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Original Article

Solving nth order Differential Equations and Polynomial Equations of nth Degree by Using Unified Formulas Composed of Radical Expressions

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Abstract - *This paper presents new theorems solving differential equations of nth-order where the possibility of calculating solutions nearly in parallel. These theorems are based on an engineering methodology which forward the concept of solutions architecting according to an engineering approach, where the process of developing expressions and sub-expressions of solutions is based on requirements engineering, analysis, design, and then developing the complete algebraic formulas of solutions to be scalable and projectable on any orders or degrees of equations. The new engineering methodology in this paper is initially developed to solve nth degree polynomial equations in general forms while using specific new unified formulas composed of radical expressions, which allow calculating the roots nearly in parallel. Then, this paper forwards this engineering methodology by using the roots of nth degree polynomial equations in general forms to solve differential equations of nth order. This methodology is presenting specific logic, statements, conditions, mathematical expressions and new unified formulas that allow calculating the solutions of nth degree polynomials and nth-order differential equations. In addition, this paper presents the results of applying this engineered methodology on differential and polynomial equations of fourth degree, fifth degree and sixth degree. This methodology is built on precise details that are providing step-by-step logic and formulas to calculate the solutions, which allow concretizing multiple theorems formulating the algebraic expressions of all solutions for different orders and degrees of equations where the possibility of calculating the values of these solutions nearly in parallel while relying on distributed structures of terms.*

Keywords - *Differential equations*, *new engineered methodology, polynomial equations, roots parallel calculations, solutions architecting, solving nth degree polynomials, solving nth order differential equations.*

1. Introduction

In the fields of mathematics, differential equations are specific forms of equations expressed by relying on derivatives whereas having at least one unknown function and its derivations.

Differential equations are widely used in physics, economics, biology, automation, industries and engineering, because these differential equations are formulating the relations between quantitative functions and their rates of changing, which allow studying, analyzing, controlling and even predicting the values and evolutions of these functions.

These differential equations were introduced as a part of Calculus, which was invented in the 17^{th} century by Isaac Newton [1-3] and Gottfried Leibniz [2]. Since then, there have been many books and research papers tackling the solvability of differential equations and their classifications according to the properties of equations.

Among the classes of differential equations, we have ordinary differentials [4], partial differentials [5], linear differentials [6], non-linear differentials [7], homogeneous differentials [8] and heterogeneous differentials [9]. In addition, there are other classes of differential equations that are varying depending on context and properties of equations [10-12].

Ordinary differentials are specific form of differential equations with exactly one unknown function and its derivatives basing on one unknown variable, whereas partial differentials are more composed forms of differential equations handling multiple variables by at least one unknown function and its derivatives.

Linear differentials are a category of differential equations that are linear in the unknown function and its derivatives, so it can be written in the form $\left\{\left(\sum_{i=0}^{i=u} a_i(x) * y^{(i)}\right) = b(x)\right\}$. The expressions $a_0(x), \ldots, a_n(x)$ and $b(x)$ are arbitrary differentiable functions that do not need to be linear, whereas $y', \ldots, y^{(n)}$ are the consecutive derivatives of an unknown function *y* in term of the variable *x*.

Nonlinear differentials are more composed forms of differential equations where the unknown function (or its derivatives) does not appear in a linear way. This means the equation cannot be expressed as a sum of terms where each term is a constant (or a function of independent variables) multiplied by either the unknown function of dependent variable or one of its derivatives.

A differential equation can be referred as homogeneous in two prospect scenarios. The first prospect is when we describe a first order differential equation as to be homogeneous if we can express it as follows: $\{f(x,y)dy = g(x,y)dx\}$ where *f* and *g* are homogeneous functions of the same degree of *x* and *y*. The second prospect is when we describe a differential equation as to be homogeneous if it is a homogeneous function of the unknown function and its derivatives. In simpler terms, this means that if we scale the input variables by a constant, the output of the function scales by a power of that constant $\{f(tx_1, tx_2, \ldots, tx_m) = t^k f(x_1, x_2, \ldots, x_m)\}$.

Heterogeneous differentials are differential equations where the right-hand side of the equation is not equal to zero, in case we dedicate this side of equation to express only the independent variables. In simpler terms, it is a differential equation that includes a non-zero function of the independent variable(s) on one side of the equation or containing constant terms that are not multiplied by the function of dependent variable nor its derivatives.

Among the most common methods to solve differential equations is converting them according to polynomial forms of equations, then using the roots of these polynomials to express the solutions of differential equations. This method is built on the inductive relation between the order of differential equations and the degrees of polynomial equations.

In order to help solving nth-order differential equations, we need to start by converting them to polynomial forms of nth degrees, then using the roots of these polynomials to express the solutions of corresponding differentials, which usually lead toward using numerical analysis algorithms to find these roots especially when the equations are not having symmetries and the nth degree is higher than four.

Solving nth order differential equations and nth degree polynomials has been challenging over centuries to mathematicians, scientists and researchers, peculiarly when looking for algebraic terms that may help expressing the roots of equations. This encountered challenge is due to the complexity of calculations that can outstrip the previsions of human mind especially when orders and degrees of equations are making them transcending above the quartic form.

The complexity of mathematical calculations during the attempts of solving polynomial equations of high degrees is majorly due to using radicalities while adopting particular approaches. In addition, reaching a point of having highly complex outcomes where the form of resulted equation is far from foreseeing a simplification or reduction; will lead toward conducting exhaustive change on used approach or even replacing it by adopting a different one.

Adopting a specific approach to solve particular degrees or orders of equations lead toward having limitations in the resulted forms of equations where simplifications and reductions are harder to be conducted by comparison to the starting point. In addition, conducted calculations may expand in high rate when searching for the solution while trying different approaches that may impose restarting calculations from scratch. Furthermore, the induced complexity of resulted forms of equations may lure toward adopting a narrowed solution to be used on a specific form of polynomials or differential equations that have particular conditions stated by the values of included coefficients.

As a result, discovering unified solution formulas for polynomials and differential equations in general forms and in complete forms is more challenging when relying on a research methodology where calculations and logic may to be restarted from scratch when the approach is drastically modified or even replaced.

Therefore, we rely on an engineering methodology where we construct the appropriate approach bit-by-bit basing on patterns and characteristics that should be met. Then, we use this built approach to architect the adequate roots and to structure their involved terms and sub-terms. Then, we forward logic and calculations toward engineering the mathematical formulas of all roots, in order to allow the calculation of all expected solutions nearly in parallel.

Relying on this engineering methodology to solve polynomials and differential equations avoid us restarting the logic and calculations from scratch, and allow us to keep relying on the exact approach and results of calculations while conducting only slight modifications, when necessary, toward reaching the final forms of unified formulas. In addition, this engineering methodology allows us to project the exact approach and extend the same logic toward solving higher orders of differentials and higher degrees of polynomials.

In this engineering methodology, the axe of focus is building and architecting the necessary formulas according to a scalable logic where we start from requirements engineering toward designing the starting point, the path, the destination and the structure of expected final results. As a result, conducted calculations and adopted reasoning follow a pr-designed path toward structuring the unified formulas of niched roots.

The advantage of this paper is presenting specific theorems listing algebraic formulas designed to solve fourth degree, fifth degree and sixth degree polynomial equations in general forms by using radical expressions where the possibility of calculating the values of all roots nearly in parallel. Then, this paper forwards the presented theorems toward solving differential equations of fourth order, fifth order and sixth order while providing the same convenient of calculating solutions nearly in parallel.

This paper is also presenting the engineered requirements and techniques that lead to architect the results of proposed theorems, whereas allowing to scale the roots of nth degree polynomial equations toward calculating the solutions of nth order differential equations.

This paper is a principal step in our work of solving nth order differential equations and nth degree polynomials basing on projecting the presented methods and results in this paper on other equations where orders and degrees are higher than six, which will be presented in other articles.

The proposed concept in this article about architecting solutions according to a distributed structure of terms can extend itself to different problems in geometry, numbers theory and algebra in general; because this concept is introducing an engineering methodology based on identifying patterns and characteristics that allow forwarding calculations and expressions toward specific converging points where the results are built step-by-step and not only searched.

This engineering methodology was first used to develop the solutions of quartic polynomial equations in general forms [13], by building the unified formulas of the four roots of any quartic equation, which allow calculating the four solutions nearly simultaneously. Then, the same engineering methodology was scaled to solve quantic equations [14] in complete forms by proposing the necessary unified formulas of roots to calculate the five solutions nearly in parallel. In addition, this engineering methodology was used to solve sixth degree polynomial equations [15] in complete forms whereas providing the possibility of calculating the six solutions nearly in parallel. Therefore, this paper is allowing to scale this engineering methodology and its results on nth order differential equations and nth degree polynomials.

In addition, this paper presents new theorems developed to solve nth order differential equations and nth degree polynomial equations while providing necessary logic, conditions, parameters and formulas to calculate the solutions nearly in parallel by extending the proposed methodologies.

Furthermore, this paper presents new additional solutions for nth order differential equations that allow interconnecting many arbitrary points which open the way to scall up the range of using these differential equations in business analytics, data analytics predictive analysis and systems control.

Because the content of this paper is original, and it is embedding many new proposed formulas, mathematical expressions and theorems which are built on extendable logic; this paper will focus on presenting these theorems and their formulas in a scaling manner whereas relying on relevant proofs from the papers [13], [14], and [15].

The contents of this paper are structured as follow: Section 2 presenting the used engineering methodology and its developed requirements and techniques to solve nth order differential equations. Section 3, presenting the used methodology to solve nth degree polynomial equations. Section 4, presenting six theorems developed to solve fourth degree polynomial equations whereas allowing to determine the amount of complex solutions. Section 5, presenting developed theorems and formulas to solve fourth order differential equations whereas allowing to determine the amount of expected complex functions among the solution. Section 6, presenting two theorems developed to solve fifth degree polynomial equations. Section 7, presenting developed theorems and formulas to solve fifth order differential equations. Section 8, presenting two theorems developed to solve sixth degree polynomial equations. Section 9, presenting developed theorems and formulas to solve sixth order differential equations.

Section 10, presenting developed theorem to solve nth degree polynomial equations. Section 11, presenting new developed theorems and formulas to solve nth order differential equations. Finally, Section 12 for conclusion.

2. Methodology to Solve nth Order Differential Equations

This section presents the requirements, the techniques and the formulas to solve nth order differential Equations according to an organized method step by step while relying on the proposed methodology in this paper to solve nth degree polynomial equations.

In addition, this section presents new unified solutions for nth order differential equations that allow interconnecting many arbitrary points which allow scaling up their use in business analytics, data analytics and predictive analysis by mapping variables and datums of collected data to each other while tracking macro transitions and micro variations among them.

1. Expressing a differential equation of nth order according to the form $\sum c_i f_{(x)}^{(i)} = C$ where $f_{(x)}^{(i)}$ is the derivation of order (i), whereas c_i and C are arbitrary values.
2. Supposing the function $f_{(x)}$ is expressed as follows: $f_{(x)} = e^{sx+\ \ } + b$.
3. Expressing the derivative of order (i) of the function $f_{(x)}$ as follows $f_{(x)}^{(k)} = s^i e^{sx+a}$, where $(i > 0)$.
4. Converting the nth order differential equation to be presented as a nth degree polynomial form, which to be expressed as follows: $e^{sx+a} * \sum c_i s^i = C - bc_0$.
5. In order to identify the value of the variable (b), we consider the equation $(bc_0 + e^{sx+a} * \sum c_i s^i = C)$ at the point of calculated root $(s = s_k)$.
6. Solving the equation $(b * c_0 = C)$ in order to identify the value of the variable (b), which to be as follows: $\left[b = \frac{C}{c_0}\right]$ where $c_0 \neq 0$.
7. Solving the nth degree polynomial equation $\sum c_i s^i = 0$ in order to calculate *n* roots which we can present in the form of the group $\{s_1; \dots;\ s_n\}$.
8. Using the proposed engineered methodology to solve the nth degree polynomial equation $\sum c_i s^i = 0$, or using numerical analysis.
9. In order to calculate all the roots nearly in parallel for the polynomial equation $\sum c_i s^i = 0$, we use the proposed engineering methodology to solve nth degree polynomial equations.
10. We identify an initialization condition for $f_{(x)}$ where $(x = 0)$.
11. The initialization condition for $f_{(x)}$ where $(x = 0)$ should be presented as an arbitrary value, which to be expressed as follows: $f_{(x=0)} = I_0$.
12. The value of the variable (a) should be calculated by relying on the initialization condition.
13. The value of the variable (a) is to be identified by using the expression $a = \log(I_0 - b)$, which allow calculating the value of the variable (a) as follows: $a = \log\left(I_0 - \frac{C}{c_0}\right)$ where $c_0 \neq 0$.
14. The solution of the nth order differential equation should be expressed as $\mathrm{DS_k} = e^{s_k x + \log\left(I_0 - \frac{C}{c_0}\right)} + \frac{C}{c_0}$, where s_k is a calculated root for the nth degree polynomial equation $\sum c_i s^i = 0$.
15. When using the proposed engineering methodology to solve nth degree polynomials, we became able to calculate all the roots $\{s_1; \dots;\ s_n\}$ nearly in parallel, which allow calculating all the solutions $\{DS_1; \dots;\ DS_n\}$ of nth order differential equation nearly in parallel.
16. When having a differential equation of nth order $\left(\sum c_i f_{(x)}^{(i)} = C\right)$, we can identify T arbitrary values $\{I_k \mid k \in [\![0, T-1]\!]\ and\ T \in [\![1, n]\!]\}$, where each arbitrary value I_k is identified at a specific referencing point x_k by using the expression $f(x_k) = I_k$.

17. If there is an amount of T different roots $\{s_{p_1};\ s_{p_2};\ ...;\ s_{p_T}\}$ for the nth degree polynomial equation $(\sum c_i s^i = C)$ corresponding to the nth order differential equation $\left(\sum c_i f_{(x)}^{(i)} = C\right)$, then we can identify T arbitrary values $\{I_k \mid k \in [\![0, T-1]\!]\ and\ T \in [\![1, n]\!]\}$, where each arbitrary value I_k is identified at a specific referencing point x_k which to be logically relevant to the root s_{p_k} in term of distribution, behavior or ratio of change.
18. After identifying T arbitrary values $\{I_k \mid k \in [\![0, T-1]\!]\ and\ T \in [\![1, n]\!]\}$, where each arbitrary value I_k is identified at a specific referencing point x_k by using the expression $f(x_k) = I_k$, we can interconnect these arbitrary points to each other by using the solutions of the differential equation.
19. If there are two different roots $\{s_a;\ s_{b\neq a}\}$ for the nth degree polynomial equation $(\sum c_i s^i = C)$ corresponding to the nth order differential equation $\left(\sum c_i f_{(x)}^{(i)} = C\right)$, then we can calculate other new solutions for the differential equation which to be expressed as $\left\{DS'_{n+1} = R'_{(I_1)} e^{s_b x} + \left(R'_{(I_0)} - \frac{C}{c_0} - R'_{(I_1)}\right) e^{xS_a} + \frac{C}{c_0};\ DS'_{n+2} = R'_{(I_1)} e^{xS_a} + \left(R'_{(I_0)} - \frac{C}{c_0} - R_{(I_1)}\right) e^{xS_b} + \frac{C}{c_0}\right\}$.
20. If there are three different roots $\{s_a;\ s_b;\ s_c\}$ for the nth degree polynomial equation $(\sum c_i s^i = C)$ corresponding to the nth order differential equation $\left(\sum c_i f_{(x)}^{(i)} = C\right)$, then we can calculate other new solutions for the differential equation which to be expressed as $\left\{DS'_{(n+i>n)} = R'_{(I_2)} e^{xS_3} + \left(R'_{(I_1)} - R_{(I_2)}\right) e^{xS_2} + \left(R'_{(I_0)} - \frac{C}{c_0} - R'_{(I_1)}\right) e^{xS_1} + \frac{C}{c_0};\ where\ s_k \in \{s_a;\ s_b;\ s_c\}\right\}$
21. If there is an amount of T different roots $\{s_{p_1};\ s_{p_1};\ ...;\ s_{p_T}\}$ for the nth degree polynomial equation $(\sum c_i s^i = C)$ corresponding to the nth order differential equation $\left(\sum c_i f_{(x)}^{(i)} = C\right)$, then we can calculate other new solutions for the differential equation which to be expressed as $\left\{DS'_{(n+i>n)} = R'_{(I_{(T-1)})} e^{xS_T} + \sum_{L=1}^{L=T-2} \left[\left(R'_{(I_L)} - R'_{(I_{(L+1)})}\right) e^{xS_{(L+1)}}\right] + \left(R'_{(I_0)} - \frac{C}{c_0} - R'_{(I_1)}\right) e^{xS_1} + \frac{C}{c_0};\ where\ s_k \in \{s_{p_1};\ s_{p_1};\ ...;\ s_{p_T}\}\right\}$

3. Engineered Methodology to Solve nth Degree Polynomials

The presented methodology in this paper to solve general forms of nth degree polynomial equations is based on architecting the roots of these equations according to a distributed structure of terms while relying on radical expressions.

In addition, this engineering methodology is relying on developing specific patterns into the structure of niched roots in order to help converging calculations whereas eliminating degrees of polynomials.

Furthermore, this developed methodology is built on an engineering logic where roots are predesigned before being expressed according to unified mathematical formulas, which support the expressions structuring for all expected roots of aimed polynomial equation.

The presented methodology in this paper lead to define a list of engineered requirements and techniques according to a scaled logic, which is helping to develop the necessary unified formulas to calculate the roots of nth degree polynomial equations in general forms whereas enabling to calculate the values of possible roots nearly in parallel.

The results of our engineered requirements, techniques and formulas according to the developed methodology are described as follow:

1. Roots should be expressed according to a distributed structure of terms $\left\{\sum_{i=0}^{i=u} \mathrm{T_i}\right\}$, which will be multiplied by each other during calculations.
2. Each included term in the distributed structure of roots should be expressed according to the simplest possible radicality.
3. All included terms in the distributed structure of roots should either be constants or be radical expressions.
4. The included constant terms in the distributed structure of roots should allow eliminating specific parts with specific degrees from a polynomial equation.

5. We adapt a polynomial equation of nth degree $\left\{\left(\sum_{i=0}^{i=n} a_i X^i\right) = 0\right\}$ where $\{a_n \neq 0\}$ by presenting it as $\left\{\left(\sum_{i=0}^{i=n} \frac{a_i}{a_n} X^i\right) = 0\right\}$.
6. We use the expression $\left\{X = \frac{-a_{n-1}}{n a_n} + \frac{Y}{n}\right\}$ to eliminate the term of degree $(n-1)$ from a polynomial equation of nth degree $\left\{\left(\sum_{i=0}^{i=n} \frac{a_i}{a_n} X^i\right) = 0\right\}$ when $(n-1)$ is an odd value or when this elimination is simplifying calculations.
7. All included radical terms in each root should have the same radicality, in order to converge resulted expressions during calculations. Therefore, we choose them to have a radicality of square root.
8. Each included radical term in the distributed structure of a root $\{\sum_{i=0}^{i=u} \mathrm{T_i}\}$ should be expressed according to a sum of simple radical terms $\left\{\left(\sum_{i=0}^{i=u} T_i\right) = \left(\sum_{i=0}^{i=u} x_i\right) = \left(\sum_{i=0}^{i=u} \sqrt{y_i}\right)\right\}$ when the degree of polynomial equation is equal four.
9. Each included radical term in the distributed structure of a root $\{\sum_{i=0}^{i=u} T_i\}$ should be expressed according to a multiplication of at least two different sub-terms $\{(\sum_{i=0}^{i=u} T_i) = (\sum_{i\neq j} x_i\, x_j)\}$ when the degree of polynomial equation is surpassing four.
10. When the degree of polynomial equation is surpassing four, each included sub-term $\{x_l\}$ in the distributed structure of a root $\{(\sum_{i=0}^{i=u} T_i) = (\sum_{i\neq j} x_i\, x_j)\}$ should appear in multiple distributed terms in order to allow further factorizations.
11. When the degree of polynomial equation is surpassing four, each included sub-term $\{x_l\}$ in the distributed structure of terms $\{(\sum_{i=0}^{i=u} T_i) = (\sum_{i\neq j} x_i\, x_j)\}$ should be presented according to a radical expression of cubic root, quadratic root or a constant.
12. Combinations among included sub-terms in a root should allow expressing the values of involved coefficients in a polynomial equation.
13. The included sub-terms in the distributed structure of terms should allow neutralizing their contents when they are multiplied by each other in order to have simplified results.
14. The included sub-terms in the distributed structure of terms should allow eliminating radicality when they are raised to the power of higher polynomial degrees.
15. The included sub-terms in the distributed structure of terms should allow eliminating radicality when they are multiplied by each other.
16. The included sub-terms in the distributed structure of terms should allow forwarded calculations to supress terms that are having odd values of polynomial degrees.
17. The included sub-terms in the distributed structure of terms should allow forwarded calculations to either supressing terms of the highest degrees or supressing terms of the lowest degrees.
18. The distributed structure of terms $\{(\sum_{i=0}^{i=u} T_i) = (\sum_{i\neq j} x_i\, x_j)\}$ should include a sub-term $\{x_1\}$ presented according to a radical expression of cubic root where $\left(x_1 = \sqrt{\frac{-b}{3} + \frac{1}{3}\sqrt[3]{-\frac{\mathrm{D}}{2} + \sqrt{\left(\frac{\mathrm{D}}{2}\right)^2 + \left(\frac{\mathrm{C}}{3}\right)^3}} + \frac{1}{3}\sqrt[3]{-\frac{\mathrm{D}}{2} - \sqrt{\left(\frac{\mathrm{D}}{2}\right)^2 + \left(\frac{\mathrm{C}}{3}\right)^3}}}\right)$.
19. The distributed structure of terms should include two sub-terms $\{x_2, x_3\}$ presented according to radical expressions of quadratic roots where $\left(x_2 = \sqrt{-\frac{\frac{P}{2}+x_1}{2} + \sqrt{\left(\frac{\frac{P}{2}+x_1}{2}\right)^2 - \frac{Q^2}{64x_1}}}\right)$ and $\left(x_3 = \sqrt{-\frac{\frac{P}{2}+x_1}{2} - \sqrt{\left(\frac{\frac{P}{2}+x_1}{2}\right)^2 - \frac{Q^2}{64\ {}_1}}}\right)$.

