method. Present research results can be especially valuable in connections between Contraction mapping, distance mapping, and self-mappings. The concept of metric fixed point theory in this present research paper creates a great idea to generalize the classical concepts. The objective of this research paper is to provide a comprehensive study of cone and cone rectangular metric spaces, examining their fundamental properties, exploring their theoretical keystones, and identifying applications. The objective of this research is to provide a complete study of metric spaces and fixed-point conditions, examining their fundamental properties.

4. Conclusion

In this present research paper, some generalized fixed point theorems have been established by Huang & Zhang [3]. Here, the main technique has implemented the known novel Banach Contraction Mapping principle and the condition of Banach Fixed point property, taking the self-mappings as on its particular properties.

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