20. In order to eliminate high degree expressions in a polynomial equation whereas allowing calculations to converge, we use a constant value $\{\alpha_1\}$ expressed by using included sub-terms in the distributed structure of root where $\{\alpha_1 = \sum x_i^2\}$.
21. In order to eliminate average degree expressions in a polynomial equation whereas allowing calculations to converge, we use a constant value $\{\alpha_2\}$ expressed by using included sub-terms in the distributed structure of root where $\{\alpha_2 = \sum_{i\neq j} x_i^2 \, x_j^2\}$
22. In order to eliminate low degree expressions in a polynomial equation whereas allowing calculations to converge, we use a constant value $\{\alpha_3\}$ expressed by using included sub-terms in the distributed structure of root where $\{\alpha_3 = \sum_{i\neq j\neq k} x_i x_j x_k\}$
23. In order to eliminate the lowest degree expressions in a polynomial equation whereas allowing calculations to converge, we use a constant value $\{\alpha_4\}$ expressed by using included sub-terms in the distributed structure of root where $\{\alpha_4 = \sum_{i\neq j\neq k\neq l} x_i x_j x_k \, x_l\}$.
24. In order to eliminate odd degrees of expressions in a polynomial equation whereas allowing calculations to converge, we re-formulate the solution $\{X = (\sum_{i\neq j} x_i \, x_j)\}$ to be presented as $\{X = (\sum x_i)^2 - \sum x_i^2 = (\sum x_i)^2 - \alpha_1\}$.
25. In order to reduce degrees of expressions in a polynomial equation whereas allowing calculations to converge, we re-formulate the second-degree form $\left\{X^2 = \left(\sum_{i\neq j} x_i \, x_j\right)^2\right\}$ to be presented as $\{X^2 = \alpha_2 + 2\alpha_3(\sum x_i) + 6\alpha_4\}$.
26. In order to reduce degrees of complex expressions in a polynomial equation whereas allowing calculations to converge, we re-formulate the quartic form $\left\{X^4 = \left(\sum_{i\neq j} x_i \, x_j\right)^4\right\}$ to be presented as $\{X^4 = 4(\sum x_i)^2 \, \alpha_3^2 + 4\alpha_3(\sum x_i)[\alpha_2 + 6\alpha_4] + [\alpha_2 + 6\alpha_4]^2\}$.
27. We use the proposed constants $\{\alpha_1, \alpha_2, \alpha_3, \alpha_4\}$ and the expression $\{X = (\sum_{i\neq j} x_i \, x_j)\}$ in order re-express the polynomial equation $\left\{\left(\sum_{i=0}^{i=n} \frac{a_i}{a_n} X^i\right) = 0\right\}$ to be represented as $\left\{\left(\sum_{i=0}^{i=n} \gamma_i Z^i\right) = 0\right\}$ where $\{Z = (\sum x_i)\}$.
28. When the solution of nth degree polynomial equation is expressed as $\{X = (\sum_{i\neq j} x_i \, x_j)\}$, we adopt a constant value $\left\{\frac{\Gamma}{\alpha_3} = \frac{\Gamma}{\sum_{i\neq j\neq k} x_i x_j x_k} = V\right\}$ where Γ is expressed in function of $\{(\sum x_k)\}$; in order to converge calculations during the process of equations solving.
29. The resulted polynomial expression at the final stages of forwarded calculations should have only one unknown variable expressed by including the use of one sub-term incorporated in the distributed structure of roots which to be considered as an unknown variable.
30. The resulted polynomial form at the final stages of forwarded calculations should have less degree than the staring point, or should not have the term of constant value (the term with zero degree). Otherwise, this resulted polynomial form should not have any terms with odd degrees.
31. The included sub-terms $\{x_i\}$ in the distributed structure of a root $\{(\sum \mathrm{T_i} = \sum x_i) \; or \; (\sum \mathrm{T_i} = \sum_{i\neq j} x_i \, x_j)\}$ should also be used in the calculation of all other roots by changing signs of these sub-terms whereas exploiting the involved coefficients in the polynomial equation.
32. Reusing the included sub-terms in the structure of a root while only changing their signs should allow calculating the values of different roots $\{Solution_k = \sum \pm \mathrm{T_i}\}$ nearly in parallel.

4. Solving Fourth Degree Polynomial Equations

This section presents the developed theorems and formulas to solve fourth degree polynomial equations by using the proposed engineering methodology in this paper to solve nth degree polynomial equations in general forms and in complete forms.

4.1. First proposed theorem for fourth degree polynomials

This section presents the first developed theorem to solve fourth degree polynomial equations that are expressed according to the form: $x^4 + cx^2 + dx + e = 0$, by converting this fourth degree equation into the form of a third degree polynomial which we can express as follows: $x_0^3 + \frac{c}{2}x_0^2 + \frac{c^2-4e}{16}x_0 - \frac{d^2}{64} = 0$. The proof of this theorem is detailed in [13].

Theorem 1

A fourth-degree polynomial equation under the expression (Equation. 1), where coefficients belong to the group of numbers $\mathbb{R}$, has four solutions.

$$x^4 + cx^2 + dx + e = 0 \tag{1}$$

$$-2\left[\sqrt{x_0}^2 + \sqrt{x_1}^2 + \sqrt{x_0}^2\right] = c \tag{2}$$

$$\text{For } d \le 0\text{: } -8\sqrt{x_0}\sqrt{x_1}\sqrt{x_2} = d \tag{3}$$

$$\text{For } d \ge 0\text{: } 8\sqrt{x_0}\sqrt{x_1}\sqrt{x_2} = d \tag{4}$$

$$x_{0,1} = \frac{-B}{3} + \frac{1}{3}\sqrt[3]{-\frac{D'}{2} + \sqrt{\left(\frac{D'}{2}\right)^2 + \left(\frac{C'}{3}\right)^3}} + \frac{1}{3}\sqrt[3]{-\frac{D'}{2} - \sqrt{\left(\frac{D'}{2}\right)^2 + \left(\frac{C'}{3}\right)^3}} \mid \left\{B = \frac{c}{2}\ ;\ D' = \frac{-27d^2 - 2c^3 + 72ce}{64}\ ;\ C' = -\frac{3c^2 + 36e}{16}\right\} \tag{5}$$

$$x_{0,1} = \frac{-B}{3} + \frac{1}{3}\sqrt[3]{-\frac{D'}{2} + \sqrt{\left(\frac{D'}{2}\right)^2 + \left(\frac{C'}{3}\right)^3}} + \frac{1}{3}\sqrt[3]{-\frac{D'}{2} - \sqrt{\left(\frac{D'}{2}\right)^2 + \left(\frac{C'}{3}\right)^3}} \mid \left\{B = \frac{c}{2}\ ;\ D' = \frac{-2c^3 + 72}{64}\ ;\ C' = -\frac{3c^2 + 36e}{16}\right\} \tag{6}$$

If $d < 0$, and by using the expressions of $x_{0,1}$ in (Equation. 5), c in (Equation. 2) and d in (Equation. 3), the four solutions for (Equation. 1) are as shown in (Equation. 7), (Equation. 8), (Equation. 9) and (Equation. 10).

$$\textbf{Solution 1: } S_{1,1} = \sqrt{x_{0,1}} + \sqrt{-\frac{\frac{c}{2}+x_{0,1}}{2} + \sqrt{\left(\frac{\frac{c}{2}+x_{0,1}}{2}\right)^2 - \frac{d^2}{64x_{0,1}}}} + \sqrt{-\frac{\frac{c}{2}+x_{0,1}}{2} - \sqrt{\left(\frac{\frac{c}{2}+x_{0,1}}{2}\right)^2 - \frac{d^2}{64x_{0,1}}}} \tag{7}$$

$$\textbf{Solution 2: } S_{1,2} = -\sqrt{x_{0,1}} - \sqrt{-\frac{\frac{c}{2}+x_{0,1}}{2} + \sqrt{\left(\frac{\frac{c}{2}+x_{0,1}}{2}\right)^2 - \frac{d^2}{64x_{0,1}}}} + \sqrt{-\frac{\frac{c}{2}+x_{0,1}}{2} - \sqrt{\left(\frac{\frac{c}{2}+x_{0,1}}{2}\right)^2 - \frac{d^2}{64x_{0,1}}}} \tag{8}$$

$$\textbf{Solution 3: } S_{1,3} = -\sqrt{x_{0,1}} + \sqrt{-\frac{\frac{c}{2}+x_{0,1}}{2} + \sqrt{\left(\frac{\frac{c}{2}+x_{0,1}}{2}\right)^2 - \frac{d^2}{64x_{0,1}}}} - \sqrt{-\frac{\frac{c}{2}+x_{0,1}}{2} - \sqrt{\left(\frac{\frac{c}{2}+x_{0,1}}{2}\right)^2 - \frac{d^2}{64x_{0,1}}}} \tag{9}$$

$$\textbf{Solution 4: } S_{1,4} = \sqrt{x_{0,1}} - \sqrt{-\frac{\frac{c}{2}+x_{0,1}}{2} + \sqrt{\left(\frac{\frac{c}{2}+x_{0,1}}{2}\right)^2 - \frac{d^2}{64x_{0,1}}}} - \sqrt{-\frac{\frac{c}{2}+x_{0,1}}{2} - \sqrt{\left(\frac{\frac{c}{2}+x_{0,1}}{2}\right)^2 - \frac{d^2}{64x_{0,1}}}} \tag{10}$$

If $d > 0$, and by using the expressions of $x_{0,1}$ in (Equation. 5), c in (Equation. 2) and d in (Equation. 4), the four solutions for (Equation. 1) are as shown in (Equation. 11), (Equation. 12), (Equation. 13) and (Equation. 14).

$$\textbf{Solution 1: } S_{2,1} = -\sqrt{x_{0,1}} - \sqrt{-\frac{\frac{c}{2}+x_{0,1}}{2} + \sqrt{\left(\frac{\frac{c}{2}+x_{0,1}}{2}\right)^2 - \frac{d^2}{64x_{0,1}}}} - \sqrt{-\frac{\frac{c}{2}+x_{0,1}}{2} - \sqrt{\left(\frac{\frac{c}{2}+x_{0,1}}{2}\right)^2 - \frac{d^2}{64x_{0,1}}}} \tag{11}$$

$$\textbf{Solution 2: } S_{2,2} = -\sqrt{x_{0,1}} + \sqrt{-\frac{\frac{c}{2}+x_{0,1}}{2} + \sqrt{\left(\frac{\frac{c}{2}+x_{0,1}}{2}\right)^2 - \frac{d^2}{64\ _{0,1}}}} + \sqrt{-\frac{\frac{c}{2}+x_{0,1}}{2} - \sqrt{\left(\frac{\frac{c}{2}+x_{0,1}}{2}\right)^2 - \frac{d^2}{64x_{0,1}}}} \tag{12}$$

$$\textbf{Solution 3: } S_{2,3} = \sqrt{x_{0,1}} - \sqrt{-\frac{\frac{c}{2}+x_{0,1}}{2} + \sqrt{\left(\frac{\frac{c}{2}+x_{0,1}}{2}\right)^2 - \frac{d^2}{64x_{0,1}}}} + \sqrt{-\frac{\frac{c}{2}+x_{0,1}}{2} - \sqrt{\left(\frac{\frac{c}{2}+x_{0,1}}{2}\right)^2 - \frac{d^2}{64x_{0,1}}}} \tag{13}$$

$$\textbf{Solution 4: } S_{2,4} = \sqrt{x_{0,1}} + \sqrt{-\frac{\frac{c}{2}+x_{0,1}}{2} + \sqrt{\left(\frac{\frac{c}{2}+x_{0,1}}{2}\right)^2 - \frac{d^2}{64x_{0,1}}}} - \sqrt{-\frac{\frac{c}{2}+x_{0,1}}{2} - \sqrt{\left(\frac{\frac{c}{2}+x_{0,1}}{2}\right)^2 - \frac{d^2}{64x_{0,1}}}} \tag{14}$$

If $d = 0$, and by using the expressions of $x_{0,1}$ in (Equation. 6) and c in (Equation. 2), the four solutions for (Equation. 1) are as shown in (Equation. 15), (Equation. 16), (Equation. 17) and (Equation. 18).

$$\textbf{Solution 1: } S_{3,1} = \sqrt{x_{0,1}} + \sqrt{-\left(\frac{c}{2} + x_{0,1}\right)} \tag{15}$$

$$\textbf{Solution 2: } S_{3,2} = -\sqrt{x_{0,1}} - \sqrt{-\left(\frac{c}{2} + x_{0,1}\right)} \tag{16}$$

$$\textbf{Solution 3: } S_{3,3} = -\sqrt{x_{0,1}} + \sqrt{-\left(\frac{c}{2} + x_{0,1}\right)} \tag{17}$$

$$\textbf{Solution 4: } S_{3,4} = \sqrt{x_{0,1}} - \sqrt{-\left(\frac{c}{2} + x_{0,1}\right)} \tag{18}$$

4.2. Second proposed Theorem for Fourth Degree Polynomials

This section presents the second developed theorem for fourth degree polynomials that are expressed according to the form: $x^4 + cx^2 + dx + e = 0$. This theorem identifies whether this fourth-degree equation accepts complex solutions with imaginary parts different from zero. The proof of this theorem is detailed in [13].

Theorem 2

Considering the fourth-degree polynomial equation $x^4 + cx^2 + dx + e = 0$ where all coefficients belong to the group of numbers $\mathbb{R}$. If $e \neq 0$ and $c > 0$; then, this fourth-degree polynomial equation accepts at least two complex solutions with imaginary parts different from zero.

4.3. Third proposed theorem for fourth degree polynomials

This section presents the third developed theorem to solve fourth degree polynomial equations that are expressed according to the form: $ax^4 + bx^3 + cx^2 + dx + e = 0$ where $a \neq 0$, by converting this fourth degree equation into the form of a third degree equation which we can express as follows: $y_0^3 + \frac{P}{2}\, y_0^2 + \frac{P^2-4R}{16}\, y_0 - \frac{Q^2}{64} = 0$. The proof of this theorem is detailed in [13].

Theorem 3

A fourth-degree polynomial equation under the expressed form in (Equation. 19), where coefficients belong to the group of numbers $\mathbb{R}$ and $a \neq 0$, has four solutions.

$$ax^4 + bx^3 + cx^2 + dx + e = 0 \; with \; a \neq 0 \tag{19}$$

$$P = -6\left(\frac{b}{a}\right)^2 + \frac{16c}{a}; Q = 8\left(\frac{b}{a}\right)^3 - \frac{32cb}{a^2} + \frac{64d}{a}; R = -3\left(\frac{b}{a}\right)^4 + \frac{16cb^2}{a^3} - \frac{64db}{a^2} + \frac{256e}{a} \tag{20}$$

$$y_{0,1} = -\frac{P^{;}}{3} + \frac{1}{3}\sqrt[3]{-\frac{R^{;}}{2} + \sqrt{\left(\frac{R^{;}}{2}\right)^2 + \left(\frac{Q^{;}}{3}\right)^3}} + \frac{1}{3}\sqrt[3]{-\frac{R^{;}}{2} + \sqrt{\left(\frac{R^{;}}{2}\right)^2 + \left(\frac{Q^{;}}{3}\right)^3}} \;\Big|\; \left\{P^{;} = \frac{P}{2}\;;\; R^{;} = \frac{-27Q^2 - 2P^3 + 72PR}{64}\;;\; Q^{;} = -\frac{3P^2 + 36R}{16}\right\} \tag{21}$$

If $\left(\frac{8b^3}{a^3} - \frac{32cb}{a^2} + \frac{64d}{a}\right) < 0$, and by using $y_{0,1}$ in (Equation. 21), P in (Equation. 20) and Q in (Equation. 20); the four solutions for (Equation. 19) are as shown in (Equation. 22), (Equation. 23), (Equation. 24) and (Equation. 25).

$$\textbf{Solution 1: } S_{1,1} = -\frac{b}{4a} + \frac{1}{4}\sqrt{y_{0,1}} + \frac{1}{4}\sqrt{-\frac{\frac{P}{2}+y_{0,1}}{2} + \sqrt{\left(\frac{\frac{P}{2}+y_{0,1}}{2}\right)^2 - \frac{Q^2}{64y_{0,1}}}} + \frac{1}{4}\sqrt{-\frac{\frac{P}{2}+y_{0,1}}{2} - \sqrt{\left(\frac{\frac{P}{2}+y_{0,1}}{2}\right)^2 - \frac{Q^2}{64y_{0,1}}}} \quad (22)$$

$$\textbf{Solution 2: } S_{1,2} = -\frac{b}{4a} - \frac{1}{4}\sqrt{y_{0,1}} - \frac{1}{4}\sqrt{-\frac{\frac{P}{2}+y_{0,1}}{2} + \sqrt{\left(\frac{\frac{P}{2}+y_{0,1}}{2}\right)^2 - \frac{Q^2}{64y_{0,1}}}} + \frac{1}{4}\sqrt{-\frac{\frac{P}{2}+y_{0,1}}{2} - \sqrt{\left(\frac{\frac{P}{2}+y_{0,1}}{2}\right)^2 - \frac{Q^2}{64y_{0,1}}}} \quad (23)$$

$$\textbf{Solution 3: } S_{1,3} = -\frac{b}{4a} - \frac{1}{4}\sqrt{y_{0,1}} + \frac{1}{4}\sqrt{-\frac{\frac{P}{2}+y_{0,1}}{2} + \sqrt{\left(\frac{\frac{P}{2}+y_{0,1}}{2}\right)^2 - \frac{Q^2}{64y_{0,1}}}} - \frac{1}{4}\sqrt{-\frac{\frac{P}{2}+y_{0,1}}{2} - \sqrt{\left(\frac{\frac{P}{2}+y_{0,1}}{2}\right)^2 - \frac{Q^2}{64y_{0,1}}}} \quad (24)$$

$$\textbf{Solution 4: } S_{1,4} = -\frac{b}{4a} + \frac{1}{4}\sqrt{y_{0,1}} - \frac{1}{4}\sqrt{-\frac{\frac{P}{2}+y_{0,1}}{2} + \sqrt{\left(\frac{\frac{P}{2}+y_{0,1}}{2}\right)^2 - \frac{Q^2}{64y_{0,1}}}} - \frac{1}{4}\sqrt{-\frac{\frac{P}{2}+y_{0,1}}{2} - \sqrt{\left(\frac{\frac{P}{2}+y_{0,1}}{2}\right)^2 - \frac{Q^2}{64y_{0,1}}}} \quad (25)$$

If $\left(\frac{8b^3}{a^3} - \frac{32cb}{a^2} + \frac{64d}{a}\right) > 0$, and by using $y_{0,1}$ in (Equation. 21), P in (Equation. 20) and Q in (Equation. 20); the four solutions for (Equation. 19) are as shown in (Equation. 26), (Equation. 27), (Equation. 28) and (Equation. 29).

$$\textbf{Solution 1: } S_{2,1} = -\frac{b}{4a} - \frac{1}{4}\sqrt{y_{0,1}} - \frac{1}{4}\sqrt{-\frac{\frac{P}{2}+y_{0,1}}{2} + \sqrt{\left(\frac{\frac{P}{2}+y_{0,1}}{2}\right)^2 - \frac{Q^2}{64y_{0,1}}}} - \frac{1}{4}\sqrt{-\frac{\frac{P}{2}+y_{0,1}}{2} - \sqrt{\left(\frac{\frac{P}{2}+y_{0,1}}{2}\right)^2 - \frac{Q^2}{64y_{0,1}}}} \quad (26)$$

$$\textbf{Solution 2: } S_{2,2} = -\frac{b}{4a} - \frac{1}{4}\sqrt{y_{0,1}} + \frac{1}{4}\sqrt{-\frac{\frac{P}{2}+y_{0,1}}{2} + \sqrt{\left(\frac{\frac{P}{2}+y_{0,1}}{2}\right)^2 - \frac{Q^2}{64y_{0,1}}}} + \frac{1}{4}\sqrt{-\frac{\frac{P}{2}+y_{0,1}}{2} - \sqrt{\left(\frac{\frac{P}{2}+y_{0,1}}{2}\right)^2 - \frac{Q^2}{64y_{0,1}}}} \quad (27)$$

$$\textbf{Solution 3: } S_{2,3} = -\frac{b}{4a} + \frac{1}{4}\sqrt{y_{0,1}} - \frac{1}{4}\sqrt{-\frac{\frac{P}{2}+y_{0,1}}{2} + \sqrt{\left(\frac{\frac{P}{2}+y_{0,1}}{2}\right)^2 - \frac{Q^2}{64y_{0,1}}}} + \frac{1}{4}\sqrt{-\frac{\frac{P}{2}+y_{0,1}}{2} - \sqrt{\left(\frac{\frac{P}{2}+y_{0,1}}{2}\right)^2 - \frac{Q^2}{64y_{0,1}}}} \quad (28)$$

$$\textbf{Solution 4: } S_{2,4} = -\frac{b}{4a} + \frac{1}{4}\sqrt{y_{0,1}} + \frac{1}{4}\sqrt{-\frac{\frac{P}{2}+y_{0,1}}{2} + \sqrt{\left(\frac{\frac{P}{2}+y_{0,1}}{2}\right)^2 - \frac{Q^2}{64y_{0,1}}}} - \frac{1}{4}\sqrt{-\frac{\frac{P}{2}+y_{0,1}}{2} - \sqrt{\left(\frac{\frac{P}{2}+y_{0,1}}{2}\right)^2 - \frac{Q^2}{64y_{0,1}}}} \quad (29)$$

If $\left(\frac{8b^3}{a^3} - \frac{32cb}{a^2} + \frac{64d}{a}\right) = 0$, and by using $y_{0,1}$ in (Equation. 21) and P in (Equation. 20); the four solutions for (Equation. 19) are as shown in (Equation. 30), (Equation. 31), (Equation. 32) and (Equation. 33).

$$\textbf{Solution 1: } S_{3,1} = -\frac{b}{4a} + \frac{1}{4}\sqrt{y_{0,1}} + \frac{1}{4}\sqrt{-\left(\frac{P}{2} + y_{0,1}\right)} \quad (30)$$

$$\textbf{Solution 2: } S_{3,2} = -\frac{b}{4a} - \frac{1}{4}\sqrt{y_{0,1}} - \frac{1}{4}\sqrt{-\left(\frac{P}{2} + y_{0,1}\right)} \quad (31)$$

$$\textbf{Solution 3: } S_{3,3} = -\frac{b}{4a} - \frac{1}{4}\sqrt{y_{0,1}} + \frac{1}{4}\sqrt{-\left(\frac{P}{2} + y_{0,1}\right)} \quad (32)$$

$$\textbf{Solution 4: } S_{3,4} = -\frac{b}{4a} + \frac{1}{4}\sqrt{y_{0,1}} - \frac{1}{4}\sqrt{-\left(\frac{P}{2} + y_{0,1}\right)} \tag{33}$$

4.4. Fourth proposed Theorem for Fourth Degree Polynomials

This section presents the fourth developed theorem for fourth degree polynomials that are expressed according to the form: $ax^4 + bx^2 + cx^2 + dx + e = 0$ where $a \neq 0$. This theorem identifies whether this fourth-degree equation accepts at least two complex solutions with imaginary parts different from zero by relying on the value $\left(-\frac{6b^2}{a^2} + \frac{16}{a}\right)$. The proof of this theorem is detailed in [13].

Theorem 4

Considering the polynomial equation $ax^4 + bx^3 + cx^2 + dx + e = 0$ where all coefficients belong to the group of numbers $\mathbb{R}$, if $a \neq 0$ and $e \neq 0$ and $\left(-\frac{6b^2}{a^2} + \frac{16}{a}\right) > 0$; then, this fourth degree polynomial equation accepts at least two complex solutions, where the imaginary parts are different from zero and dependent on the group of coefficients $\{a, b, c\}$.

4.5. Fifth proposed Theorem for Fourth Degree Polynomials

This section presents the fifth developed theorem for fourth degree polynomials that are expressed according to the form: $ax^4 + bx^2 + cx^2 + dx + e = 0$ where $a \neq 0$ and $e \neq 0$. This theorem identifies whether this fourth-degree equation accepts at least two complex solutions with imaginary parts different from zero by relying on the value $\left(-\frac{6d^2}{e^2} + \frac{16c}{e}\right)$. The proof of this theorem is detailed in [13].

Theorem 5

Considering the polynomial equation $ax^4 + bx^3 + cx^2 + dx + e = 0$ where all coefficients belong to the group of numbers $\mathbb{R}$, if $a \neq 0$ and $e \neq 0$ and $\left(-\frac{6d^2}{e^2} + \frac{16c}{e}\right) > 0$; then, this fourth degree polynomial equation accepts at least two complex solutions, where the imaginary parts are different from zero and dependent on the group of coefficients $\{c, d, e\}$.

3.6. Sixth proposed Theorem for Fourth Degree Polynomials

This section presents the sixth developed theorem for fourth degree polynomials that are expressed according to the form: $ax^4 + bx^2 + cx^2 + dx + e = 0$ where $a \neq 0$ and $e \neq 0$. This theorem identifies whether this fourth-degree equation accepts four complex solutions with imaginary parts different from zero by relying on the values $\left(-\frac{6b^2}{a^2} + \frac{16c}{a}\right)$ and $\left(-\frac{6d^2}{e^2} + \frac{16}{e}\right)$. The proofs of this theorem are detailed in [13].

Theorem 6

Considering the polynomial equation $ax^4 + bx^3 + cx^2 + dx + e = 0$ where all coefficients belong to the group of numbers $\mathbb{R}$, if $a \neq 0$ and $e \neq 0$ and $\left(-\frac{6b^2}{a^2} + \frac{16c}{a}\right) > 0$ and $\left(-\frac{6d^2}{e^2} + \frac{16c}{e}\right) > 0$; then, this fourth degree polynomial equation accepts four complex solutions with imaginary parts different from zero.

5. Solving Fourth Order Differential Equations

This section presents the developed theorems and formulas to solve fourth order differential equations by using the proposed methodologies in this paper to solve nth order differential equations and nth degree polynomial equations.

5.1. First proposed Theorem for Fourth Order Differential Equations

This section presents the first developed theorem to solve fourth order differential equations that are expressed according to the form: $a * f^{(4)}(x) + b * f^{(3)}(x) + c * f^{(2)}(x) + d * f^{(1)}(x) + e * f^{(0)}(x) = K$ where $a \neq 0$, by supposing that the solution is expressed according to an exponential form, then converting the fourth order differential equation into an equivalent polynomial form of fourth degree where we use the presented theorems to solve fourth degree equations in this paper.

Theorem 7

A fourth order differential equation under the expressed form in (Equation. 34) where coefficients belong to the group of numbers $\mathbb{R}$ and $a \neq 0$, has multiple solutions presented as $f(x)$ which we can express according to the exponential form shown in (Equation. 35).

$$a * f^{(4)}(x) + b * f^{(3)}(x) + c * f^{(2)}(x) + d * f^{(1)}(x) + e * f^{(0)}(x) = K \;\text{ where a} \neq 0 \tag{34}$$

$$f(x) = e^{sx+u} + v \tag{35}$$

The value of *v*, which is included in the solution *f(x)* shown in (Equation. 35), is supposed to be an arbitrary value. We can calculate the arbitrary value of *v* by using the shown expression in (Equation. 36).

$$v = \frac{K}{e} \tag{36}$$

The value of *u,* which is included in the solution *f(x)* shown in (Equation. 35), is supposed to be an arbitrary value. We can calculate the value of *u* while relying on a condition for initialization value I_0 which to be identified at the point $x = 0$. Therefore, we can use the expression $f(x = 0) = I_0$ in order to identify the arbitrary value of *u* as shown in (Equation. 37).

$$u = log\left(I_0 - \frac{K}{e}\right) \tag{37}$$

By supposing that the solution of the fourth order differential equation is expressed according to the exponential form shown in (Equation. 35); we can convert this differential equation into the form of a fourth degree equation as shown in (Equation. 38), where we can use the proposed solutions in Theorem 3 for fourth degree polynomial equations in complete forms.

$$ax^4 + bx^3 + cx^2 + dx + e = 0 \; with \; a \neq 0 \tag{38}$$

$$P = -6\left(\frac{b}{a}\right)^2 + \frac{16c}{a}; Q = 8\left(\frac{b}{a}\right)^3 - \frac{32cb}{a^2} + \frac{64d}{a}; R = -3\left(\frac{b}{a}\right)^4 + \frac{16cb^2}{a^3} - \frac{64db}{a^2} + \frac{256e}{a} \tag{39}$$

$$y_{0,1} = -\frac{P^{;}}{3} + \frac{1}{3}\sqrt[3]{-\frac{R^{;}}{2} + \sqrt{\left(\frac{R^{;}}{2}\right)^2 + \left(\frac{Q^{;}}{3}\right)^3}} + \frac{1}{3}\sqrt[3]{-\frac{R^{;}}{2} + \sqrt{\left(\frac{R^{;}}{2}\right)^2 + \left(\frac{Q^{;}}{3}\right)^3}} \mid \left\{P^{;} = \frac{P}{2} \; ; \; R^{;} = \frac{-27Q^2 - 2P^3 + 72PR}{64} \; ; \; Q^{;} = -\frac{3P^2 + 36R}{16}\right\} \tag{40}$$

$$y_{0,2} = -\frac{\frac{P}{2} + y_{0,1}}{2} + \sqrt{\left(\frac{\frac{P}{2} + y_{0,1}}{2}\right)^2 - \frac{Q^2}{64y_{0,1}}} \tag{41}$$

$$y_{0,3} = -\frac{\frac{P}{2} + y_{0,1}}{2} - \sqrt{\left(\frac{\frac{P}{2} + y_{0,1}}{2}\right)^2 - \frac{Q^2}{64y_{0,1}}} \tag{42}$$

If $\left(\frac{8b^3}{a^3} - \frac{32cb}{a^2} + \frac{64d}{a}\right) < 0$, and by using $y_{0,1}$ in (Equation. 40), $y_{0,2}$ in (Equation. 41), $y_{0,3}$ in (Equation. 42), P in (Equation. 39) and Q in (Equation. 39); the four solutions for the fourth order differential equation show in (Equation. 34) are as expressed in (Equation. 43), (Equation. 44), (Equation. 45) and (Equation. 46).

Solution 1: $$DS_{1,1}(x) = \frac{K}{e} + \left(I_0 - \frac{K}{e}\right) e^{\left[-\frac{b}{4a} + \frac{1}{4}\sqrt{y_{0,1}} + \frac{1}{4}\sqrt{y_{0,2}} + \frac{1}{4}\sqrt{y_{0,3}}\right]x} \tag{43}$$

Solution 2: $$DS_{1,2}(x) = \frac{K}{e} + \left(I_0 - \frac{k}{e}\right) e^{\left[-\frac{b}{4a} - \frac{1}{4}\sqrt{y_{0,1}} - \frac{1}{4}\sqrt{y_{0,2}} + \frac{1}{4}\sqrt{y_{0,3}}\right]x} \tag{44}$$

Solution 3: $$DS_{1,3}(x) = \frac{K}{e} + \left(I_0 - \frac{k}{e}\right) e^{\left[-\frac{b}{4a} - \frac{1}{4}\sqrt{y_{0,1}} + \frac{1}{4}\sqrt{y_{0,2}} - \frac{1}{4}\sqrt{y_{0,3}}\right]x} \tag{45}$$

Solution 4: $$DS_{1,4}(x) = \frac{K}{e} + \left(I_0 - \frac{k}{e}\right) e^{\left[-\frac{b}{4a} + \frac{1}{4}\sqrt{y_{0,1}} - \frac{1}{4}\sqrt{y_{0,2}} - \frac{1}{4}\sqrt{y_{0,3}}\right]x} \tag{46}$$

If $\left(\frac{8b^3}{a^3} - \frac{32cb}{a^2} + \frac{64d}{a}\right) > 0$, and by using $y_{0,1}$ in (Equation. 40), $y_{0,2}$ in (Equation. 41), $y_{0,3}$ in (Equation. 42), P in (Equation. 39) and Q in (Equation. 39); the four solutions for the fourth order differential equation show in (Equation. 34) are as expressed in (Equation. 47), (Equation. 48), (Equation. 49) and (Equation. 50).

Solution 1: $$DS_{2,1}(x) = \frac{K}{e} + \left(I_0 - \frac{K}{e}\right) e^{\left[-\frac{b}{4a} - \frac{1}{4}\sqrt{y_{0,1}} - \frac{1}{4}\sqrt{y_{0,2}} - \frac{1}{4}\sqrt{y_{0,3}}\right]x} \tag{47}$$

Solution 2: $$DS_{2,2}(x) = \frac{K}{e} + \left(I_0 - \frac{K}{e}\right) e^{\left[-\frac{b}{4a} - \frac{1}{4}\sqrt{y_{0,1}} + \frac{1}{4}\sqrt{y_{0,2}} + \frac{1}{4}\sqrt{y_{0,3}}\right]x} \tag{48}$$

$$\textbf{Solution 3: } DS_{2,3}(x) = \frac{K}{e} + \left(I_0 - \frac{K}{e}\right) e^{\left[-\frac{b}{4a}+\frac{1}{4}\sqrt{y_{0,1}}-\frac{1}{4}\sqrt{y_{0,2}}+\frac{1}{4}\sqrt{y_{0,3}}\right]x} \quad (49)$$

$$\textbf{Solution 4: } DS_{2,4}(x) = \frac{K}{e} + \left(I_0 - \frac{K}{e}\right) e^{\left[-\frac{b}{4a}+\frac{1}{4}\sqrt{y_{0,1}}+\frac{1}{4}\sqrt{y_{0,2}}-\frac{1}{4}\sqrt{y_{0,3}}\right]x} \quad (50)$$

If $\left(\frac{8b^3}{a^3} - \frac{32cb}{a^2} + \frac{64}{a}\right) = 0$, and by using $y_{0,1}$ in (Equation. 40) and P in (Equation. 39); the four solutions for the fourth order differential equation show in (Equation. 34) are as expressed in (Equation. 51), (Equation. 52), (Equation. 53) and (Equation. 54).

$$\textbf{Solution 1: } DS_{3,1}(x) = \frac{K}{e} + \left(I_0 - \frac{K}{e}\right) e^{\left[-\frac{b}{4a}+\frac{1}{4}\sqrt{y_{0,1}}+\frac{1}{4}\sqrt{-\left(\frac{P}{2}+y_{0,1}\right)}\right]x} \quad (51)$$

$$\textbf{Solution 2: } DS_{3,2}(x) = \frac{K}{e} + \left(I_0 - \frac{K}{e}\right) e^{\left[-\frac{b}{4a}-\frac{1}{4}\sqrt{y_{0,1}}-\frac{1}{4}\sqrt{-\left(\frac{P}{2}+y_{0,1}\right)}\right]x} \quad (52)$$

$$\textbf{Solution 3: } DS_{3,3}(x) = \frac{K}{e} + \left(I_0 - \frac{K}{e}\right) e^{\left[-\frac{b}{4a}-\frac{1}{4}\sqrt{y_{0,1}}+\frac{1}{4}\sqrt{-\left(\frac{P}{2}+y_{0,1}\right)}\right]x} \quad (53)$$

$$\textbf{Solution 4: } DS_{3,4}(x) = \frac{K}{e} + \left(I_0 - \frac{K}{e}\right) e^{\left[-\frac{b}{4a}+\frac{1}{4}\sqrt{y_{0,1}}-\frac{1}{4}\sqrt{-\left(\frac{P}{2}+y_{0,1}\right)}\right]x} \quad (54)$$

5.2. Second proposed theorem for fourth order differential equations

This section presents the second developed theorem to solve fourth order differential equations that are expressed according to the form: $a * f^{(4)}(x) + b * f^{(3)}(x) + c * f^{(2)}(x) + d * f^{(1)}(x) + e * f^{(0)}(x) = K$ where $a \neq 0$. This theorem identifies whether this fourth order differential equation accepts at least two complex functions as solution where the imaginary parts are different from zero by relying on the value $\left(-\frac{6b^2}{a^2} + \frac{16c}{a}\right)$. The proof of this theorem is relying on the proof of Theorem 4 of this paper which is detailed in [13].

Theorem 8

Considering the differential equation $a * f^{(4)}(x) + b * f^{(3)}(x) + c * f^{(2)}(x) + d * f^{(1)}(x) + e * f^{(0)}(x) = K$ where all coefficients belong to the group of numbers $\mathbb{R}$, if $a \neq 0$ and $e \neq 0$ and $\left(-\frac{6b^2}{a^2} + \frac{16c}{a}\right) > 0$; then, this fourth order differential equation accepts at least two complex functions as solutions, where the imaginary parts of these functions are different from zero and principally dependent on the group of coefficients $\{a, b, c\}$.

5.3. Third proposed theorem for fourth order differential equations

This section presents the third developed theorem to solve fourth order differential equations that are expressed according to the form: $a * f^{(4)}(x) + b * f^{(3)}(x) + c * f^{(2)}(x) + d * f^{(1)}(x) + e * f^{(0)}(x) = K$ where $a \neq 0$ and $e \neq 0$. This theorem identifies whether this fourth order differential equation accepts at least two complex functions as solutions where the imaginary parts are different from zero by relying on the value $\left(-\frac{6d^2}{e^2} + \frac{16}{e}\right)$. The proof of this theorem is relying on the proof of Theorem 5 of this paper which is detailed in [13].

Theorem 9

Considering the differential equation $a * f^{(4)}(x) + b * f^{(3)}(x) + c * f^{(2)}(x) + d * f^{(1)}(x) + e * f^{(0)}(x) = K$ where all coefficients belong to the group of numbers $\mathbb{R}$, if $a \neq 0$ and $e \neq 0$ and $\left(-\frac{6d^2}{e^2} + \frac{16c}{e}\right) > 0$; then, this fourth order differential equation accepts at least two complex functions as solutions, where the imaginary parts of these functions are different from zero and principally dependent on the group of coefficients $\{c, d, e\}$.

5.4. Fourth Proposed Theorem for Fourth Order Differential Equations

This section presents the fourth developed theorem to solve fourth order differential equations that are expressed according to the form: $a * f^{(4)}(x) + b * f^{(3)}(x) + c * f^{(2)}(x) + d * f^{(1)}(x) + e * f^{(0)}(x) = K$ where $a \neq 0$ and $e \neq 0$. This theorem identifies whether this fourth order differential equation accepts four complex functions as solutions where the imaginary parts are different from zero by relying on the values $\left(-\frac{6b^2}{a^2}+\frac{16c}{a}\right)$ and $\left(-\frac{6d^2}{e^2}+\frac{16c}{e}\right)$. The proof of this theorem is relying on the proofs of Theorem 6 of this paper which are detailed in [13].

Theorem 10

Considering the differential equation $a*f^{(4)}(x)+b*f^{(3)}(x)+c*f^{(2)}(x)+d*f^{(1)}(x)+e*f^{(0)}(x)=K$ where all coefficients belong to the group of numbers $\mathbb{R}$, if $a\neq 0$ and $e\neq 0$ and $\left(-\frac{6b^2}{a^2}+\frac{16c}{a}\right)>0$ and $\left(-\frac{6d^2}{e^2}+\frac{16c}{e}\right)>0$; then, this fourth order differential equation accepts four complex functions as solutions, where the imaginary parts of these functions are different from zero.

6. Solving Fifth Degree Polynomial Equations

This section presents the developed theorems and formulas to solve fifth degree polynomial equations by using the proposed engineering methodology in this paper to solve nth degree polynomial equations.

6.1. First proposed Theorem for Fifth Degree Polynomials

This section presents the first developed theorem to solve fifth degree polynomial equations that are expressed according to the form: $\mathrm{A}w^5+\mathrm{B}w^4+\mathrm{C}w^3+\mathrm{D}w^2+\mathrm{E}w+\mathrm{F}=0$ where $\mathrm{A}\neq 0$, by converting this quantic equation into the form of a fourth degree equation which we can express as follows: $z^4+\Gamma_3z^3+\Gamma_2z^2+\Gamma_1z+\Gamma_0=0$. The proof of this theorem is detailed in [14].

$$\mathrm{A}w^5+\mathrm{B}w^4+\mathrm{C}w^3+\mathrm{D}w^2+\mathrm{E}w+\mathrm{F}=0 \text{ with } \mathrm{A}\neq 0 \tag{55}$$

$$x^5+cx^3+dx^2+ex+f=0 \tag{56}$$

$$c=-10\frac{\mathrm{B}^2}{\mathrm{A}^2}+25\frac{\mathrm{C}}{\mathrm{A}} \tag{57}$$

$$d=20\frac{\mathrm{B}^3}{\mathrm{A}^3}-75\frac{\mathrm{CB}}{\mathrm{A}^2}+125\frac{\mathrm{D}}{\mathrm{A}} \tag{58}$$

$$e=-15\frac{\mathrm{B}^4}{\mathrm{A}^4}+75\frac{\mathrm{CB}^2}{\mathrm{A}^3}-250\frac{\mathrm{DB}}{\mathrm{A}^2}+625\frac{\mathrm{E}}{\mathrm{A}} \tag{59}$$

$$f=4\frac{\mathrm{B}^5}{\mathrm{A}^5}-25\frac{\mathrm{CB}^3}{\mathrm{A}^4}+125\frac{\mathrm{DB}^2}{\mathrm{A}^3}-625\frac{\mathrm{EB}}{\mathrm{A}^2}+3125\frac{\mathrm{F}}{\mathrm{A}} \tag{60}$$

$$z^4+\Gamma_3z^3+\Gamma_2z^2+\Gamma_1z+\Gamma_0=0 \tag{61}$$

Theorem 11

After reducing the form of fifth degree polynomial shown in (Equation. 55) to the presented form in (Equation. 56) where coefficients are expressed in (Equation. 57), (Equation. 58), (Equation. 59) and (Equation. 60); the fifth degree polynomial equation shown in (Equation. 56), where coefficients belong to the group of numbers $\mathbb{R}$, can be reduced to a fourth degree polynomial equation, which may be expressed as shown in (Equation. 61). The reduction from quantic polynomial to quartic polynomial is conducted by supposing $x=x_0x_1+x_0x_2+x_0x_3+x_1x_2+x_1x_3+x_2x_3$, whereas supposing $z=(x_0+x_1+x_2+x_3)$ is the solution for fourth degree polynomial equation in (Equation. 61) by using Theorem 3 to solve quartic polynomials and by relying on the expression $x_3=-\frac{\Gamma_3}{4}$. The variable Γ_3 is the solution for the polynomial equation shown in (Equation. 62), whereas the coefficients of this equation are shown in (Equation. 63), (Equation. 64) and (Equation. 65). The coefficients Γ_2, Γ_1 and Γ_0 of quartic equation (Equation. 61) are determined by using calculated values of Γ_3 and using the shown expressions in (Equation. 67), (Equation. 68) and (Equation. 69). As a result, we have eight calculated values as potential solutions for fifth degree polynomial equation shown in (Equation. 56), where many of them are only redundancies of others, because there are only five official solutions to determine.

The eight solutions to calculate for quantic equation (Equation. 56) are as shown in the groups (Equation. 70) and (Equation. 71).

The proposed five values as official solutions for fifth degree polynomial equation shown in (Equation. 56) are as presented in (Equation. 72), (Equation. 73), (Equation. 74), (Equation. 75) and (Equation. 76).

The proposed five values as official solutions for fifth degree polynomial equation shown in (Equation. 55) are as presented in (Equation. 77), (Equation. 78), (Equation. 79), (Equation. 80) and (Equation. 81).

The group of expressions $\{\Gamma_{3,1}\ ;\ \Gamma_{3,2}\ ; -\Gamma_{3,1}\ ;\ -\Gamma_{3,2}\}$ are the identified four values of the variable Γ_3 by using the presented expressions in (Equation. 66), which are calculated as solutions for the fourth-degree polynomial equation shown in (Equation. 62).

We use the expression $\left\{\alpha_{(1,\Gamma_{3,i})} = \frac{\Gamma_{3,i}^4 - (16d)^2/(e-c^2/4)}{4\Gamma_{3,i}^2}\right\}$ in order to simplify calculations, which allow obtaining the quartic equation shown in (Equation. 62).

The group of expressions $\left\{S^{;}_{(\Gamma_{3,1},1)}\ ;\ S^{;}_{(\Gamma_{3,1},2)}\ ;\ S^{;}_{(\Gamma_{3,1},3)}\ ;\ S^{;}_{(\Gamma_{3,1},4)}\right\}$ are the identified four solutions for the fourth degree polynomial equation shown in (Equation. 61) by using Theorem 3 to calculate these four roots nearly in parallel.

$$\lambda_2\Gamma_3^4 + \lambda_1\Gamma_3^2 + \lambda_0 = 0 \tag{62}$$

$$\lambda_2 = 1024\frac{(e-\frac{c^2}{4})}{16}; \tag{63}$$

$$\lambda_1 = 512c - 40\frac{(16d)^2}{e-\frac{c^2}{4}}; \tag{64}$$

$$\lambda_0 = -128\frac{(16\ \ )^2}{\left(e-\frac{c^2}{4}\right)^2}[f - \frac{cd}{2}]; \tag{65}$$

$$\Gamma_3 = \pm\sqrt{-\frac{P^{;}}{2} \pm \sqrt{\left(\frac{P^{;}}{2}\right)^2 - 4Q^{;}}} \mid P^{;} = \frac{\lambda_1}{\lambda_2}\ \ and\ Q^{;} = \frac{\lambda_0}{\lambda_2} \tag{66}$$

$$\Gamma_2 = \frac{(16d)^2}{2\Gamma_3^2\left(e-\frac{c^2}{4}\right)} \tag{67}$$

$$\Gamma_1 = -\frac{1}{4}\Gamma_3^3 + \frac{3(16d)^2}{16\left(e-\frac{c^2}{4}\right)\Gamma_3} \tag{68}$$

$$\Gamma_0 = -\frac{1}{16}\Gamma_3^4 - \frac{(16d)^2}{32\left(e-\frac{c^2}{4}\right)} + \frac{\left(f-\frac{cd}{2}\right)(16d)^2}{2\Gamma_3^2\left(e-\frac{c^2}{4}\right)^2} + \frac{(16d)^4}{16\Gamma_3^4\left(e-\frac{c^2}{4}\right)^2} \tag{69}$$

$$N_{\{\Gamma_{3,1}\}} = \left\{\frac{1}{2}\left[S^{;2}_{(\Gamma_{3,1},1)} - \alpha_{(1,\Gamma_{3,1})}\right], \frac{1}{2}\left[S^{;2}_{(\Gamma_{3,1},2)} - \alpha_{(1,\Gamma_{3,1})}\right], \frac{1}{2}\left[S^{;2}_{(\Gamma_{3,1},3)} - \alpha_{(1,\Gamma_{3,1})}\right], \frac{1}{2}\left[S^{;2}_{(\Gamma_{3,1},4)} - \alpha_{(1,\Gamma_{3,1})}\right]\right\} \tag{70}$$

$$N_{\{\Gamma_{3,2}\}} = \left\{\frac{1}{2}\left[S^{;2}_{(\Gamma_{3,2},1)} - \alpha_{(1,\Gamma_{3,2})}\right], \frac{1}{2}\left[S^{;2}_{(\Gamma_{3,2},2)} - \alpha_{(1,\Gamma_{3,2})}\right], \frac{1}{2}\left[S^{;2}_{(\Gamma_{3,2},3)} - \alpha_{(1,\Gamma_{3,2})}\right], \frac{1}{2}\left[S^{;2}_{(\Gamma_{3,2},4)} - \alpha_{(1,\Gamma_{3,2})}\right]\right\} \tag{71}$$

$$s_1 = \frac{1}{2}[S^{;2}_{(\Gamma_{3,1},1)} - \alpha_{(1,\Gamma_{3,1})}] \tag{72}$$

$$s_2 = \frac{1}{2}[S^{;2}_{(\Gamma_{3,1},2)} - \alpha_{(1,\Gamma_{3,1})}] \tag{73}$$

$$s_3 = \frac{1}{2}[S'^{2}_{(\Gamma_{3,1},3)} - \alpha_{(1,\Gamma_{3,1})}] \quad (74)$$

$$s_4 = \frac{1}{2}[S'^{2}_{(\Gamma_{3,1},4)} - \alpha_{(1,\Gamma_{3,1})}] \quad (75)$$

$$s_5 = -\frac{f}{s_1 s_2 s_3 s_4} \quad (76)$$

$$\text{Solution 1: } S_1 = -\frac{B}{5A} + \frac{1}{10}\left[S'^{2}_{(\Gamma_{3,1},1)} - \alpha_{(1,\Gamma_{3,1})}\right] \quad (77)$$

$$\text{Solution 2: } S_2 = -\frac{B}{5A} + \frac{1}{10}\left[S'^{2}_{(\Gamma_{3,1},2)} - \alpha_{(1,\Gamma_{3,1})}\right] \quad (78)$$

$$\text{Solution 3: } S_3 = -\frac{B}{5A} + \frac{1}{10}\left[S'^{2}_{(\Gamma_{3,1},3)} - \alpha_{(1,\Gamma_{3,1})}\right] \quad (79)$$

$$\text{Solution 4: } S_4 = -\frac{B}{5A} + \frac{1}{10}\left[S'^{2}_{(\Gamma_{3,1},4)} - \alpha_{(1,\Gamma_{3,1})}\right] \quad (80)$$

$$\text{Solution 5: } S_5 = -\frac{F}{AS_1S_2S_3S_4} \quad (81)$$

6.2. Second Proposed Theorem for Fifth Degree Polynomials

This section presents the second developed theorem to solve fifth degree polynomial equations that are expressed according to the form: $Ax^5 + Bx^4 + Cx^3 + Dx^2 + Ex + F = 0$ where $A \neq 0$, by converting this quantic equation into the form of a fourth degree equation which we can express as follows: $z^4 + \Upsilon_3 z^3 + \Upsilon_2 z^2 + \Upsilon_1 z + \Upsilon_0 = 0$. The proof of this theorem is detailed in [14]. The axe of difference in this theorem is conducting calculations on the fifth-degree polynomial shown in (Equation. 82) without eliminating the fourth-degree part by avoiding the use of the expression $x = \frac{-b+y}{5}$.

$$x^5 + bx^4 + cx^3 + dx^2 + ex + f = 0 \quad (82)$$

$$b = \frac{B}{A}; c = \frac{C}{A}; d = \frac{D}{A}; e = \frac{E}{A}; f = \frac{F}{A}.$$

Theorem 12

The fifth-degree polynomial equation shown in (Equation. 82) is reducible to the quartic equation shown in (Equation. 83), where coefficients belong to the group of numbers $\mathbb{R}$ without the need to eliminate the fourth-degree part. The reduction from fifth degree to fourth degree is conducted by supposing $x = x_0x_1 + x_0x_2 + x_0x_3 + x_1x_2 + x_1x_3 + x_2x_3$, whereas supposing $z = (x_0 + x_1 + x_2 + x_3)$ is the solution for fourth degree polynomial equation shown in (Equation. 83) by using Theorem 3. The variable Υ_3 is the solution for the polynomial equation shown in (Equation. 87) by using the expression of quadratic solution, whereas $x_3 = -\frac{\Upsilon_3}{4}$. The coefficients of shown polynomial in (Equation. 87) are as expressed in (Equation. 88), (Equation. 89) and (Equation. 90). The value of Υ_3 is equal to $\Upsilon_{3,1}$, which is presented in (Equation. 91). The coefficients Υ_2, Υ_1 and Υ_0 are determined by using calculated value of Υ_3 in (Equation. 91) and using the shown expressions in (Equation. 84), (Equation. 85) and (Equation. 86).

The five proposed solutions for polynomial equation (Equation. 82) are as shown in (Equation. 92), (Equation. 93), (Equation. 94), (Equation. 95) and (Equation. 96).

We use the expression $\left\{\alpha_{(1,\Upsilon_{3,1})} = \frac{\Upsilon_{3,1}^4 - 8b\Upsilon_{3,1}^2 - [16(d-bc)]^2/(e-\frac{c^2}{4})}{4\Upsilon_{3,1}^2}\right\}$ in order to simplify calculations, which allow obtaining the quartic equation shown in (Equation. 87).

The group of expressions $\left\{\xi_{(\Upsilon_{3,1},1)} ; \xi_{(\Upsilon_{3,1},2)} ; \xi_{(\Upsilon_{3,1},3)} ; \xi_{(\Upsilon_{3,1},4)}\right\}$ are the identified four solutions for the fourth-degree polynomial equation shown in (Equation. 83) by using Theorem 3 to calculate these four roots nearly in parallel.

$$z^4 + \Upsilon_3 z^3 + \Upsilon_2 z^2 + \Upsilon_1 z + \Upsilon_0 = 0 \tag{83}$$

$$\Upsilon_2 = \frac{[16(d-bc)]^2}{2\Upsilon_3^2\left(e-\frac{c^2}{4}\right)} + 4b \tag{84}$$

$$\Upsilon_1 = -\frac{1}{4}\Upsilon_3^3 + \frac{3[16(d-bc)]^2}{16\left(e-\frac{c^2}{4}\right)\Upsilon_3} + 4b\Upsilon_3 \tag{85}$$

$$\Upsilon_0 = -\frac{1}{16}\Upsilon_3^4 + b\Upsilon_3^2 - \frac{[16(d-bc)]^2}{32\left(e-\frac{c^2}{4}\right)} + \frac{\left(f-\frac{cd}{2}+\frac{bc^2}{4}\right)[16(d-bc)]^2}{2\Upsilon_3^2\left(e-\frac{c^2}{4}\right)^2} + \frac{b[16(d-bc)]^2}{2\Upsilon_3^2\left(e-\frac{c^2}{4}\right)} + \frac{[16(d-bc)]^4}{16\ {}_3^4\left(e-\frac{c^2}{4}\right)^2} \tag{86}$$

$$\beta_2\Upsilon_3^4 + \beta_1\Upsilon_3^2 + \beta_0 = 0 \tag{87}$$

$$\beta_2 = 1024\frac{\left(e-\frac{c^2}{4}\right)}{16(d-bc)} \tag{88}$$

$$\beta_1 = 512c - 40\frac{[16(d-bc)]^2}{e-\frac{c^2}{4}} + 1024b^2 \tag{89}$$

$$\beta_0 = -128\frac{[16(d-bc)]^2}{\left(e-\frac{c^2}{4}\right)^2}[f - \frac{cd}{2} + \frac{bc^2}{4}] + 128b\frac{[16(d-bc)]^2}{\left(e-\frac{c^2}{4}\right)} \tag{90}$$

$$\Gamma_{3,1} = \pm\sqrt{-\frac{M^{;}}{2} \pm \sqrt{\left(\frac{M^{;}}{2}\right)^2 - 4N^{;}}} \mid M^{;} = \frac{\beta_1}{\beta_2} \ \ and \ \ N^{;} = \frac{\beta_0}{\beta_2} \tag{91}$$

Solution 1: $S_1 = \frac{1}{2}\left[\xi^2_{(\Upsilon_{3,1},1)} - \alpha_{(1,\Upsilon_{3,1})}\right]$ (92)

Solution 2: $S_2 = \frac{1}{2}\left[\xi^2_{(\Upsilon_{3,1},2)} - \alpha_{(1,\Upsilon_{3,1})}\right]$ (93)

Solution 3: $S_3 = \frac{1}{2}\left[\xi^2_{(\Upsilon_{3,1},3)} - \alpha_{(1,\Upsilon_{3,1})}\right]$ (94)

Solution 4: $S_4 = \frac{1}{2}\left[\xi^2_{(\Upsilon_{3,1},4)} - \alpha_{(1,\Upsilon_{3,1})}\right]$ (95)

Solution 5: $S_5 = -\frac{f}{S_1S_2S_3S_4}$ (96)

7. Solving Fifth Order Differential Equations

This section presents the developed theorems and formulas to solve fifth order differential equations by using the proposed methodologies in this paper to solve nth order differential equations and nth degree polynomial equations.

7.1. First proposed Theorem for Fifth Order Differential Equation

This section presents the first developed theorem to solve fifth order differential equations that are expressed according to the form: $A * g^{(5)}(x) + B * g^{(4)}(x) + C * g^{(3)}(x) + D * g^{(2)}(x) + E * g^{(1)}(x) + F * g^{(0)}(x) = K$ where A $\neq 0$, by supposing that the solution is expressed according to an exponential form, then converting the fifth order differential equation into an equivalent polynomial form of fifth degree where we use the presented theorems in this paper to solve fifth degree equations.

Theorem 13

A fifth order differential equation under the expressed form in (Equation. 97) where coefficients belong to the group of numbers $\mathbb{R}$ and $A \neq 0$, has multiple solutions presented as $g_{(x)}$ which we can express according to the exponential form shown in (Equation. 98).

$$A * g^{(5)}(x) + B * g^{(4)}(x) + C * g^{(3)}(x) + D * g^{(2)}(x) + E * g^{(1)}(x) + F * g^{(0)}(x) = K \text{ with } A \neq 0 \tag{97}$$

$$g_{(x)} = e^{sx+u} + v \tag{98}$$

The value of *v*, which is included in the solution *g(x)* shown in (Equation. 98), is considered as an arbitrary value. We can calculate the arbitrary value of *v* by using shown expression in (Equation. 99).

$$v = \frac{K}{F} \tag{99}$$

The value of *u,* which is included in the solution *g(x)* shown in (Equation. 98), is considered as an arbitrary value. We can calculate the arbitrary value of *u* while relying on a condition of initialization value I_0 which to be identified at the point $x = 0$. Therefore, we can use the expression $g(x = 0) = I_0$ in order to identify the arbitrary value of *u* as shown in (Equation. 100).

$$u = log\left(I_0 - \frac{K}{F}\right) \tag{100}$$

By supposing that the solution of the fifth order differential equation is expressed according to the exponential form shown in (Equation. 98); we can convert this differential equation into the form of a fifth degree polynomial equation as shown in (Equation. 101), where we can use the proposed solutions in Theorem 11 for fifth degree polynomial equations in complete forms.

$$\mathrm{Aw^5 + Bw^4 + Cw^3 + Dw^2 + Ew + F = 0 \text{ with } A \neq 0} \tag{101}$$

$$x^5 + cx^3 + dx^2 + ex + f = 0 \tag{102}$$

$$c = -10\frac{\mathrm{B^2}}{\mathrm{A^2}} + 25\frac{\mathrm{C}}{\mathrm{A}} \tag{103}$$

$$d = 20\frac{\mathrm{B^3}}{\mathrm{A^3}} - 75\frac{\mathrm{CB}}{\mathrm{A^2}} + 125\frac{\mathrm{D}}{\mathrm{A}} \tag{104}$$

$$e = -15\frac{\mathrm{B^4}}{\mathrm{A^4}} + 75\frac{\mathrm{CB^2}}{\mathrm{A^3}} - 250\frac{\mathrm{DB}}{\mathrm{A^2}} + 625\frac{\mathrm{E}}{\mathrm{A}} \tag{105}$$

$$f = 4\frac{\mathrm{B^5}}{\mathrm{A^5}} - 25\frac{\mathrm{CB^3}}{\mathrm{A^4}} + 125\frac{\mathrm{DB^2}}{\mathrm{A^3}} - 625\frac{\mathrm{EB}}{\mathrm{A^2}} + 3125\frac{\mathrm{F}}{\mathrm{A}} \tag{106}$$

$$z^4 + \Gamma_3 z^3 + \Gamma_2 z^2 + \Gamma_1 z + \Gamma_0 = 0 \tag{107}$$

We use Theorem 11 in this paper to solve the fifth degree polynomial equation shown in (Equation. 101).

After reducing the form of fifth degree polynomial shown in (Equation. 101) to the presented form in (Equation. 102) where coefficients are expressed in (Equation. 103), (Equation. 104), (Equation. 105) and (Equation. 106); the fifth degree polynomial equation shown in (Equation. 102), where coefficients belong to the group of numbers ℝ, can be reduced to a fourth degree polynomial equation, which may be expressed as shown in (Equation. 107). The reduction from quantic polynomial to quartic polynomial is conducted by supposing $x = x_0x_1 + x_0x_2 + x_0x_3 + x_1x_2 + x_1x_3 + x_2x_3$, whereas supposing $z = (x_0 + x_1 + x_2 + x_3)$ is the solution for fourth degree polynomial equation in (Equation. 107) by using Theorem 3 to solve quartic polynomials and by relying on the expression $x_3 = -\frac{\Gamma_3}{4}$. The variable Γ_3 is the solution for the polynomial equation shown in (Equation. 108), whereas the coefficients of this equation are shown in (Equation. 109), (Equation. 110) and (Equation. 111). The coefficients Γ_2, Γ_1 and Γ_0 of quartic equation (Equation. 107) are determined by using calculated values of Γ_3 and using the shown expressions in (Equation. 113), (Equation. 114) and (Equation. 115).

The proposed five values as official solutions for fifth degree polynomial equation shown in (Equation. 101) are as presented in (Equation. 116), (Equation. 117), (Equation. 118), (Equation. 119) and (Equation. 120).

The proposed five functions as official solutions for fifth order differential equation shown in (Equation. 97) are as presented in (Equation. 121), (Equation. 122), (Equation. 123), (Equation. 124) and (Equation. 125).

The group of expressions $\{\Gamma_{3,1}\ ;\ \Gamma_{3,2}\ ; -\Gamma_{3,1}\ ;\ -\Gamma_{3,2}\}$ are the identified four values of the variable Γ_3 by using the presented expressions in (Equation. 112), which are calculated as solutions for the fourth-degree polynomial equation shown in (Equation. 108).

We use the expression $\left\{\alpha_{(1,\Gamma_{3,i})} = \frac{\Gamma_{3,i}^4 - (16d)^2/(e-c^2/4)}{4\Gamma_{3,i}^2}\right\}$ in order to simplify calculations, which allow obtaining the quartic equation shown in (Equation. 108).

The group of expressions $\left\{S^{;}_{(\Gamma_{3,1},1)}\ ;\ S^{;}_{(\Gamma_{3,1},2)}\ ;\ S^{;}_{(\Gamma_{3,1},3)}\ ;\ S^{;}_{(\Gamma_{3,1},4)}\right\}$ are the identified four solutions for the fourth degree polynomial equation shown in (Equation. 107) by using Theorem 3 to calculate these four roots nearly in parallel.

$$\lambda_2\Gamma_3^4 + \lambda_1\Gamma_3^2 + \lambda_0 = 0 \tag{108}$$

$$\lambda_2 = 1024\frac{(e-\frac{c^2}{4})}{16d}; \tag{109}$$

$$\lambda_1 = 512c - 40\frac{(16d)^2}{e-\frac{c^2}{4}}; \tag{110}$$

$$\lambda_0 = -128\frac{(16d)^2}{\left(e-\frac{c^2}{4}\right)^2}[f - \frac{cd}{2}]; \tag{111}$$

$$\Gamma_3 = \pm\sqrt{-\frac{P^{;}}{2} \pm \sqrt{\left(\frac{P^{;}}{2}\right)^2 - 4Q^{;}}}\ |\ P^{;} = \frac{\lambda_1}{\lambda_2}\ \ and\ Q^{;} = \frac{\lambda_0}{\lambda_2} \tag{112}$$

$$\Gamma_2 = \frac{(16d)^2}{2\Gamma_3^2\left(e-\frac{c^2}{4}\right)} \tag{113}$$

$$\Gamma_1 = -\frac{1}{4}\Gamma_3^3 + \frac{3(16d)^2}{16\left(e-\frac{c^2}{4}\right)\Gamma_3} \tag{114}$$

$$\Gamma_0 = -\frac{1}{16}\Gamma_3^4 - \frac{(16d)^2}{32\left(e-\frac{c^2}{4}\right)} + \frac{\left(f-\frac{cd}{2}\right)(16d)^2}{2\Gamma_3^2\left(e-\frac{c^2}{4}\right)^2} + \frac{(16d)^4}{16\Gamma_3^4\left(e-\frac{c^2}{4}\right)^2} \tag{115}$$

$$S_1 = -\frac{B}{5A} + \frac{1}{10}\left[S^{;2}_{(\Gamma_{3,1},1)} - \alpha_{(1,\Gamma_{3,1})}\right] \tag{116}$$

$$S_2 = -\frac{B}{5A} + \frac{1}{10}\left[S^{;2}_{(\Gamma_{3,1},2)} - \alpha_{(1,\Gamma_{3,1})}\right] \tag{117}$$

$$S_3 = -\frac{B}{5A} + \frac{1}{10}\left[S^{;2}_{(\Gamma_{3,1},3)} - \alpha_{(1,\Gamma_{3,1})}\right] \tag{118}$$

$$S_4 = -\frac{B}{5A} + \frac{1}{10}\left[S^{;2}_{(\Gamma_{3,1},4)} - \alpha_{(1,\Gamma_{3,1})}\right] \tag{119}$$

$$S_5 = -\frac{F}{AS_1S_2S_3S_4} \tag{120}$$

$$\text{Solution 1: } DS_1 = \frac{K}{F} + \left(I_0 - \frac{K}{F}\right) e^{\left[-\frac{\mathrm{B}}{5\mathrm{A}} + \frac{1}{10}\left[\mathrm{S}^{;2}_{(\Gamma_{3,1},1)} - \alpha_{(1,\Gamma_{3,1})}\right]\right]x} \tag{121}$$

$$\text{Solution 2: } DS_2 = \frac{K}{F} + \left(I_0 - \frac{K}{F}\right) e^{\left[-\frac{\mathrm{B}}{5\mathrm{A}} + \frac{1}{10}\left[\mathrm{S}^{;2}_{(\Gamma_{3,1},2)} - \alpha_{(1,\Gamma_{3,1})}\right]\right]x} \tag{122}$$

$$\text{Solution 3: } DS_3 = \frac{K}{F} + \left(I_0 - \frac{K}{F}\right) e^{\left[-\frac{\mathrm{B}}{5\mathrm{A}} + \frac{1}{10}\left[\mathrm{S}^{;2}_{(\Gamma_{3,1},3)} - \alpha_{(1,\Gamma_{3,1})}\right]\right]x} \tag{123}$$

$$\text{Solution 4: } DS_4 = \frac{K}{F} + \left(I_0 - \frac{K}{F}\right) e^{\left[-\frac{\mathrm{B}}{5\mathrm{A}} + \frac{1}{10}\left[\mathrm{S}^{;2}_{(\Gamma_{3,1},4)} - \alpha_{(1,\Gamma_{3,1})}\right]\right]x} \tag{124}$$

$$\text{Solution 5: } DS_5 = \frac{K}{F} + \left(I_0 - \frac{K}{F}\right) e^{\left[\frac{-\mathrm{F}}{\mathrm{A}\mathrm{S}_1\mathrm{S}_2\mathrm{S}_3\mathrm{S}_4}\right]x} \tag{125}$$

7.2. Second proposed Theorem for Fifth Order Differential Equation

This section presents the second developed theorem to solve fifth order differential equations that are expressed according to the form: $A * g^{(5)}(x) + B * g^{(4)}(x) + C * g^{(3)}(x) + D * g^{(2)}(x) + E * g^{(1)}(x) + F * g^{(0)}(x) = K$ where $\mathrm{A} \neq 0$, by supposing that the solution is expressed according to an exponential form, then converting the fifth order differential equation into an equivalent polynomial form of fifth degree where we use the presented theorems to solve fifth degree equations in this paper. The advantage of this second theorem is avoiding the elimination of the fourth order part from the equation.

Theorem 14

A fifth order differential equation under the expressed form in (Equation. 126) where coefficients belong to the group of numbers $\mathbb{R}$ and $A \neq 0$, has multiple solutions presented as $g(x)$ which we can express according to the exponential form shown in (Equation. 127).

$$A * g^{(5)}(x) + B * g^{(4)}(x) + C * g^{(3)}(x) + D * g^{(2)}(x) + E * g^{(1)}(x) + F * g^{(0)}(x) = K \text{ with } \mathrm{A} \neq 0 \tag{126}$$

$$g(x) = e^{sx+u} + v \tag{127}$$

The value of *v*, which is included in the solution *g(x)* shown in (Equation. 127), is considered as an arbitrary value. We can calculate the arbitrary value of *v* by using shown expression in (Equation. 128).

$$v = \frac{K}{F} \tag{128}$$

The value of *u,* which is included in the solution *g(x)* shown in (Equation. 127), is considered as an arbitrary value. We can calculate the arbitrary value of *u* while relying on a condition of initialization value I_0 which to be identified at the point $x = 0$. Therefore, we can use the expression $g(x = 0) = I_0$ in order to identify the arbitrary value of *u* as shown in (Equation. 129).

$$u = log\left(I_0 - \frac{K}{F}\right) \tag{129}$$

By supposing that the solution of the fifth order differential equation is expressed according to the exponential form shown in (Equation. 127); we can convert this differential equation into the form of a fifth degree polynomial equation as shown in (Equation. 130), where we can use the proposed solutions in Theorem 12 for fifth degree polynomial equations in complete forms without eliminating the fourth degree part by avoiding the use of the expression $x = (-b + y)/5$.

$$x^5 + bx^4 + cx^3 + dx^2 + ex + f = 0 \tag{130}$$

$$b = \frac{\mathrm{B}}{\mathrm{A}}; c = \frac{\mathrm{C}}{\mathrm{A}}; d = \frac{\mathrm{D}}{\mathrm{A}}; e = \frac{\mathrm{E}}{\mathrm{A}}; f = \frac{\mathrm{F}}{\mathrm{A}}.$$

We use Theorem 12 in this paper to solve the fifth degree polynomial equation shown in (Equation. 130).

The fifth-degree polynomial equation shown in (Equation. 130) is reducible to the quartic equation shown in (Equation. 131), where coefficients belong to the group of numbers $\mathbb{R}$ without the need to eliminate the fourth-degree part. The reduction from fifth degree to fourth degree is conducted by supposing $x = x_0x_1 + x_0x_2 + x_0x_3 + x_1x_2 + x_1x_3 + x_2x_3$, whereas supposing $z = (x_0 + x_1 + x_2 + x_3)$ is the solution for fourth degree polynomial equation shown in (Equation. 131) by using Theorem 3. The variable Υ_3 is the solution for the polynomial equation shown in (Equation. 135) by using the expression of quadratic solution, whereas $x_3 = -\frac{\Upsilon_3}{4}$. The coefficients of shown polynomial in (Equation. 135) are as expressed in (Equation. 136), (Equation. 137) and (Equation. 138). The value of Υ_3 is equal to $\Upsilon_{3,1}$, which is presented in (Equation. 139). The coefficients Υ_2, Υ_1 and Υ_0 are determined by using calculated value of Υ_3 in (Equation. 139) and using the shown expressions in (Equation. 132), (Equation. 133) and (Equation. 134).

The five solutions for polynomial equation (Equation. 130) are as shown in (Equation. 140), (Equation. 141), (Equation. 142), (Equation. 143) and (Equation. 144).

The five proposed solutions for fifth order differential equation (Equation. 126) are as shown in (Equation. 145), (Equation. 146), (Equation. 147), (Equation. 148) and (Equation. 149).

We use the expression $\left\{\alpha_{(1,\Upsilon_{3,1})} = \frac{\Upsilon_{3,1}^4 - 8b\Upsilon_{3,1}^2 - [16(d-bc)]^2/(e-\frac{c^2}{4})}{4\Upsilon_{3,1}^2}\right\}$ in order to simplify calculations, which allow obtaining the quartic equation shown in (Equation. 135).

The group of expressions $\left\{\xi_{(\Upsilon_{3,1},1)}\ ;\ \xi_{(\Upsilon_{3,1},2)}\ ;\ \xi_{(\Upsilon_{3,1},3)}\ ;\ \xi_{(\Upsilon_{3,1},4)}\right\}$ are the identified four solutions for the fourth-degree polynomial equation shown in (Equation. 131) by using Theorem 3 to calculate these four roots nearly in parallel.

$$z^4 + \Upsilon_3 z^3 + \Upsilon_2 z^2 + \Upsilon_1 z + \Upsilon_0 = 0 \tag{131}$$

$$\Upsilon_2 = \frac{[16(d-bc)]^2}{2\Upsilon_3^2\left(e-\frac{c^2}{4}\right)} + 4b \tag{132}$$

$$\Upsilon_1 = -\frac{1}{4}\Upsilon_3^3 + \frac{3[16(d-bc)]^2}{16\left(e-\frac{c^2}{4}\right)\Upsilon_3} + 4b\Upsilon_3 \tag{133}$$

$$\Upsilon_0 = -\frac{1}{16}\Upsilon_3^4 + b\Upsilon_3^2 - \frac{[16(d-bc)]^2}{32\left(e-\frac{c^2}{4}\right)} + \frac{\left(f-\frac{cd}{2}+\frac{bc^2}{4}\right)[16(d-bc)]^2}{2\Upsilon_3^2\left(e-\frac{c^2}{4}\right)^2} + \frac{b[16(d-bc)]^2}{2\Upsilon_3^2\left(e-\frac{c^2}{4}\right)} + \frac{[16(d-bc)]^4}{16\ {}_3^4\left(e-\frac{c^2}{4}\right)^2} \tag{134}$$

$$\beta_2\Upsilon_3^4 + \beta_1\Upsilon_3^2 + \beta_0 = 0 \tag{135}$$

$$\beta_2 = 1024\frac{\left(e-\frac{c^2}{4}\right)}{16(d-bc)} \tag{136}$$

$$\beta_1 = 512c - 40\frac{[16(d-bc)]^2}{e-\frac{c^2}{4}} + 1024b^2 \tag{137}$$

$$\beta_0 = -128\frac{[16(d-bc)]^2}{\left(e-\frac{c^2}{4}\right)^2}\left[f - \frac{cd}{2} + \frac{bc^2}{4}\right] + 128b\frac{[16(d-bc)]^2}{\left(e-\frac{c^2}{4}\right)} \tag{138}$$

$$\Gamma_{3,1} = \sqrt{-\frac{M^{;}}{2} + \sqrt{\left(\frac{M^{;}}{2}\right)^2 - 4N^{;}}}\ \mid M^{;} = \frac{\beta_1}{\beta_2}\ \ and\ \ N^{;} = \frac{\beta_0}{\beta_2} \tag{139}$$

$$S_1 = \frac{1}{2}\left[\xi^2_{(\Upsilon_{3,1},1)} - \alpha_{(1,\Upsilon_{3,1})}\right] \quad (140)$$

$$S_2 = \frac{1}{2}\left[\xi^2_{(\Upsilon_{3,1},2)} - \alpha_{(1,\Upsilon_{3,1})}\right] \quad (141)$$

$$S_3 = \frac{1}{2}\left[\xi^2_{(\Upsilon_{3,1},3)} - \alpha_{(1,\Upsilon_{3,1})}\right] \quad (142)$$

$$S_4 = \frac{1}{2}\left[\xi^2_{(\Upsilon_{3,1},4)} - \alpha_{(1,\Upsilon_{3,1})}\right] \quad (143)$$

$$S_5 = -\frac{f}{S_1 S_2 S_3 S_4} \quad (144)$$

Solution 1: $$DS_1 = \frac{K}{F} + \left(I_0 - \frac{K}{F}\right) e^{\frac{1}{2}\left[\xi^2_{(\Upsilon_{3,1},1)} - \alpha_{(1,\Upsilon_{3,1})}\right]x} \quad (145)$$

Solution 2: $$DS_2 = \frac{K}{F} + \left(I_0 - \frac{K}{F}\right) e^{\frac{1}{2}\left[\xi^2_{(\Upsilon_{3,1},1)} - \alpha_{(1,\Upsilon_{3,1})}\right]x} \quad (146)$$

Solution 3: $$DS_3 = \frac{K}{F} + \left(I_0 - \frac{K}{F}\right) e^{\frac{1}{2}\left[\xi^2_{(\Upsilon_{3,1},1)} - \alpha_{(1,\Upsilon_{3,1})}\right]x} \quad (147)$$

Solution 4: $$DS_4 = \frac{K}{F} + \left(I_0 - \frac{K}{F}\right) e^{\frac{1}{2}\left[\xi^2_{(\Upsilon_{3,1},1)} - \alpha_{(1,\Upsilon_{3,1})}\right]x} \quad (148)$$

Solution 5: $$DS_5 = \frac{K}{F} + \left(I_0 - \frac{K}{F}\right) e^{\left[\frac{-\mathrm{f}}{\mathrm{S_1 S_2 S_3 S_4}}\right]x} \quad (149)$$

8. Solving Sixth Degree Polynomial Equations

This section presents the developed theorems and formulas to solve sixth degree polynomial equations by using the proposed engineering methodology in this paper to solve nth degree polynomial equations.

8.1. First proposed Theorem for Sixth Degree Polynomials

This section presents the first developed theorem to solve sixth degree polynomial equations that are expressed according to the form: $\mathrm{A}x^6 + \mathrm{B}x^5 + \mathrm{C}x^4 + \mathrm{D}x^3 + \mathrm{E}x^2 + \mathrm{F}x + \mathrm{G} = 0$ where $\mathrm{A} \neq 0$ and $\mathrm{B} \neq 0$, by converting the sixth degree polynomial into the form of a fourth degree which we can express as follows: $z^4 + \Gamma_3 z^3 + \Gamma_2 z^2 + \Gamma_1 z + \Gamma_0 = 0$. The proof of this theorem is detailed in [15].

$$\mathrm{A}x^6 + \mathrm{B}x^5 + \mathrm{C}x^4 + \mathrm{D}x^3 + \mathrm{E}x^2 + \mathrm{F}x + \mathrm{G} = 0 \; with \; \mathrm{A} \neq 0 \; and \; \mathrm{B} \neq 0 \quad (150)$$

$$x^6 + bx^5 + cx^4 + dx^3 + ex^2 + fx + g = 0 \; with \; b \neq 0 \quad (151)$$

$$b = \frac{\mathrm{B}}{\mathrm{A}};\; c = \frac{\mathrm{C}}{\mathrm{A}};\; d = \frac{\mathrm{D}}{\mathrm{A}};\; e = \frac{\mathrm{E}}{\mathrm{A}};\; f = \frac{\mathrm{F}}{\mathrm{A}};\; g = \frac{\mathrm{G}}{\mathrm{A}}; \quad (152)$$

$$z^4 + \Gamma_3 z^3 + \Gamma_2 z^2 + \Gamma_1 z + \Gamma_0 = 0 \quad (153)$$

Theorem 15

After reducing the form of sixth degree polynomial shown in (Equation. 150) to the presented form in (Equation. 151) where coefficients are as expressed in (Equation. 152); the sixth-degree polynomial equation shown in (Equation. 151), where coefficients belong to the group of numbers ℝ, can be reduced to a fourth-degree polynomial equation, which may be expressed as shown in (Equation. 153). The reduction from sixth degree polynomial to quartic polynomial is conducted by supposing $x = x_0x_1 + x_0x_2 + x_0x_3 + x_1x_2 + x_1x_3 + x_2x_3$, whereas supposing $z = (x_0 + x_1 + x_2 + x_3)$ is the solution for fourth degree polynomial equation in (Equation. 153) by using Theorem 3 and relying on the expression $x_3 = -\frac{\Gamma_3}{4}$. The variable Γ_3 is defined as shown in (Equation. 154) where α_3 is presented in (Equation. 155) and Γ_4 is the solution for the polynomial equation (Equation. 156), which relies on the coefficients (Equation. 157), (Equation. 158), (Equation. 159) and (Equation. 160). The shown coefficients in (Equation. 157), (Equation. 158), (Equation. 159) and (Equation. 160) are expressed by using the constant V which is presented in (Equation. 161). The coefficients Γ_3, Γ_2, Γ_1 and Γ_0 of quartic equation (Equation. 153), which is used to calculate z, are determined by using the shown expressions in (Equation. 154), (Equation. 163), (Equation. 164) and (Equation. 165) while using calculated values of Γ_4 and V. As a result, we have twelve calculated values as potential solutions for sixth degree polynomial equation shown in (Equation. 151), where many of them are only redundancies of others, because there are only six official solutions to determine.

The twelve solutions to calculate for sixth degree equation (Equation. 151) are as shown in the groups (Equation. 166), (Equation. 167) and (Equation. 168). The proposed six values as official solutions for sixth degree polynomial equation shown in (Equation. 151) are as presented in (Equation. 169), (Equation. 170), (Equation. 171), (Equation. 172), (Equation. 173) and (Equation. 174).

The group of expressions $\{\pm\Gamma_{4,1}\ ;\ \pm\Gamma_{4,2}\ ;\ \pm\Gamma_{4,3}\}$ are the identified values of the variable Γ_4, which are calculated as solutions for the sixth-degree polynomial equation shown in (Equation. 156) by using the solution of third degree equations which is presented in (Equation. 162).

We use the expressions $\left\{\alpha_{(1,\Gamma_{4,i})} = \frac{\Gamma_{4,i}^4 + \frac{32\Gamma_{4,i}^4}{V^2b^2} - \frac{8\Gamma_{4,i}^4}{Vb} + \frac{12d\Gamma_{4,i}^2}{b^2} - \frac{8c\Gamma_{4,i}^2}{b} - \frac{V^2\left(f - \frac{d^2}{4b}\right)}{b}}{4\Gamma_{4,i}^2}\right\}$ in order to simplify calculations, which allow obtaining the shown equation in (Equation. 156).

The value of V shown in (Equation. 161) is used to simplify expressing the formulas during calculations where $\frac{\Upsilon_4}{\alpha_3} = V$.

The group of expressions $\left\{\dot{S}_{(\Gamma_{4,1},1)}\ ;\ \dot{S}_{(\Gamma_{4,1},2)}\ ;\ \dot{S}_{(\Gamma_{4,1},3)}\ ;\ \dot{S}_{(\Gamma_{4,1},4)}\right\}$ are the identified four solutions for the fourth-degree polynomial equation shown in (Equation. 153) by using Theorem 3 to calculate these four roots nearly in parallel.

$$\Gamma_3 = \frac{4\alpha_3}{b} + \Gamma_4 \tag{154}$$

$$\alpha_3 = -\frac{\frac{4\Gamma_4\left(f - \frac{d^2}{4b}\right)}{b}}{\frac{32f}{b^2} + \frac{40d^2}{b^3} - \frac{64cd}{b^2} + \frac{64e}{b}} \tag{155}$$

$$\lambda_3\Gamma_4^6 + \lambda_2\Gamma_4^4 + \lambda_1\Gamma_4^2 + \lambda_0 = 0 \tag{156}$$

$$\lambda_3 = -\frac{40960}{V^4b^4} + \frac{16384}{V^3b^3} - \frac{1536}{V^2b^2} \tag{157}$$

$$\lambda_2 = -\frac{245}{V^2b^4} + \frac{16384c}{V^2b^3} + \frac{3072d}{Vb^3} - \frac{2048c}{Vb^2} + \frac{1024}{V} \tag{158}$$

$$\lambda_1 = -\frac{512d}{b} + \frac{1536f}{b^3} + \frac{28V^2f}{b} - \frac{7V^2d^2}{b^2} + \frac{96Vf}{b^2} - \frac{168d^2V}{b^3} + \frac{192cdV}{b^2} - \frac{192Ve}{b} - \frac{3456\ \ ^2}{b^4} + \frac{4096cd}{b^3} - \frac{1024e}{b^2} - \frac{1024c^2}{b^2} \tag{159}$$

$$\lambda_0 = -\frac{64\ \ ^2d^3}{b^4} + \frac{64cd^2V^2}{b^3} - \frac{64eV^2d}{b^2} + \frac{128V^2g}{b} + \frac{192V^2df}{b^3} - \frac{128V^2cf}{b^2} \tag{160}$$

$$V = -\frac{\frac{32f}{b^2}+\frac{40d^2}{b^3}-\frac{64cd}{b^2}+\frac{64e}{b}}{4\frac{\left(f-\frac{d^2}{4b}\right)}{b}} \tag{161}$$

$$\Gamma_{4,1}^2 = \frac{-b^{;}}{3} + \frac{1}{3}\sqrt[3]{-\frac{D^{;}}{2}+\sqrt{\left(\frac{D^{;}}{2}\right)^2+\left(\frac{C^{;}}{3}\right)^3}} + \frac{1}{3}\sqrt[3]{-\frac{D^{;}}{2}-\sqrt{\left(\frac{D^{;}}{2}\right)^2+\left(\frac{C^{;}}{3}\right)^3}} \mid \left\{b^{;} = \frac{\lambda_2}{\lambda_3}, c^{;} = \frac{\lambda_1}{\lambda_3} \ and \ d^{;} = \frac{\lambda_0}{\lambda_3}\right\}\left\{D^{;} = 27d^{;} + 2b^{;3} - 9c^{;}\, b^{;} \ and \ C^{;} = 9c^{;} - 3b^{;2}\right\} \tag{162}$$

$$\Gamma_2 = \frac{8\Gamma_4^2}{Vb} - \frac{6d}{b^2} + \frac{4c}{b} + \frac{\left(f-\frac{d^2}{4b}\right)V^2}{2b\Gamma_4^2} - \frac{8\Gamma_4^2}{V^2b^2} \tag{163}$$

$$\Gamma_1 = \frac{5\Gamma_4^3}{Vb} + \frac{3Vd^2}{4b^3\Gamma_4} - \frac{6d\Gamma_4}{b^2} + \frac{4c\Gamma_4}{b} - \frac{dcV}{\mathrm{b}^2\Gamma_4} + \frac{eV}{b\Gamma_4} - \frac{\Gamma_4^3}{4} - \frac{8\Gamma_4^3}{V^2b^2} + \frac{f-\frac{d^2}{4b}}{4\Gamma_4 b}V^2 \tag{164}$$

$$\Gamma_0 = \frac{\Gamma_4^4}{2Vb} - \frac{V^2d^3}{16\ {}^4\Gamma_4^2} + \frac{3Vd^2}{8b^3} - \frac{3d\Gamma_4^2}{4b^2} + \frac{c\Gamma_4^2}{2b} + \frac{cd^2V^2}{8\mathrm{b}^3\Gamma_4^2} - \frac{cdV}{2b^2} + \frac{eV}{2b} - \frac{eV^2d}{4b^2\Gamma_4^2} + \frac{gV^2}{2b\Gamma_4^2} - \left(\frac{\Gamma_4^2}{4} + V^2\frac{f-\frac{d^2}{4b}}{4b\Gamma_4^2}\right)\left(\frac{\Gamma_4^2}{4} + \frac{8\Gamma_4^2}{V^2b^2} - \frac{2\Gamma_4^2}{VB} + \frac{3d}{b^2} - \frac{2c}{b} - \frac{\left(f-\frac{d^2}{4b}\right)}{4b\Gamma_4^2}V^2\right) \tag{165}$$

$$N_{\{\Gamma_{4,1}\}} = \left\{\frac{1}{2}\left[\dot{S}^2_{(\Gamma_{4,1},1)} - \alpha_{(1,\Gamma_{4,1})}\right], \frac{1}{2}\left[\dot{S}^2_{(\Gamma_{4,1},2)} - \alpha_{(1,\Gamma_{4,1})}\right], \frac{1}{2}\left[\dot{S}^2_{(\Gamma_{4,1},3)} - \alpha_{(1,\Gamma_{4,1})}\right], \frac{1}{2}\left[\dot{S}^2_{(\Gamma_{4,1},4)} - \alpha_{(1,\Gamma_{4,1})}\right]\right\} \tag{166}$$

$$N_{\{\Gamma_{4,2}\}} = \left\{\frac{1}{2}\left[\dot{S}^2_{(\Gamma_{4,2},1)} - \alpha_{(1,\Gamma_{4,2})}\right], \frac{1}{2}\left[\dot{S}^2_{(\Gamma_{4,2},2)} - \alpha_{(1,\Gamma_{4,2})}\right], \frac{1}{2}\left[\dot{S}^2_{(\Gamma_{4,2},3)} - \alpha_{(1,\Gamma_{4,2})}\right], \frac{1}{2}\left[\dot{S}^2_{(\Gamma_{4,2},4)} - \alpha_{(1,\Gamma_{4,2})}\right]\right\} \tag{167}$$

$$N_{\{\Gamma_{4,3}\}} = \left\{\frac{1}{2}\left[\dot{S}^2_{(\Gamma_{4,3},1)} - \alpha_{(1,\Gamma_{4,3})}\right], \frac{1}{2}\left[\dot{S}^2_{(\Gamma_{4,3},2)} - \alpha_{(1,\Gamma_{4,3})}\right], \frac{1}{2}\left[\dot{S}^2_{(\Gamma_{4,3},3)} - \alpha_{(1,\Gamma_{4,3})}\right], \frac{1}{2}\left[\dot{S}^2_{(\Gamma_{4,3},4)} - \alpha_{(1,\Gamma_{4,3})}\right]\right\} \tag{168}$$

$$S_1 = \frac{1}{2}\left[\dot{S}^2_{(\Gamma_{4,1},1)} - \alpha_{(1,\Gamma_{4,1})}\right] \tag{169}$$

$$S_2 = \frac{1}{2}\left[\dot{S}^2_{(\Gamma_{4,1},2)} - \alpha_{(1,\Gamma_{4,1})}\right] \tag{170}$$

$$S_3 = \frac{1}{2}\left[\dot{S}^2_{(\Gamma_{4,1},3)} - \alpha_{(1,\Gamma_{4,1})}\right] \tag{171}$$

$$S_4 = \frac{1}{2}\left[\dot{S}^2_{(\Gamma_{4,1},4)} - \alpha_{(1,\Gamma_{4,1})}\right] \tag{172}$$

$$S_5 = -\frac{b+S_1+S_2+S_3+S_4}{2} - \sqrt{\left(\frac{b+S_1+S_2+S_3+S_4}{2}\right)^2 - \frac{g}{S_1S_2S_3S_4}} \tag{173}$$

$$S_6 = -\frac{b+S_1+S_2+S_3+S_4}{2} + \sqrt{\left(\frac{b+S_1+S_2+S_3+S_4}{2}\right)^2 - \frac{g}{S_1S_2S_3S_4}} \tag{174}$$

8.2. Second Proposed Theorem for Sixth Degree Polynomials

This section presents the second developed theorem to solve sixth degree polynomial equations that are expressed according to the form: $Aw^6 + Cw^4 + Dw^3 + Ew^2 + Fw + G = 0$ whith $\mathrm{A} \neq 0$ whereas the coefficient of the fifth-degree part is equal zero. Solving this sixth degree equation is based on using the expression $w = \sqrt{\frac{-C}{15}} + x$ to induce a fifth degree part, then converting the result into the form of a fourth degree equation which we can express as follows: $z^4 + \Upsilon_3 z^3 + \Upsilon_2 z^2 + \Upsilon_1 z + \Upsilon_0 = 0$. The proof of this theorem is detailed in [15].

$$x^6 + bx^5 + dx^3 + ex^2 + fx + g = 0 \tag{175}$$

$$b = 6\sqrt{\frac{-C}{15A}} \tag{176}$$

$$d = \frac{8C}{3A}\sqrt{\frac{-C}{15A}} + \frac{D}{A} \tag{177}$$

$$e = \frac{-C^2}{3A^2} + \frac{3D}{A}\sqrt{\frac{-C}{15A}} + \frac{E}{A} \tag{178}$$

$$f = -\frac{18C^2}{5A^2}\sqrt{\frac{-C}{15A}} - \frac{DC}{5A^2} + \frac{2E}{A}\sqrt{\frac{-C}{15}} + \frac{F}{A} \tag{179}$$

$$g = \frac{-16C^3}{3375A^3} - \frac{DC}{15A^2}\sqrt{\frac{-C}{15A}} - \frac{EC}{15A^2} + \frac{F}{A}\sqrt{\frac{-C}{15A}} + \frac{G}{A} \tag{180}$$

Theorem 16

In order to reduce the sixth degree polynomial equation $Aw^6 + Cw^4 + Dw^3 + Ew^2 + Fw + G = 0 \; with \; A \neq 0$ to the quartic equation shown in (Equation. 181), where coefficients belong to the group of numbers $\mathbb{R}$ we first replace w with $\left(w = \sqrt{\frac{-C}{15A}} + x\right)$ in order to obtain the fifth degree equation shown in (Equation. 175) where the coefficients are as expressed in (Equation. 176), (Equation. 177), (Equation. 178), (Equation. 179) and (Equation. 180). Then, the reduction from sixth degree to fourth degree is conducted by supposing $x = (x_0x_1 + x_0x_2 + x_0x_3 + x_1x_2 + x_1x_3 + x_2x_3)$, whereas supposing $z = (x_0 + x_1 + x_2 + x_3)$ is the solution for fourth degree polynomial equation in (Equation. 181) by using Theorem 3 and relying on the expression $x_3 = -\frac{\Upsilon_3}{4}$. The variable Υ_3 is defined as shown in (Equation. 182) where α_3 is presented in (Equation. 186) and Υ_4 is the solution for the polynomial equation (Equation. 187), which relies on the coefficients (Equation. 188), (Equation. 189), (Equation. 190) and (Equation. 191). The shown coefficients in (Equation. 188), (Equation. 189), (Equation. 190) and (Equation. 191) are expressed by using the constant V, which is defined in (Equation. 192). The coefficients Υ_3, Υ_2, Υ_1 and Υ_0 of quartic equation (Equation. 181) are determined by using calculated value of Υ_4 and using the shown expressions in (Equation. 182), (Equation. 183), (Equation. 184) and (Equation. 185).

The six proposed solutions for polynomial equation $x^6 + bx^5 + dx^3 + ex^2 + fx + g = 0$ shown in (Equation. 175) are as shown in (Equation. 194), (Equation. 195), (Equation. 196), (Equation. 197), (Equation. 198) and (Equation. 199).

The six proposed solutions for polynomial equation $Aw^6 + Cw^4 + Dw^3 + Ew^2 + Fw + G = 0 \; with \; A \neq 0$ are as shown in (Equation. 200), (Equation. 201), (Equation. 202), (Equation. 203), (Equation. 204) and (Equation. 205).

The group of expressions $\{\pm\Gamma_{4,1} \; ; \; \pm\Gamma_{4,2} \; ; \; \pm\Gamma_{4,3}\}$ are the identified values of the variable Γ_4, which are calculated as solutions for the sixth-degree polynomial equation presented in (Equation. 187) by using the solution of third degree equations shown in (Equation. 193).

We use the expressions $\left\{\alpha_1 = \frac{\Upsilon_4^4 + \frac{32\Upsilon_4^4}{V^2b^2} - \frac{8\Upsilon_4^4}{Vb} + \frac{12d\Upsilon_4^2}{b^2} - \frac{V^2\left(f - \frac{d^2}{4b}\right)}{b}}{4\Upsilon_4^2}\right\}$ in order to simplify calculations, which allow obtaining the quartic equation shown in (Equation. 187).

The value of V shown in (Equation. 192) is used to simplify expressing the formulas during calculations where $\frac{\Upsilon_4}{\alpha_3} = V$.

The group of expressions $\left\{\xi_{(\Upsilon_{4,1},1)} \,;\, \xi_{(\Upsilon_{4,1},2)} \,;\, \xi_{(\Upsilon_{4,1},3)}, \xi_{(\Upsilon_{4,1},4)}\right\}$ are the identified four solutions for the fourth-degree polynomial equation shown in (Equation. 181) by using Theorem 3 to calculate these four roots nearly in parallel.

$$z^4 + \Upsilon_3 z^3 + \Upsilon_2 z^2 + \Upsilon_1 z + \Upsilon_0 = 0 \tag{181}$$

$$\Upsilon_3 = \frac{4\alpha_3}{b} + \Upsilon_4 \tag{182}$$

$$\Upsilon_2 = \frac{8\Upsilon_4^2}{Vb} - \frac{6d}{b^2} + \frac{\left(f - \frac{d^2}{4b}\right)V^2}{2b\Upsilon_4^2} - \frac{8\Upsilon_4^2}{V^2 b^2} \tag{183}$$

$$\Upsilon_1 = \frac{5\Gamma_4^3}{Vb} + \frac{3Vd^2}{4b^3\Upsilon_4} - \frac{6d\Upsilon_4}{b^2} + \frac{eV}{b\Upsilon_4} - \frac{\Upsilon_4^3}{4} - \frac{8\Upsilon_4^3}{V^2 b^2} + \frac{f - \frac{d^2}{4b}}{4\Upsilon_4 b} V^2 \tag{184}$$

$$\Upsilon_0 = \frac{\Upsilon_4^4}{2V} - \frac{V^2 d^3}{16b^4\Upsilon_4^2} + \frac{3Vd^2}{8b^3} - \frac{3d\Upsilon_4^2}{4b^2} + \frac{eV}{2b} - \frac{eV^2 d}{4b^2\Upsilon_4^2} + \frac{gV^2}{2b\Upsilon_4^2} - \left(\frac{\Upsilon_4^2}{4} + V^2\frac{f - \frac{d^2}{4b}}{4b\Upsilon_4^2}\right)\left(\frac{\Upsilon_4^2}{4} + \frac{8\Upsilon_4^2}{V^2 b^2} - \frac{2\Upsilon_4^2}{VB} + \frac{3d}{b^2} - \frac{\left(f - \frac{d^2}{4b}\right)}{4b\Upsilon_4^2} V^2\right) \tag{185}$$

$$\alpha_3 = -\frac{\Upsilon_4 \frac{4\left(f - \frac{d^2}{4b}\right)}{b}}{\frac{32f}{b^2} + \frac{40d^2}{b^3} + \frac{64e}{b}} \tag{186}$$

$$\beta_3\Upsilon_4^6 + \beta_2\Upsilon_4^4 + \beta_1\Upsilon_4^2 + \beta_0 = 0 \tag{187}$$

$$\beta_3 = -\frac{40960}{V^4 b^4} + \frac{16384}{V^3 b^3} - \frac{1536}{V^2 b^2} \tag{188}$$

$$\beta_2 = -\frac{24576d}{V^2 b^4} + \frac{3072d}{Vb^3} + \frac{1024}{V} \tag{189}$$

$$\beta_1 = -\frac{512d}{b} + \frac{1536f}{b^3} + \frac{28V^2 f}{b} - \frac{7V^2 d^2}{b^2} + \frac{96Vf}{b^2} - \frac{168d^2 V}{b^3} - \frac{192Ve}{b} - \frac{3456d^2}{b^4} - \frac{1024e}{b^2} \tag{190}$$

$$\beta_0 = -\frac{64V^2 d^3}{b^4} - \frac{64eV^2 d}{b^2} + \frac{128V^2 g}{b} + \frac{192V^2 df}{b^3} \tag{191}$$

$$V = -\frac{\frac{32f}{b^2} + \frac{40d^2}{b^3} + \frac{64e}{b}}{\frac{4\left(f - \frac{d^2}{4b}\right)}{b}} \tag{192}$$

$$\Upsilon_{4,1}^2 = \frac{-b^{;}}{3} + \frac{1}{3}\sqrt[3]{-\frac{D^{;}}{2} + \sqrt{\left(\frac{D^{;}}{2}\right)^2 + \left(\frac{C^{;}}{3}\right)^3}} + \frac{1}{3}\sqrt[3]{-\frac{D^{;}}{2} - \sqrt{\left(\frac{D^{;}}{2}\right)^2 + \left(\frac{C^{;}}{3}\right)^3}} \mid \left\{b^{;} = \frac{\beta_2}{\beta_3}, c^{;} = \frac{\beta_1}{\beta_3} \text{ and } d^{;} = \frac{\beta_0}{\beta_3}\right\}; \{D\hat{};= 27d^{;} + 2b^{;3} - 9c^{;}b^{;} \text{ and } C^{;} = 9c^{;} - 3b^{;2}\} \tag{193}$$

$$s_1 = \frac{1}{2}\left[\xi^2_{(\Upsilon_{4,1},1)} - \alpha_{(1,\Upsilon_{4,1})}\right] \tag{194}$$

$$s_2 = \frac{1}{2}\left[\xi^2_{(\Upsilon_{4,1},2)} - \alpha_{(1,\Upsilon_{4,1})}\right] \tag{195}$$

$$s_3 = \frac{1}{2}\left[\xi^2_{(\Upsilon_{4,1},3)} - \alpha_{(1,\Upsilon_{4,1})}\right] \tag{196}$$

$$s_4 = \frac{1}{2}[\xi^2_{(\Upsilon_{4,1},4)} - \alpha_{(1,\Upsilon_{4,1})}] \quad (197)$$

$$s_5 = -\frac{b+s_1+s_2+s_3+s_4}{2} - \sqrt{\left(\frac{b+S_1+S_2+S_3+S_4}{2}\right)^2 - \frac{g}{S_1 S_2 S_3 S_4}} \quad (198)$$

$$s_6 = -\frac{b+s_1+s_2+s_3+s_4}{2} + \sqrt{\left(\frac{b+S_1+S_2+S_3+S_4}{2}\right)^2 - \frac{g}{S_1 S_2 S_3 S_4}} \quad (199)$$

$$\text{Solution 1 : } S_1' = \sqrt{\frac{-C}{15}} + \frac{1}{2}[\xi^2_{(\Upsilon_{4,1},1)} - \alpha_{(1,\Upsilon_{4,1})}] \quad (200)$$

$$\text{Solution 2 : } S_2' = \sqrt{\frac{-C}{15}} + \frac{1}{2}[\xi^2_{(\Upsilon_{4,1},2)} - \alpha_{(1,\Upsilon_{4,1})}] \quad (201)$$

$$\text{Solution 3 : } S_3' = \sqrt{\frac{-C}{15A}} + \frac{1}{2}[\xi^2_{(\Upsilon_{4,1},3)} - \alpha_{(1,\Upsilon_{4,1})}] \quad (202)$$

$$\text{Solution 4 : } S_4' = \sqrt{\frac{-C}{15}} + \frac{1}{2}[\xi^2_{(\Upsilon_{4,1},4)} - \alpha_{(1,\Upsilon_{4,1})}] \quad (203)$$

$$\text{Solution 5 : } S_5' = \sqrt{\frac{-C}{15A}} - \frac{b+s_1+s_2+s_3+S_4}{2} - \sqrt{\left(\frac{b+S_1+S_2+S_3+S_4}{2}\right)^2 - \frac{g}{S_1 S_2 S_3 S_4}} \quad (204)$$

$$\text{Solution 6 : } S_6' = \sqrt{\frac{-C}{15}} - \frac{b+s_1+s_2+s_3+s_4}{2} + \sqrt{\left(\frac{b+S_1+S_2+S_3+S_4}{2}\right)^2 - \frac{g}{S_1 S_2 S_3 S_4}} \quad (205)$$

9. Solving Sixth Order Differential Equations

This section presents the developed theorems and formulas to solve sixth order differential equations by using the proposed methodologies in this paper to solve nth order differential equations and nth degree polynomial equations.

9.1. First proposed Theorem for Sixth Order Differential Equations

This section presents the first developed theorem to solve sixth order differential equations that are expressed according to the form: $A * H^{(6)}(x) + B * H^{(5)}(x) + C * H^{(4)}(x) + D * H^{(3)}(x) + E * H^{(2)}(x) + F * H^{(1)}(x) + G * H^{(0)}(x) = K$ where $A \neq 0$, by supposing that the solution is expressed according to an exponential form, then converting the sixth order differential equation into an equivalent polynomial form of sixth degree where we use the presented theorems to solve polynomial equations in this paper.

Theorem 17

A sixth order differential equation under the expressed form in (Equation. 206) where coefficients belong to the group of numbers $\mathbb{R}$ and $A \neq 0$, has multiple solutions presented as $H(x)$ which we can express according to the exponential form shown in (Equation. 207).

$$A * H^{(6)}(x) + B * H^{(5)}(x) + C * H^{(4)}(x) + D * H^{(3)}(x) + E * H^{(2)}(x) + F * H^{(1)}(x) + G * H^{(0)}(x) = K \text{ with } A \neq 0 \quad (206)$$

$$H_{(x)} = e^{sx+u} + v \quad (207)$$

The value of *v*, which is included in the solution *H(x)* shown in (Equation. 207), is considered as an arbitrary value. We can calculate the arbitrary value of *v* by using shown expression in (Equation. 208).

$$v = \frac{K}{G} \quad (208)$$

The value of *u,* which is included in the solution *H(x)* shown in (Equation. 207), is considered as an arbitrary value. We can calculate the arbitrary value of *u* while relying on a condition of initialization value I_0 which to be identified at the point $x = 0$. Therefore, we can use the expression $H(x = 0) = I_0$ in order to identify the arbitrary value of *u* as shown in (Equation.

209).

$$u = log\left(I_0 - \frac{K}{G}\right) \quad (209)$$

By supposing that the solution of the sixth order differential equation is expressed according to the exponential form shown in (Equation. 207); we can convert this differential equation into the form of a sixth degree polynomial equation as shown in (Equation. 210), where we can use the proposed solutions in Theorem 15 for sixth degree polynomial equations in general forms.

$$\mathrm{A}x^6 + \mathrm{B}x^5 + \mathrm{C}x^4 + \mathrm{D}x^3 + \mathrm{E}x^2 + \mathrm{F}x + \mathrm{G} = 0 \; with \; \mathrm{A} \neq 0 \; and \; \mathrm{B} \neq 0 \quad (210)$$

$$x^6 + bx^5 + cx^4 + dx^3 + ex^2 + fx + g = 0 \; with \; b \neq 0 \quad (211)$$

$$b = \frac{\mathrm{B}}{\mathrm{A}};\; c = \frac{\mathrm{C}}{\mathrm{A}};\; d = \frac{\mathrm{D}}{\mathrm{A}};\, e = \frac{\mathrm{E}}{\mathrm{A}};\; f = \frac{\mathrm{F}}{\mathrm{A}};\; g = \frac{\mathrm{G}}{\mathrm{A}}; \quad (212)$$

$$z^4 + \Gamma_3 z^3 + \Gamma_2 z^2 + \Gamma_1 z + \Gamma_0 = 0 \quad (213)$$

We use Theorem 15 in this paper to solve the sixth degree polynomial shown in (Equation. 210).

After reducing the form of sixth degree polynomial shown in (Equation. 210) to the presented form in (Equation. 211) where coefficients are as expressed in (Equation. 212); the sixth-degree polynomial equation shown in (Equation. 211), where coefficients belong to the group of numbers ℝ, can be reduced to a fourth-degree polynomial equation, which may be expressed as shown in (Equation. 213). The reduction from sixth degree polynomial to quartic polynomial is conducted by supposing $x = x_0x_1 + x_0x_2 + x_0x_3 + x_1x_2 + x_1x_3 + x_2x_3$, whereas supposing $z = (x_0 + x_1 + x_2 + x_3)$ is the solution for fourth degree polynomial equation in (Equation. 213) by using Theorem 3 and relying on the expression $x_3 = -\frac{\Gamma_3}{4}$. The variable Γ_3 is defined as shown in (Equation. 214) where α_3 is presented in (Equation. 215) and Γ_4 is the solution for the polynomial equation (Equation. 216), which relies on the coefficients (Equation. 217), (Equation. 218), (Equation. 219) and (Equation. 220). The shown coefficients in (Equation. 217), (Equation. 218), (Equation. 219) and (Equation. 220) are expressed by using the constant V which is presented in (Equation. 221). The coefficients Γ_3, Γ_2, Γ_1 and Γ_0 of quartic equation (Equation. 213), which is used to calculate z, are determined by using the shown expressions in (Equation. 214), (Equation. 223), (Equation. 224) and (Equation. 225) while using calculated values of Γ_4 and V.

The proposed six values as official solutions for sixth degree polynomial equation shown in (Equation. 211) are as presented in (Equation. 226), (Equation. 227), (Equation. 228), (Equation. 229), (Equation. 230) and (Equation. 231).

The proposed six functions as official solutions for sixth order differential equation shown in (Equation. 206) are as presented in (Equation. 232), (Equation. 233), (Equation. 234), (Equation. 235), (Equation. 236) and (Equation. 237).

The group of expressions $\{\pm\Gamma_{4,1} \,;\, \pm\Gamma_{4,2} \,;\, \pm\Gamma_{4,3}\}$ are the identified values of the variable Γ_4, which are calculated as solutions for the sixth-degree polynomial equation presented in (Equation. 216) by using the solution of third degree equations shown in (Equation. 222).

We use the expressions $\left\{\alpha_{(1,\Gamma_{4,i})} = \frac{\Gamma_{4,i}^4 + \frac{32\Gamma_{4,i}^4}{V^2b^2} - \frac{8\Gamma_{4,i}^4}{Vb} + \frac{12d\Gamma_{4,i}^2}{b^2} - \frac{8c\Gamma_{4,i}^2}{b} - \frac{V^2\left(f - \frac{d^2}{4b}\right)}{b}}{4\Gamma_{4,i}^2}\right\}$ in order to simplify calculations, which allow obtaining the shown equation in (Equation. 216).

The value of V shown in (Equation. 221) is used to simplify expressing the formulas during calculations where $\frac{\Upsilon_4}{\alpha_3} = V$.

The group of expressions $\left\{\dot{\mathcal{S}}_{(\Gamma_{4,1},1)} \,;\, \dot{\mathcal{S}}_{(\Gamma_{4,1},2)} \,;\, \dot{\mathcal{S}}_{(\Gamma_{4,1},3)} \,;\, \dot{\mathcal{S}}_{(\Gamma_{4,1},4)}\right\}$ are the identified four solutions for the fourth-degree polynomial equation shown in (Equation. 213) by using Theorem 3 to calculate these four roots nearly in parallel.

$$\Gamma_3 = \frac{4\alpha_3}{b} + \Gamma_4 \tag{214}$$

$$\alpha_3 = -\frac{\frac{4\Gamma_4\left(f-\frac{d^2}{4b}\right)}{b}}{\frac{32f}{b^2}+\frac{40d^2}{b^3}-\frac{64cd}{b^2}+\frac{64e}{b}} \tag{215}$$

$$\lambda_3\Gamma_4^6 + \lambda_2\Gamma_4^4 + \lambda_1\Gamma_4^2 + \lambda_0 = 0 \tag{216}$$

$$\lambda_3 = -\frac{40960}{V^4b^4} + \frac{16384}{V^3b^3} - \frac{1536}{V^2b^2} \tag{217}$$

$$\lambda_2 = -\frac{24576d}{V^2b^4} + \frac{163}{V^2b^3} + \frac{3072d}{Vb^3} - \frac{2048c}{Vb^2} + \frac{1024}{V} \tag{218}$$

$$\lambda_1 = -\frac{512d}{b} + \frac{1536f}{b^3} + \frac{28V^2f}{b} - \frac{7V^2d^2}{b^2} + \frac{96Vf}{b^2} - \frac{168d^2V}{b^3} + \frac{192cdV}{b^2} - \frac{192Ve}{b} - \frac{3456d^2}{b^4} + \frac{4096cd}{b^3} - \frac{1024e}{b^2} - \frac{1024c^2}{b^2} \tag{219}$$

$$\lambda_0 = -\frac{64V^2d^3}{b^4} + \frac{64cd^2V^2}{b^3} - \frac{64eV^2d}{b^2} + \frac{128V^2g}{b} + \frac{192V^2df}{b^3} - \frac{128V^2cf}{b^2} \tag{220}$$

$$V = -\frac{\frac{32f}{b^2}+\frac{40d^2}{b^3}-\frac{64cd}{b^2}+\frac{64e}{b}}{4\frac{\left(f-\frac{d^2}{4b}\right)}{b}} \tag{221}$$

$$\Gamma_{4,1}^2 = \frac{-b^{;}}{3} + \frac{1}{3}\sqrt[3]{-\frac{D^{;}}{2}+\sqrt{\left(\frac{D^{;}}{2}\right)^2+\left(\frac{C^{;}}{3}\right)^3}} + \frac{1}{3}\sqrt[3]{-\frac{D^{;}}{2}-\sqrt{\left(\frac{D^{;}}{2}\right)^2+\left(\frac{C^{;}}{3}\right)^3}} \mid \left\{b^{;} = \frac{\lambda_2}{\lambda_3}, c^{;} = \frac{\lambda_1}{\lambda_3} \text{ and } d^{;} = \frac{\lambda_0}{\lambda_3}\right\}\left\{D^{;} = 27d^{;} + 2b^{;3} - 9c^{;}\, b^{;} \text{ and } C^{;} = 9c^{;} - 3b^{;2}\right\} \tag{222}$$

$$\Gamma_2 = \frac{8\Gamma_4^2}{Vb} - \frac{6d}{b^2} + \frac{4c}{b} + \frac{\left(f-\frac{d^2}{4b}\right)V^2}{2b\Gamma_4^2} - \frac{8\Gamma_4^2}{V^2b^2} \tag{223}$$

$$\Gamma_1 = \frac{5\Gamma_4^3}{Vb} + \frac{3Vd^2}{4b^3\Gamma_4} - \frac{6d\Gamma_4}{b^2} + \frac{4c\Gamma_4}{b} - \frac{dcV}{b^2\Gamma_4} + \frac{eV}{b\Gamma_4} - \frac{\Gamma_4^3}{4} - \frac{8\Gamma_4^3}{V^2b^2} + \frac{f-\frac{d^2}{4b}}{4\Gamma_4 b}V^2 \tag{224}$$

$$\Gamma_0 = \frac{\Gamma_4^4}{2Vb} - \frac{V^2d^3}{16b^4\Gamma_4^2} + \frac{3Vd^2}{8b^3} - \frac{3d\Gamma_4^2}{4b^2} + \frac{c\Gamma_4^2}{2b} + \frac{cd^2V^2}{8b^3\Gamma_4^2} - \frac{cdV}{2b^2} + \frac{eV}{2b} - \frac{eV^2d}{4b^2\Gamma_4^2} + \frac{gV^2}{2b\Gamma_4^2} - \left(\frac{\Gamma_4^2}{4} + V^2\frac{f-\frac{d^2}{4b}}{4b\Gamma_4^2}\right)\left(\frac{\Gamma_4^2}{4} + \frac{8\Gamma_4^2}{V^2b^2} - \frac{2\Gamma_4^2}{VB} + \frac{3d}{b^2} - \frac{2c}{b} - \frac{\left(f-\frac{d^2}{4b}\right)}{4b\Gamma_4^2}V^2\right) \tag{225}$$

$$S_1 = \frac{1}{2}\left[\dot{S}^2_{(\Gamma_{4,1},1)} - \alpha_{(1,\Gamma_{4,1})}\right] \tag{226}$$

$$S_2 = \frac{1}{2}\left[\dot{S}^2_{(\Gamma_{4,1},2)} - \alpha_{(1,\Gamma_{4,1})}\right] \tag{227}$$

$$S_3 = \frac{1}{2}\left[\dot{S}^2_{(\Gamma_{4,1},3)} - \alpha_{(1,\Gamma_{4,1})}\right] \tag{228}$$

$$S_4 = \frac{1}{2}\left[\dot{S}^2_{(\Gamma_{4,1},4)} - \alpha_{(1,\Gamma_{4,1})}\right] \tag{229}$$

$$S_5 = -\frac{b+S_1+S_2+S_3+S_4}{2} - \sqrt{\left(\frac{b+S_1+S_2+S_3+S_4}{2}\right)^2 - \frac{g}{S_1S_2S_3S_4}} \tag{230}$$

$$S_6 = -\frac{b+S_1+S_2+S_3+S_4}{2} + \sqrt{\left(\frac{b+S_1+S_2+S_3+S_4}{2}\right)^2 - \frac{g}{S_1S_2S_3S_4}} \quad (231)$$

Solution 1: $DS_1 = \frac{K}{G} + \left(I_0 - \frac{K}{G}\right) e^{\frac{1}{2}\left[\dot{S}^2_{(\Gamma_{4,1,1})} - \alpha_{(1,\Gamma_{4,1})}\right]x}$ (232)

Solution 2: $DS_2 = \frac{K}{G} + \left(I_0 - \frac{K}{G}\right) e^{\frac{1}{2}\left[\dot{S}^2_{(\Gamma_{4,1,2})} - \alpha_{(1,\Gamma_{4,1})}\right]x}$ (233)

Solution 3: $DS_3 = \frac{K}{G} + \left(I_0 - \frac{K}{G}\right) e^{\frac{1}{2}\left[\dot{S}^2_{(\Gamma_{4,1,3})} - \alpha_{(1,\Gamma_{4,1})}\right]x}$ (234)

Solution 4: $DS_4 = \frac{K}{G} + \left(I_0 - \frac{K}{G}\right) e^{\frac{1}{2}\left[\dot{S}^2_{(\Gamma_{4,1}41)} - \alpha_{(1,\Gamma_{4,1})}\right]x}$ (235)

Solution 5: $DS_5 = \frac{K}{G} + \left(I_0 - \frac{K}{G}\right) e^{\left[-\frac{b+S_1+S_2+S_3+S_4}{2} - \sqrt{\left(\frac{b+S_1+S_2+S_3+S_4}{2}\right)^2 - \frac{g}{S_1S_2S_3S_4}}\right]x}$ (236)

Solution 6: $DS_6 = \frac{K}{G} + \left(I_0 - \frac{K}{G}\right) e^{\left[-\frac{b+S_1+S_2+S_3+S_4}{2} + \sqrt{\left(\frac{b+S_1+S_2+S_3+S_4}{2}\right)^2 - \frac{g}{S_1S_2S_3S_4}}\right]x}$ (237)

9.2. Second proposed Theorem for Sixth Order Differential Equation

This section presents the second developed theorem to solve sixth order differential equations that are expressed according to the form: $A * H^{(6)}(x) + C * H^{(4)}(x) + D * H^{(3)}(x) + E * H^{(2)}(x) + F * H^{(1)}(x) + G * H^{(0)}(x) = K$ where $A \neq 0$, by supposing that the solution is expressed according to an exponential form, then converting the sixth order differential equation into an equivalent polynomial form of sixth degree where we use the presented theorems to solve polynomial equations in this paper. The axe of difference in this form of sixth order differential equation is having a value of zero for the coefficient of fifth order part.

Theorem 18

A sixth order differential equation under the expressed form in (Equation. 238) where coefficients belong to the group of numbers $\mathbb{R}$ and $A \neq 0$, has multiple solutions presented as $H(x)$ which we can express according to the exponential form shown in (Equation. 239).

$$A * H^{(6)}(x) + C * H^{(4)}(x) + D * H^{(3)}(x) + E * H^{(2)}(x) + F * H^{(1)}(x) + G * H^{(0)}(x) = K \text{ with } A \neq 0 \quad (238)$$

$$H(x) = e^{sx+u} + v \quad (239)$$

The value of *v*, which is included in the solution *H(x)* shown in (Equation. 239), is considered as an arbitrary value. We can calculate the arbitrary value of *v* by using shown expression in (Equation. 240).

$$v = \frac{K}{G} \quad (240)$$

The value of *u,* which is included in the solution *H(x)* shown in (Equation. 239), is considered as an arbitrary value. We can calculate the arbitrary value of *u* while relying on a condition of initialization value I_0 which to be identified at the point $x = 0$. Therefore, we can use the expression $H(x = 0) = I_0$ in order to identify the arbitrary value of *u* as shown in (Equation. 241).

$$u = log\left(I_0 - \frac{K}{G}\right) \quad (241)$$

By supposing that the solution of the sixth order differential equation is expressed according to the exponential form shown in (Equation. 239); we can convert this differential equation into the form of a sixth degree polynomial equation as shown in (Equation. 242), where we can use the proposed solutions in Theorem 16 for sixth degree polynomial equations in general forms.

$$Aw^6 + Cw^4 + Dw^3 + Ew^2 + Fw + G = 0 \; with \; A \neq 0 \quad (242)$$

$$x^6 + bx^5 + dx^3 + ex^2 + fx + g = 0 \tag{243}$$

$$b = 6\sqrt{\frac{-C}{15}} \tag{244}$$

$$d = \frac{8C}{3A}\sqrt{\frac{-C}{15}} + \frac{D}{A} \tag{245}$$

$$e = \frac{-C^2}{3A^2} + \frac{3D}{A}\sqrt{\frac{-C}{15}} + \frac{E}{A} \tag{246}$$

$$f = -\frac{18\ \ ^2}{5A^2}\sqrt{\frac{-C}{15A}} - \frac{DC}{5A^2} + \frac{2E}{A}\sqrt{\frac{-C}{15}} + \frac{F}{A} \tag{247}$$

$$g = \frac{-16C^3}{3375A^3} - \frac{DC}{15A^2}\sqrt{\frac{-C}{15}} - \frac{EC}{15\ \ ^2} + \frac{F}{A}\sqrt{\frac{-C}{15A}} + \frac{G}{A} \tag{248}$$

We use Theorem 16 in this paper to solve the sixth degree polynomial shown in (Equation. 242).

In order to reduce the sixth degree polynomial equation $Aw^6 + Cw^4 + Dw^3 + Ew^2 + Fw + G = 0\ with\ A \neq\ 0$ to the quartic equation shown in (Equation. 249), where coefficients belong to the group of numbers $\mathbb{R}$ we first replace w with $w = \sqrt{\frac{-C}{15A}} + x$ in order to obtain the shown equation in (Equation. 243) where coefficients are presented in (Equation. 244), (Equation. 245), (Equation. 246), (Equation. 247) and (Equation. 248). Then, the reduction from sixth degree to fourth degree is conducted by supposing $x = (x_0x_1 + x_0x_2 + x_0x_3 + x_1x_2 + x_1x_3 + x_2x_3)$, whereas supposing $z = (x_0 + x_1 + x_2 + x_3)$ is the solution for fourth degree polynomial equation in (Equation. 249) by using Theorem 3 and relying on the expression $x_3 = -\frac{\Upsilon_3}{4}$. The variable Υ_3 is defined as shown in (Equation. 250) where α_3 is presented in (Equation. 254) and Υ_4 is the solution for the polynomial equation (Equation. 255), which relies on the coefficients (Equation. 256), (Equation. 257), (Equation. 258) and (Equation. 259). The shown coefficients in (Equation. 256), (Equation. 257), (Equation. 258) and (Equation. 259) are expressed by using the constant V, which is defined in (Equation. 260). The coefficients Υ_3, Υ_2, Υ_1 and Υ_0 of quartic equation (Equation. 249) are determined by using calculated value of Υ_4 and using the shown expressions in (Equation. 250), (Equation. 251), (Equation. 252) and (Equation. 253).

The six proposed solutions for polynomial equation $x^6 + bx^5 + dx^3 + ex^2 + fx + g = 0$ shown in (Equation. 243) are as shown in (Equation. 262), (Equation. 263), (Equation. 264), (Equation. 265), (Equation. 266) and (Equation. 267).

The six proposed solutions for polynomial equation $Aw^6 + Cw^4 + Dw^3 + Ew^2 + Fw + G = 0\ with\ A \neq\ 0$ are as shown in (Equation. 268), (Equation. 269), (Equation. 270), (Equation. 271), (Equation. 272) and (Equation. 273).

The six solutions for sixth order differential equation are as shown in (Equation. 274), (Equation. 275), (Equation. 276), (Equation. 277), (Equation. 278) and (Equation. 279).

The group of expressions $\{\pm\Gamma_{4,1}\ ;\ \pm\Gamma_{4,2}\ ;\ \pm\Gamma_{4,3}\}$ are the identified values of the variable Γ_4, which are calculated as solutions for the sixth-degree polynomial equation presented in (Equation. 255) by using the solution of third degree equations shown in (Equation. 261).

We use the expressions $\left\{\alpha_1 = \frac{\Upsilon_4^4 + \frac{32\Upsilon_4^4}{V^2b^2} - \frac{8\Upsilon_4^4}{Vb} + \frac{12d\Upsilon_4^2}{b^2} - \frac{V^2\left(f - \frac{d^2}{4b}\right)}{b}}{4\Upsilon_4^2}\right\}$ in order to simplify calculations, which allow obtaining the sixth degree equation shown in (Equation. 255).

The value of V shown in (Equation. 260) is used to simplify expressing the formulas during calculations where $\frac{\Upsilon_4}{\alpha_3} = V$.

The group of expressions $\left\{\xi_{(\Upsilon_{4,1},1)}\ ;\ \xi_{(\Upsilon_{4,1},2)}\ ;\ \xi_{(\Upsilon_{4,1},3)}\ ;\ \xi_{(\Upsilon_{4,1},4)}\right\}$ are the identified four solutions for the fourth-degree polynomial equation shown in (Equation. 249) by using Theorem 3 to calculate these four roots nearly in parallel.

$$z^4 + \Upsilon_3 z^3 + \Upsilon_2 z^2 + \Upsilon_1 z + \Upsilon_0 = 0 \tag{249}$$

$$\Upsilon_3 = \frac{4\alpha_3}{b} + \Upsilon_4 \tag{250}$$

$$\Upsilon_2 = \frac{8\Upsilon_4^2}{Vb} - \frac{6d}{b^2} + \frac{\left(f-\frac{d^2}{4b}\right)V^2}{2b\Upsilon_4^2} - \frac{8\Upsilon_4^2}{V^2b^2} \tag{251}$$

$$\Upsilon_1 = \frac{5\Gamma_4^3}{Vb} + \frac{3Vd^2}{4b^3\Upsilon_4} - \frac{6d\Upsilon_4}{b^2} + \frac{eV}{b\Upsilon_4} - \frac{\Upsilon_4^3}{4} - \frac{8\Upsilon_4^3}{V^2b^2} + \frac{f-\frac{d^2}{4b}}{4\Upsilon_4 b}V^2 \tag{252}$$

$$\Upsilon_0 = \frac{\Upsilon_4^4}{2Vb} - \frac{V^2d^3}{16b^4\Upsilon_4^2} + \frac{3Vd^2}{8b^3} - \frac{3d\Upsilon_4^2}{4b^2} + \frac{eV}{2b} - \frac{eV^2d}{4b^2\Upsilon_4^2} + \frac{gV^2}{2b\Upsilon_4^2} - \left(\frac{\Upsilon_4^2}{4} + V^2\frac{f-\frac{d^2}{4b}}{4b\Upsilon_4^2}\right)\left(\frac{\Upsilon_4^2}{4} + \frac{8\Upsilon_4^2}{V^2b^2} - \frac{2\Upsilon_4^2}{VB} + \frac{3d}{b^2} - \frac{\left(f-\frac{d^2}{4b}\right)}{4b\Upsilon_4^2}V^2\right) \tag{253}$$

$$\alpha_3 = -\frac{\Upsilon_4\frac{4\left(f-\frac{d^2}{4b}\right)}{b}}{\frac{32f}{b^2}+\frac{40d^2}{b^3}+\frac{64e}{b}} \tag{254}$$

$$\beta_3\Upsilon_4^6 + \beta_2\Upsilon_4^4 + \beta_1\Upsilon_4^2 + \beta_0 = 0 \tag{255}$$

$$\beta_3 = -\frac{40960}{V^4b^4} + \frac{16384}{V^3b^3} - \frac{1536}{V^2b^2} \tag{256}$$

$$\beta_2 = -\frac{24576d}{V^2b^4} + \frac{3072d}{Vb^3} + \frac{1024}{V} \tag{257}$$

$$\beta_1 = -\frac{512d}{b} + \frac{1536f}{b^3} + \frac{28V^2f}{b} - \frac{7V^2d^2}{b^2} + \frac{96Vf}{b^2} - \frac{168d^2V}{b^3} - \frac{192Ve}{b} - \frac{3456d^2}{b^4} - \frac{1024e}{b^2} \tag{258}$$

$$\beta_0 = -\frac{64V^2d^3}{b^4} - \frac{64eV^2d}{b^2} + \frac{128V^2g}{b} + \frac{192V^2df}{b^3} \tag{259}$$

$$V = -\frac{\frac{32f}{b^2}+\frac{40d^2}{b^3}+\frac{64}{b}}{\frac{4\left(f-\frac{d^2}{4b}\right)}{b}} \tag{260}$$

$$\Upsilon_{4,1}^2 = \frac{-b^{;}}{3} + \frac{1}{3}\sqrt[3]{-\frac{D^{;}}{2} + \sqrt{\left(\frac{D^{;}}{2}\right)^2 + \left(\frac{C^{;}}{3}\right)^3}} + \frac{1}{3}\sqrt[3]{-\frac{D^{;}}{2} - \sqrt{\left(\frac{D^{;}}{2}\right)^2 + \left(\frac{C^{;}}{3}\right)^3}} \mid \left\{b^{;} = \frac{\beta_2}{\beta_3}, c^{;} = \frac{\beta_1}{\beta_3} \text{ and } d^{;} = \frac{\beta_0}{\beta_3}\right\}; \{D\hat{};= 27d^{;} + 2b^{;3} - 9c^{;}\, b^{;} \text{ and } C^{;} = 9c^{;} - 3b^{;2}\} \tag{261}$$

$$s_1 = \frac{1}{2}[\xi_{(\Upsilon_{4,1},1)}^2 - \alpha_{(1,\Upsilon_{4,1})}] \tag{262}$$

$$s_2 = \frac{1}{2}[\xi_{(\Upsilon_{4,1},2)}^2 - \alpha_{(1,\Upsilon_{4,1})}] \tag{263}$$

$$s_3 = \frac{1}{2}[\xi_{(\Upsilon_{4,1},3)}^2 - \alpha_{(1,\Upsilon_{4,1})}] \tag{264}$$

$$s_4 = \frac{1}{2}[\xi^2_{(\Upsilon_{4,1},4)} - \alpha_{(1,\Upsilon_{4,1})}] \quad (265)$$

$$s_5 = -\frac{b+S_1+S_2+S_3+S_4}{2} - \sqrt{\left(\frac{b+S_1+S_2+S_3+S_4}{2}\right)^2 - \frac{g}{S_1S_2S_3S_4}} \quad (266)$$

$$s_6 = -\frac{b+S_1+S_2+S_3+S_4}{2} + \sqrt{\left(\frac{b+S_1+S_2+S_3+S_4}{2}\right)^2 - \frac{g}{S_1S_2S_3S_4}} \quad (267)$$

$$S_1' = \sqrt{\frac{-C}{15A}} + \frac{1}{2}[\xi^2_{(\Upsilon_{4,1},1)} - \alpha_{(1,\Upsilon_{4,1})}] \quad (268)$$

$$S_2' = \sqrt{\frac{-C}{15A}} + \frac{1}{2}[\xi^2_{(\Upsilon_{4,1},2)} - \alpha_{(1,\Upsilon_{4,1})}] \quad (269)$$

$$S_3' = \sqrt{\frac{-C}{15A}} + \frac{1}{2}[\xi^2_{(\Upsilon_{4,1},3)} - \alpha_{(1,\Upsilon_{4,1})}] \quad (270)$$

$$S_4' = \sqrt{\frac{-C}{15A}} + \frac{1}{2}[\xi^2_{(\Upsilon_{4,1},4)} - \alpha_{(1,\Upsilon_{4,1})}] \quad (271)$$

$$S_5' = \sqrt{\frac{-C}{15A}} - \frac{b+S_1+S_2+S_3+S_4}{2} - \sqrt{\left(\frac{b+S_1+S_2+S_3+S_4}{2}\right)^2 - \frac{g}{S_1S_2S_3S_4}} \quad (272)$$

$$S_6' = \sqrt{\frac{-C}{15A}} - \frac{b+S_1+S_2+S_3+S_4}{2} + \sqrt{\left(\frac{b+S_1+S_2+S_3+S_4}{2}\right)^2 - \frac{g}{S_1S_2S_3S_4}} \quad (273)$$

Solution 1: $$DS_1' = \frac{K}{G} + \left(I_0 - \frac{K}{G}\right) e^{\left[\sqrt{\frac{-C}{15A}} + \frac{1}{2}[\xi^2_{(\Upsilon_{4,1},1)} - \alpha_{(1,\Upsilon_{4,1})}]\right]x} \quad (274)$$

Solution 2: $$DS_2' = \frac{K}{G} + \left(I_0 - \frac{K}{G}\right) e^{\left[\sqrt{\frac{-C}{15A}} + \frac{1}{2}[\xi^2_{(\Upsilon_{4,1},2)} - \alpha_{(1,\Upsilon_{4,1})}]\right]x} \quad (275)$$

Solution 3: $$DS_3' = \frac{K}{G} + \left(I_0 - \frac{K}{G}\right) e^{\left[\sqrt{\frac{-C}{15A}} + \frac{1}{2}[\xi^2_{(\Upsilon_{4,1},3)} - \alpha_{(1,\Upsilon_{4,1})}]\right]x} \quad (276)$$

Solution 4: $$DS_4' = \frac{K}{G} + \left(I_0 - \frac{K}{G}\right) e^{\left[\sqrt{\frac{-C}{15}} + \frac{1}{2}[\xi^2_{(\Upsilon_{4,1},4)} - \alpha_{(1,\Upsilon_{4,1})}]\right]x} \quad (277)$$

Solution 5: $$DS_5' = \frac{K}{G} + \left(I_0 - \frac{K}{G}\right) e^{\left[\sqrt{\frac{-C}{15A}} - \frac{b+S_1+S_2+S_3+S_4}{2} - \sqrt{\left(\frac{b+S_1+S_2+S_3+S_4}{2}\right)^2 - \frac{g}{S_1S_2S_3S_4}}\right]x} \quad (278)$$

Solution 6: $$DS_6' = \frac{K}{G} + \left(I_0 - \frac{K}{G}\right) e^{\left[\sqrt{\frac{-C}{15}} - \frac{b+S_1+S_2+S_3+S_4}{2} + \sqrt{\left(\frac{b+S_1+S_2+S_3+S_4}{2}\right)^2 - \frac{g}{S_1S_2S_3S_4}}\right]x} \quad (279)$$

10. Solving nth Degree Polynomial Equations

This section presents the developed theorem and formulas to solve nth degree polynomial equations by using the proposed engineering methodology in this paper.

10.1. Proposed theorem for nth degree polynomials

This subsection presents the developed theorem to solve nth degree polynomial equations that are expressed according to the form: $\left\{\left(\sum_{i=0}^{i=N} A_i X^i\right) = 0\right\}$ where $A_N \neq 0$, by converting this nth degree polynomial into the form of a reduced polynomial equation with inferior degree which we can express as follows: $\left\{\left(\sum_{i=0}^{i=M} \Gamma_i Z^i\right) = 0\right\}$ where $N>M$.

$$\left\{\left(\sum_{i=0}^{i=N} A_i X^i\right) = 0\right\} \; with\; A_n \neq 0 \tag{280}$$

$$\left\{\left(\sum_{i=0}^{i=N} \frac{A_i}{A_n} X^i\right) = 0\right\} \; with\; A_n \neq 0 \tag{281}$$

$$if\,(N \geq 7 \; and \; N \equiv 1\; MOD[2]) \Longrightarrow \left\{\left(\sum_{j=0}^{j=n} \frac{a_j}{a_n} X^j\right) = 0\right\} \; where \; \left\{(n = N+1);\; (\,a_n \neq 0);\; (\,a_0 = 0)\; and \; \left(a_{j+1} = A_{j\geq 0}\right)\right\} \tag{282}$$

$$if\; [N < 7 \; or \; (N \equiv 0 \; MOD[2])] \Longrightarrow \left\{\left(\sum_{j=0}^{j=n} \frac{a_j}{a_n} X^j\right) = 0\right\} \; where \; \left\{(n = N)\; and \; (\,a_n \neq 0)\; and \; \left(a_j = A_{j\geq 0}\right)\right\} \tag{283}$$

$$\left\{X = \frac{-a_{(n-1)}}{n a_n} + \frac{x}{n}\right\} \; to\; eleminate\; the\; part\; with\; the\; degree\; (n-1) \tag{284}$$

$$\left\{X = \sqrt{\frac{-2a_{(n-2)}}{n(n-1)a_n}} + x\right\} \; to\; create\; the\; part\; with\; the\; degree\; (n-1) \tag{285}$$

$$\{X = x\} \; to\; keep\; the\; same\; form\; of\; polynomial \tag{286}$$

$$\left\{\left(\sum_{i=0}^{i=n} b_i x^i\right) = 0\right\} \; with\; b_n = 1 \tag{287}$$

$$\left\{x = \sum_{i=0}^{i=u} T_i = \sum_{i=0}^{i=u} x_i\right\} \; if\; the\; degree\; of\; polynomial\; equation\; is\; n = 4 \tag{288}$$

$$\left\{x = \sum_{i=0}^{i=u} T_i = \sum_{i\neq j} x_i\, x_j\right\} \; if\; the\; degree\; of\; polynomial\; equation\; is\; n \geq 5 \tag{289}$$

$$\left\{\sum_{i=0}^{i=n'} \Gamma_i\, z^i = 0\right\} \; where\; (n' < n\,)\; and\; (\Gamma_{n'} = 1) \tag{290}$$

$$z = \sum_{i=0}^{i=u'} x_i = \sum_{i=0}^{i=u'} \sqrt{y_i} \tag{291}$$

$$x_1 = \sqrt{\frac{-b}{3} + \frac{1}{3}\sqrt[3]{-\frac{D}{2} + \sqrt{\left(\frac{D}{2}\right)^2 + \left(\frac{C}{3}\right)^3}} + \frac{1}{3}\sqrt[3]{-\frac{D}{2} - \sqrt{\left(\frac{D}{2}\right)^2 + \left(\frac{C}{3}\right)^3}}} \tag{292}$$

$$x_2 = \sqrt{-\frac{\frac{P}{2}+x_1}{2} + \sqrt{\left(\frac{\frac{P}{2}+x_1}{2}\right)^2 - \frac{Q^2}{64\;_1}}} \tag{293}$$

$$x_3 = \sqrt{-\frac{\frac{P}{2}+x_1}{2} - \sqrt{\left(\frac{\frac{P}{2}+x_1}{2}\right)^2 - \frac{Q^2}{64x_1}}} \tag{294}$$

$$\{\alpha_1 = \sum x_i^2\}\,;\ \{\alpha_2 = \sum_{i\neq j} x_i^2\, x_j^2\}\,;\ \{\alpha_3 = \sum_{i\neq j\neq k} x_i x_j x_k\}\,;\ \{\alpha_4 = \sum_{i\neq j\neq k\neq l} x_i x_j x_k\, x_l\} \tag{295}$$

$$\{X = (\sum x_i)^2 - \alpha_1\}\,;\ \{X^2 = \alpha_2 + 2\alpha_3(\sum x_i) + 6\alpha_4\}\,;\ \{X^4 = 4(\sum x_i)^2\,\alpha_3^2 + 4\alpha_3(\sum x_i)[\alpha_2 + 6\alpha_4] + [\alpha_2 + 6\alpha_4]^2\} \tag{296}$$

$$\left\{\sum_{i=0}^{i=v'} \lambda_i\,\Gamma^{2i} = 0\right\} where\ \left(v' \leq \frac{n}{2}\right) \tag{297}$$

$$\left\{V = \frac{\Gamma}{\alpha_3}\right\} \tag{298}$$

$$\lambda_i = g_i(V) \tag{299}$$

$$\Gamma_i = f_i(\Gamma, V) \tag{300}$$

$$Group\ K = \left\{\dot{S}_{\left(\Gamma_{r_k},1\right)}\,;\ \dot{S}_{\left(\Gamma_{r_k},2\right)};\ldots;\dot{S}_{\left(\Gamma_{r_k},n'\right)}\right\} \tag{301}$$

$$\left\{S_1 = \frac{1}{2}\left[\dot{S}^2_{(\Gamma,1)} - \alpha_{(1,\Gamma)}\right]\,;\ S_2 = \frac{1}{2}\left[\dot{S}^2_{(\Gamma,2)} - \alpha_{(1,\Gamma)}\right]\,;\ldots;\ S_{n'} = \frac{1}{2}\left[\dot{S}^2_{(\Gamma,n')} - \alpha_{(1,\Gamma)}\right]\right\} \tag{302}$$

$$\frac{\left(\sum_{i=0}^{i=n} b_i x^i\right)}{\prod_{j=1}^{j=n'}(x - S_j)} = 0 \tag{303}$$

$$\{S_{(n'+1)}\,;\ S_{(n'+2)};\ldots;\ S_{(n)}\} \tag{304}$$

$$\{S_1' = S_1\,;\ S_2' = S_2\,;\ldots;\ S_{n'}' = S_{n'};\ldots;\ S_n' = S_n\} \tag{305}$$

$$\left\{S_1' = \frac{-a_{n-1}}{na_n} + \frac{S_1}{n}\,;\ S_2' = \frac{-a_{n-1}}{na_n} + \frac{S_2}{n}\,;\ldots;\ S_{n'}' = \frac{-a_{n-1}}{na_n} + \frac{S_{n'}}{n}\,;\ldots;\ S_n' = \frac{-a_{n-1}}{na_n} + \frac{S_n}{n}\right\} \tag{306}$$

$$\left\{S_1' = \sqrt{\frac{-a_{n-1}}{n(n-2)a_n}} + S_1\,;\ S_2' = \sqrt{\frac{-a_{n-1}}{n(n-2)a_n}} + S_2\,;\ldots;\ S_{n'}' = \sqrt{\frac{-a_{n-1}}{n(n-2)a_n}} + S_{n'}\,;\ldots;\ S_n' = \sqrt{\frac{-a_{n-1}}{n(n-2)a_n}} + S_n\right\} \tag{307}$$

Theorem 19

1. We Consider the nth degree polynomial equation $\left\{\left(\sum_{i=0}^{i=N} A_i X^i\right) = 0\right\}$ where $A_n \neq 0$ and $N \geq 4$ and all coefficients belong to the group of numbers $\mathbb{R}$ as shown in (Equation. 280).
2. We first adapt the nth degree polynomial equation $\left\{\left(\sum_{i=0}^{i=N} A_i X^i\right) = 0\right\}$ shown in (Equation. 280) to be as presented in (Equation. 281) by dividing it on the coefficient A_n where $A_n \neq 0$.
3. If the degree N of the polynomial equation $\left\{\left(\sum_{i=0}^{i=N} \frac{A_i}{A_n} X^i\right) = 0\right\}$ is an odd number and if it is equal or superior than seven, then we can multiply this polynomial form by X to obtain the polynomial equation $\left\{\left(\sum_{j=0}^{j=n} \frac{a_j}{a_n} X^j\right) = 0\right\}$ shown in (Equation. 282) where $\{(n = N+1);\ (a_n \neq 0);\ (a_0 = 0)\ and\ (a_{j+1} = A_{j\geq 0})\}$.
4. If the degree N of the polynomial equation $\left\{\left(\sum_{i=0}^{i=N} \frac{A_i}{A_n} X^i\right) = 0\right\}$ is less than seven or if it is an even number, then we can adapt this polynomial form to be presented as $\left\{\left(\sum_{j=0}^{j=n} \frac{a_j}{a_n} X^j\right) = 0\right\}$ where $\{(n = N) and\ (a_n \neq 0)\ and\ (a_j = A_{j\geq 0})\}$ as presented in (Equation. 283)

5. If the degree *n* of the polynomial equation $\left\{\left(\sum_{j=0}^{j=n} \frac{a_j}{a_n} X^j\right) = 0\right\}$ is even, then there is the possibility to eliminate the part of the degree *(n-1)* by using the expression $\left\{X = \frac{-a_{(n-1)}}{na_n} + \frac{x}{n}\right\}$ shown in (Equation. 284).

6. It is optional to create the part of the degree *(n-1)* in the polynomial equation $\left\{\left(\sum_{j=0}^{j=n} \frac{a_j}{a_n} X^j\right) = 0\right\}$ by using the expression $\left\{X = \sqrt{\frac{-2a_{(n-2)}}{n(n-1)a_n}} + x\right\}$ shown in (Equation. 285), which also allows eliminating the part with degree *(n-2)*.

7. If we do not use the expression $\left\{X = \frac{-a_{(n-1)}}{na_n} + \frac{x}{n}\right\}$ nor the expression $\left\{X = \sqrt{\frac{-2a_{(n-2)}}{n(n-1)a_n}} + x\right\}$, then we rely on the use of the expression $\{X = x\}$ as shown in (Equation. 286), in order to reach the presented form in (Equation. 287).

8. We adapt the nth degree polynomial equation $\left\{\left(\sum_{j=0}^{j=n} \frac{a_j}{a_n} X^j\right) = 0\right\}$ to be presented as $\left\{\left(\sum_{i=0}^{i=n} b_i x^i\right) = 0\right\}$ where $b_n = 1$ and all coefficients belong to the group of numbers $\mathbb{R}$ as shown in (Equation. 287).

9. Considering the resulted nth degree polynomial equation $\left\{\left(\sum_{i=0}^{i=n} b_i x^i\right) = 0\right\}$ where $b_n = 1$ and all coefficients belong to the group of numbers $\mathbb{R}$ as shown in (Equation. 287); we can reduce this polynomial equation into an inferior polynomial degree $\left\{\sum_{i=0}^{i=n'} \Gamma_{\mathrm{i}}\, z^{\mathrm{i}} = 0\right\}$ where $(n' < n)$ *and* $(\Gamma_{n'} = 1)$ as shown in (Equation. 290).

10. The reduction of the polynomial equation from the nth degree to the inferior degree n' is conducted by supposing $\left\{x = \sum_{i=0}^{i=u} T_i = \sum_{i=0}^{i=u} x_i\right\}$ when $n = 4$ as shown in (Equation. 288), or supposing $\left\{x = \sum_{i=0}^{i=u} T_i = \sum_{i\neq j} x_i\, x_j\right\}$ when $n \geq 5$ as presented in (Equation. 289), whereas supposing the expression $\left\{z = \sum_{i=0}^{i=u'} x_i\right\}$ shown in (Equation. 291) is the solution for the polynomial equation of degree n' shown in (Equation. 290) by relying on the expression $x_{u'} = -\frac{\Gamma_{(u'-1)}}{u'}$ which will eventually lead to use the solutions of quartic equations.

11. The value of x_1 is expressed according to the solution of third degree polynomial equations where $x_1 = \sqrt{\frac{-b}{3} + \frac{1}{3}\sqrt[3]{-\frac{\mathrm{D}}{2} + \sqrt{\left(\frac{\mathrm{D}}{2}\right)^2 + \left(\frac{\mathrm{C}}{3}\right)^3}} + \frac{1}{3}\sqrt[3]{-\frac{\mathrm{D}}{2} - \sqrt{\left(\frac{\mathrm{D}}{2}\right)^2 + \left(\frac{\mathrm{C}}{3}\right)^3}}}$ as presented in (Equation. 292)

12. The value of x_2 is expressed according to the solution of quadratic polynomial equations where $x_2 = \sqrt{-\frac{\frac{P}{2}+x_1}{2} + \sqrt{\left(\frac{\frac{P}{2}+x_1}{2}\right)^2 - \frac{Q^2}{64x_1}}}$ as presented in (Equation. 293)

13. The value of x_3 is expressed according to the solution of quadratic polynomial equations where $x_3 = \sqrt{-\frac{\frac{P}{2}+x_1}{2} - \sqrt{\left(\frac{\frac{P}{2}+x_1}{2}\right)^2 - \frac{Q^2}{64\ _1}}}$ as presented in (Equation. 294)

14. We rely on using the constant values $\{\alpha_1 = \sum x_i^2\}$; $\{\alpha_2 = \sum_{i\neq j} x_i^2\, x_j^2\}$; $\{\alpha_3 = \sum_{i\neq j\neq k} x_i x_j x_k\}$ *and* $\{\alpha_4 = \sum_{i\neq j\neq k\neq l} x_i x_j x_k\, x_l\}$, which are shown in (Equation. 295), in order to converge calculations toward reducing the degree of polynomial.

15. We rely on using the expressions $\{X = (\sum x_i)^2 - \alpha_1\}$; $\{X^2 = \alpha_2 + 2\alpha_3(\sum x_i) + 6\alpha_4\}$ *and* $\{X^4 = 4(\sum x_i)^2\, \alpha_3^2 + 4\alpha_3(\sum x_i)[\alpha_2 + 6\alpha_4] + [\alpha_2 + 6\alpha_4]^2\}$, which are shown in (Equation. 296), in order to converge calculations toward having simplified forms.

16. The variable Γ is the solution for the polynomial equation (Equation. 297), which relies on the coefficients $\{\lambda_0;\ \lambda_1; \ldots;\ \lambda_u\}$.

17. The value of V shown in (Equation. 298) is used to simplify expressing the formulas during calculations where $\frac{\Gamma}{\alpha_3} = V$.

18. Each coefficient $\{\lambda_i\}$ is among the group $\{\lambda_0;\ \lambda_1; \dots;\ \lambda_u\}$ and it is calculated according to the shown expression in (Equation. 299) by relying only on the coefficients $\{b_0; b_1; \dots; b_n\}$ and the calculated constant value of V which is presented in (Equation. 298).

19. The polynomial equation $\left\{\sum_{i=0}^{i=v'} \lambda_{\mathrm{i}}\, \Gamma^{2\mathrm{i}} = 0\right\}$ where $\left(v' \leq \frac{n}{2}\right)$ has $(2 * v')$ roots which we can express as $\{\pm\Gamma_{\mathrm{r}_1};\ \pm\Gamma_{\mathrm{r}_2}; \dots;\ \pm\Gamma_{\mathrm{r}_{v'}}\}$.

20. We can select one root among the group $\{\pm\Gamma_{\mathrm{r}_1};\ \pm\Gamma_{\mathrm{r}_2}; \dots;\ \pm\Gamma_{\mathrm{r}_{v'}}\}$ to be considered as the principal root value Γ.

21. Each coefficient $\{\Gamma_i\}$ is among the group $\{\Gamma_0;\ \Gamma_1; \dots;\ \Gamma_{u'}\}$ and it is calculated according to the shown expression in (Equation. 300) by relying only on the coefficients $\{b_0; b_1; \dots; b_n\}$ and the calculated values of Γ and V.

22. We use the expressions $\{\alpha_{(1,\Gamma)} = \mathrm{L}(V, \Gamma)\}$ shown in (Equation. 301) to calculate the value of α_1 only by using the coefficients $\{b_0; b_1; \dots; b_n\}$ and the calculated values of Γ and V, which allow simplifying calculations toward obtaining the shown equation in (Equation. 297).

23. The group of roots $\{\pm\Gamma_{\mathrm{r}_1};\ \pm\Gamma_{\mathrm{r}_2}; \dots;\ \pm\Gamma_{\mathrm{r}_{v'}}\}$ identified for the polynomial equation $\left\{\sum_{i=0}^{i=v'} \lambda_{\mathrm{i}}\, \Gamma^{2\mathrm{i}} = 0\right\}$ will allow to calculate an amount of $(2 * v')$ groups of roots for the polynomial equation $\left\{\sum_{i=0}^{i=n'} \Gamma_{\mathrm{i}}\, z^{\mathrm{i}} = 0\right\}$ where each group of roots will be consisting of n' roots as shown in (Equation. 301)

24. Each group of roots for the polynomial equation $\left\{\sum_{i=0}^{i=n'} \Gamma_{\mathrm{i}}\, z^{\mathrm{i}} = 0\right\}$ is calculated while relying on a specific value of root $\{\pm\Gamma_{\mathrm{r}_\mathrm{k}}\}$ for the polynomial equation $\left\{\sum_{i=0}^{i=v'} \lambda_{\mathrm{i}}\, \Gamma^{2\mathrm{i}} = 0\right\}$; whereas all groups of roots of the polynomial equation $\left\{\sum_{i=0}^{i=n'} \Gamma_{\mathrm{i}}\, z^{\mathrm{i}} = 0\right\}$ will have redundancies among them.

25. In order to identify all roots, we can eliminate the redundancies of values among calculated groups of roots for the polynomial equation $\left\{\sum_{i=0}^{i=n'} \Gamma_{\mathrm{i}}\, z^{\mathrm{i}} = 0\right\}$ where each group of roots is calculated by using a different value of $\{\pm\Gamma_{\mathrm{r}_\mathrm{k}}\}$ among the identified group of roots for the polynomial equation $\left\{\sum_{i=0}^{i=v'} \lambda_{\mathrm{i}}\, \Gamma^{2\mathrm{i}} = 0\right\}$

26. We calculate the group of roots $\{\dot{\mathrm{S}}_{(\Gamma,1)};\ \dot{\mathrm{S}}_{(\Gamma,2)}; \dots;\ \dot{\mathrm{S}}_{(\Gamma,n')}\}$ to be the solutions of the polynomial equation $\left\{\sum_{i=0}^{i=n'} \Gamma_{\mathrm{i}}\, z^{\mathrm{i}} = 0\right\}$ shown in (Equation. 290)

27. We calculate the group of roots $\{\dot{\mathrm{S}}_{(\Gamma,1)};\ \dot{\mathrm{S}}_{(\Gamma,2)}; \dots;\ \dot{\mathrm{S}}_{(\Gamma,n')}\}$ nearly in parallel to be the solutions of the polynomial equation $\left\{\sum_{i=0}^{i=n'} \Gamma_{\mathrm{i}}\, z^{\mathrm{i}} = 0\right\}$ by changing the signs of the included subterms in one solution $\left\{\dot{\mathrm{S}}_{(\Gamma,k)} = \sum \pm \mathrm{T}_{\mathrm{i}}\right\}$

28. We calculate a group of n' roots $\left\{S_1 = \frac{1}{2}\left[\dot{\mathrm{S}}^2_{(\Gamma,1)} - \alpha_{(1,\Gamma)}\right];\ S_2 = \frac{1}{2}\left[\dot{\mathrm{S}}^2_{(\Gamma,2)} - \alpha_{(1,\Gamma)}\right]; \dots;\ S_{n'} = \frac{1}{2}\left[\dot{\mathrm{S}}^2_{(\Gamma,\mathrm{n}')} - \alpha_{(1,\Gamma)}\right]\right\}$ as expressed in (Equation. 302) to solve the polynomial equation $\left\{\left(\sum_{i=0}^{i=n} b_i x^i\right) = 0\right\}$, which is presented in (Equation. 287)

29. We can calculate the rest of roots for the polynomial equation $\left\{\left(\sum_{i=0}^{i=n} b_i x^i\right) = 0\right\}$ by solving the polynomial equation $\frac{\left(\sum_{i=0}^{i=n} b_i x^i\right)}{\prod_{j=1}^{j=n'}(x - S_j)} = 0$ shown in (Equation. 303) while relying on the calculated group of roots $\left\{S_1 = \frac{1}{2}\left[\dot{\mathrm{S}}^2_{(\Gamma,1)} - \alpha_{(1,\Gamma)}\right];\ S_2 = \frac{1}{2}\left[\dot{\mathrm{S}}^2_{(\Gamma,2)} - \alpha_{(1,\Gamma)}\right]; \dots;\ S_{n'} = \frac{1}{2}\left[\dot{\mathrm{S}}^2_{(\Gamma,\mathrm{n}')} - \alpha_{(1,\Gamma)}\right]\right\}$ which is presented in (Equation. 302)

30. We solve the polynomial equation $\frac{\left(\sum_{i=0}^{i=n} b_i x^i\right)}{\prod_{j=1}^{j=n'}(x - S_j)} = 0$, which has a degree of $(n - n')$, in order to identify the group of roots $\{S_{(n'+1)};\ S_{(n'+2)}; \dots;\ S_{(n)}\}$ shown in (Equation. 304).

31. We solve the polynomial equation $\frac{\left(\sum_{i=0}^{i=n} b_i x^i\right)}{\prod_{j=1}^{j=n'}(x-s_j)} = 0$ shown in (Equation. 303) by using quadratic terms if this polynomial equation is expressed according to a second degree form.
32. We solve the polynomial equation $\frac{\left(\sum_{i=0}^{i=n} b_i x^i\right)}{\prod_{j=1}^{j=n'}(x-s_j)} = 0$ shown in (Equation. 303) by using cubic terms if this polynomial equation is expressed according to a third degree form.
33. We solve the polynomial equation $\frac{\left(\sum_{i=0}^{i=n} b_i x^i\right)}{\prod_{j=1}^{j=n'}(x-s_j)} = 0$ shown in (Equation. 303) by repeating the same engineered methodology to solve nth degree polynomial equations if the degree of the equation $\frac{\left(\sum_{i=0}^{i=n} b_i x^i\right)}{\prod_{j=1}^{j=n'}(x-s_j)} = 0$ is equal or higher than four.
34. By identifying the group of roots $\{S_{(1)}\ ;\ S_{(2)}; \ldots ;\ S_{(n')}\}$ and the group of roots $\{S_{(n'+1)}\ ;\ S_{(n'+2)}; \ldots ;\ S_{(n)}\}$, we will have all the n roots for the nth degree polynomial equation $\left\{\left(\sum_{i=0}^{i=n} b_i x^i\right) = 0\right\}$ shown in (Equation. 287)
35. The group of roots for the polynomial equation $\left\{\left(\sum_{i=0}^{i=N} A_i X^i\right) = 0\right\}$ shown in (Equation. 280) will be $\{S'_{(1)}\ ;\ S'_{(2)}; \ldots ;\ S'_{(n)}\}$ where $\left\{S'_{(i)} = S_{(i)}\right\}$ as presented in (Equation. 305) in case we used the expression $(X = x)$ shown in (Equation. 286)
36. The group of roots for the polynomial equation $\left\{\left(\sum_{i=0}^{i=N} A_i X^i\right) = 0\right\}$ shown in (Equation. 280) will be $\{S'_{(1)}\ ;\ S'_{(2)}; \ldots ;\ S'_{(n)}\}$ where $\left\{S'_{(i)} = \frac{-a_{n-1}}{na_n} + \frac{1}{n} S_{(i)}\right\}$ as presented in (Equation. 306) in case we used the expression $\left\{X = \frac{-a_{(n-1)}}{na_n} + \frac{x}{n}\right\}$ shown in (Equation. 284)
37. The group of roots for the polynomial equation $\left\{\left(\sum_{i=0}^{i=N} A_i X^i\right) = 0\right\}$ shown in (Equation. 280) will be $\{S'_{(1)}\ ;\ S'_{(2)}; \ldots ;\ S'_{(n)}\}$ where $\left\{S'_{(i)} = \sqrt{\frac{-2a_{n-2}}{n(n-1)a_n}} + S_{(i)}\right\}$ as presented in (Equation. 307) in case we used the expression $\left\{X = \sqrt{\frac{-2a_{(n-2)}}{n(n-1)a_n}} + x\right\}$ shown in (Equation. 285)

11. Solving nth Order Differential Equations

This section presents the developed theorems and formulas to solve nth order differential equations by using the proposed methodologies in this paper.

11.1. First proposed Theorem for nth Order Differential Equations

This subsection presents the first developed theorem to solve nth order differential equations that are expressed according to the form: $\left\{\left(\sum_{i=0}^{i=N} A_i * H^{(i)}(x)\right) = K\right\}$ where $A_N \neq 0$, by supposing that the solution is expressed according to an exponential form, then converting the nth order differential equation into an equivalent polynomial form of nth degree where we use the presented theorems to solve polynomial equations in this paper.

Theorem 20

The nth order differential equation under the expressed form in (Equation. 308) where coefficients belong to the group of numbers $\mathbb{R}$ and $A_N \neq 0$, has multiple solutions presented as $H(x)$ which we can express according to the exponential form shown in (Equation. 309).

$$\left(\sum_{i=0}^{i=N} A_i H^{(i)}(x)\right) = K \text{ with } A_N \neq 0 \tag{308}$$

$$H_{(x)} = e^{sx+u} + v \tag{309}$$

The value of *v*, which is included in the solution *H(x)* shown in (Equation. 309), is considered as an arbitrary value. We can calculate the arbitrary value of *v* by using shown expression in (Equation. 310).

$$v = \frac{K}{A_0} \tag{310}$$

The value of u, which is included in the solution *H(x)* shown in (Equation. 309), is considered as an arbitrary value. We can calculate the arbitrary value of u while relying on a condition of initialization value I_0 which to be identified at the point $x = 0$. Therefore, we can use the expression $H(x = 0) = I_0$ in order to identify the arbitrary value of u as shown in (Equation. 311).

$$u = log\left(I_0 - \frac{K}{A_0}\right) \tag{311}$$

By supposing that the solution of the nth order differential equation is expressed according to the exponential form shown in (Equation. 309); we can convert this differential equation into the form of a nth degree polynomial equation as shown in (Equation. 312), where we can use the proposed solutions in Theorem 20 for nth degree polynomial equations in general forms.

$$\left\{\left(\sum_{i=0}^{i=N} A_i X^i\right) = 0\right\} \ with\ A_n \neq\ 0 \tag{312}$$

We use Theorem 20 to solve the nth degree polynomial equation shown in (Equation. 312) in order to calculate all n roots nearly in parallel. Otherwise, we can use numerical analysis to calculate all these roots.

After identifying the group of n roots $\{S'_{(1)}\ ;\ S'_{(2)}; \dots ;\ S'_{(n)}\}$ for the nth degree polynomial equation shown in (Equation. 312), we calculate the group of n solutions $\{DS''_{(1)}\ ;\ DS''_{(2)}; \dots ;\ DS''_{(n)}\}$ for the nth order differential equation by relying on the identified roots and the shown expression in (Equation. 309), which allow calculating each solution for the differential equation (Equation. 308) as shown in (Equation. 313).

$$DS'_{(i)} = \left(I_0 - \frac{K}{A_0}\right) e^{S'_i x} + \frac{K}{A_0} \tag{313}$$

11.2. Second proposed Theorem for nth Order Differential Equations

This subsection presents the second developed theorem to solve nth order differential equations that are expressed according to the form: $\left\{\left(\sum_{i=0}^{i=N} A_i * H^{(i)}(x)\right) = K\right\}$ where $A_N \neq 0$. This Theorem is identifying new additional solutions for nth order differential equations by combining the use of two different roots to express the new solutions, which allow interconnecting two arbitrary points $\{(x_0, I_0)\ ;\ (x_1, I_1)\}$.

Theorem 21

Supposing having the nth order differential equation $\left\{\left(\sum_{i=0}^{i=N} A_i H^{(i)}(x)\right) = K\right\}$ which is characterized by the nth degree polynomial equation $\left\{\left(\sum_{i=0}^{i=N} A_i X^i\right) = 0\right\}$ where $A_n \neq\ 0$ and all coefficient are from the group of numbers $\mathbb{R}$.

Supposing the group of n roots of the nth degree polynomial equation $\left\{\left(\sum_{i=0}^{i=N} A_i X^i\right) = 0\right\}$ which to be expressed as $\{S'_{(1)}\ ;\ S'_{(2)}; \dots ;\ S'_{(n)}\}$, whereas the group of n solutions for the corresponding nth order differential equation $\left\{\left(\sum_{i=0}^{i=N} A_i H^{(i)}(x)\right) = K\right\}$ is to be expressed as $\{DS'_{(1)}\ ;\ DS'_{(2)}; \dots ;\ DS'_{(n)}\}$ where each solution of the differential equation is calculated by using the identified roots as follows: $DS'_{(i)} = \left(I_0 - \frac{K}{A_0}\right) e^{S'_i x} + \frac{K}{A_0}$.

If there are two different roots $\{S_a\ ;\ S_b\ \}$ among the group of n roots $\{S'_{(1)}\ ;\ S'_{(2)}; \dots ;\ S'_{(n)}\}$ of the nth degree polynomial equation, then we can use the nth order differential equation to interconnect two arbitrary values $\{I_0\ ;\ I_1\ \}$ identified at two arbitrary points $\{(x_0, I_0)\ ;\ (x_1, I_1)\}$. The new solutions of the differential equation are determined by using the new functions expressed in (Equation. 314) where the coefficients $\left\{R'_{(I_0)}\ ;\ R'_{(I_1)}\ \right\}$ are as expressed in (Equation. 315) and (Equation. 316).

$$\left\{DS'_{(n+i>n)} = R'_{(I_1)} e^{S_2 x} + \left(R'_{(I_0)} - \frac{K}{A_0} - R'_{(I_1)}\right) e^{x S_1} + \frac{K}{A_0}\ ; Where\ S_k \in \{S_a;\ S_b\}\right\} \tag{314}$$

$$R'_{(I_0)} = \frac{I_0 - \frac{K}{A_0}(1 - e^{S_1 x_0}) - R'_{(I_1)}(e^{S_2 x_0} - e^{S_1 x_0})}{e^{S_1 x_0}} \tag{315}$$

$$R'_{(I_1)} = \frac{I_1 - \frac{K}{A_0}(1 - e^{S_1 x_1}) - R'_{(I_0)} e^{S_1 x_1}}{e^{S_2 x_1} - e^{S_1 x_1}} \tag{316}$$

11.3. Third proposed Theorem for nth Order Differential Equations

This subsection presents the third developed theorem to solve nth order differential equations that are expressed according to the form: $\left\{\left(\sum_{i=0}^{i=N} A_i H^{(i)}(x)\right) = K\right\}$ where $A_N \neq 0$. This Theorem is identifying new additional solutions for nth order differential equations by combining the use of three different roots to express the new solutions, which allow interconnecting three arbitrary points $\{(x_0, I_0)\ ;\ (x_1, I_1)\ ;\ (x_2, I_2)\}$.

Theorem 22

Supposing having the nth order differential equation $\left\{\left(\sum_{i=0}^{i=N} A_i H^{(i)}(x)\right) = K\right\}$ which is characterized by the nth degree polynomial equation $\left\{\left(\sum_{i=0}^{i=N} A_i X^i\right) = 0\right\}$ where $A_n \neq 0$ and all coefficient are from the group of numbers $\mathbb{R}$.

Supposing the group of n roots of the nth degree polynomial equation $\left\{\left(\sum_{i=0}^{i=N} A_i X^i\right) = 0\right\}$ which to be expressed as $\{S'_{(1)} ; S'_{(2)}; \dots ; S'_{(n)}\}$, whereas the group of n solutions for the corresponding nth order differential equation $\left\{\left(\sum_{i=0}^{i=N} A_i H^{(i)}(x)\right) = K\right\}$ is to be expressed as $\{DS'_{(1)} ; DS'_{(2)}; \dots ; DS'_{(n)}\}$ where each solution of the differential equation is calculated by using the identified roots as follows: $DS'_{(i)} = \left(I_0 - \frac{K}{A_0}\right) e^{S'_i x} + \frac{K}{A_0}$.

If there are three different roots $\{S_a ; S_b ; S_c\}$ among the group of n roots $\{S'_{(1)} ; S'_{(2)}; \dots ; S'_{(n)}\}$ of the nth degree polynomial equation, then we can use the nth order differential equation to interconnect three arbitrary values $\{I_0 ; I_1 ; I_2\}$ identified at three arbitrary points $\{(x_0, I_1) ; (x_1, I_1) ; (x_2, I_2)\}$. The new solutions of the differential equation are determined by using the new functions expressed in (Equation. 317) where the coefficients $\left\{R'_{(I_0)} ; R'_{(I_1)} ; R'_{(I_2)}\right\}$ are calculated by using shown expressions in (Equation. 318), (Equation. 319) and (Equation. 320).

$$\left\{DS'_{(n+i>n)} = R'_{(I_2)} e^{xS_3} + \left(R'_{(I_1)} - R'_{(I_2)}\right) e^{xS_2} + \left(R'_{(I_0)} - \frac{K}{A_0} - R'_{(I_1)}\right) e^{xS_1} + \frac{K}{A_0} ; where\ S_k \in \{S_a ; S_b ; S_c\}\right\} \quad (317)$$

$$R'_{(I_0)} = \frac{I_0 - \frac{K}{A_0}\left(1 - e^{S_1 x_0}\right) - R'_{(I_1)}\left(e^{S_2 x_0} - e^{S_1 x_0}\right) - R'_{(I_2)}\left(e^{S_3 x_0} - e^{S_2 x_0}\right)}{e^{S_1 x_0}} \quad (318)$$

$$R'_{(I_1)} = \frac{I_1 - \frac{K}{A_0}\left(1 - e^{S_1 x_1}\right) - R'_{(I_0)} e^{S_1 x_1} - R'_{(I_2)}\left(e^{S_3 x_1} - e^{S_2 x_1}\right)}{e^{S_2 x_1} - e^{S_1 x_1}} \quad (319)$$

$$R'_{(I_2)} = \frac{I_2 - \frac{K}{A_0}\left(1 - e^{S_1 x_2}\right) - R'_{(I_0)} e^{S_1 x_2} - R'_{(I_1)}\left(e^{S_2 x_2} - e^{S_1 x_2}\right)}{e^{S_3 x_2} - e^{S_2 x_2}} \quad (320)$$

11.4. Fourth proposed Theorem for nth Order Differential Equations

This subsection presents the fourth developed theorem to solve nth order differential equations that are expressed according to the form: $\left\{\left(\sum_{i=0}^{i=N} A_i H^{(i)}(x)\right) = K\right\}$ where $A_N \neq 0$. This Theorem is identifying new additional solutions for nth order differential equations by combining the use of four different roots to express the new solutions, which allow interconnecting four arbitrary points $\{(x_0, I_0); (x_1, I_1) ; (x_2, I_2) ; (x_3, I_3)\}$.

Theorem 23

Supposing having the nth order differential equation $\left\{\left(\sum_{i=0}^{i=N} A_i H^{(i)}(x)\right) = K\right\}$ which is characterized by the nth degree polynomial equation $\left\{\left(\sum_{i=0}^{i=N} A_i X^i\right) = 0\right\}$ where $A_n \neq 0$ and all coefficient are from the group of numbers $\mathbb{R}$.

Supposing the group of n roots of the nth degree polynomial equation $\left\{\left(\sum_{i=0}^{i=N} A_i X^i\right) = 0\right\}$ which to be expressed as $\{S'_{(1)} ; S'_{(2)}; \dots ; S'_{(n)}\}$, whereas the group of n solutions for the corresponding nth order differential equation $\left\{\left(\sum_{i=0}^{i=N} A_i H^{(i)}(x)\right) = K\right\}$ is to be expressed as $\{DS'_{(1)} ; DS'_{(2)}; \dots ; DS'_{(n)}\}$ where each solution of the differential equation is calculated by using the identified roots as follows: $DS'_{(i)} = \left(I_0 - \frac{K}{A_0}\right) e^{S'_i x} + \frac{K}{A_0}$.

If there are four different roots $\{S_a ; S_b ; S_c ; S_d\}$ among the group of n roots $\{S'_{(1)} ; S'_{(2)}; \dots ; S'_{(n)}\}$ of the nth degree polynomial equation, then we can use the nth order differential equation to interconnect four arbitrary values $\{I_0 ; I_1 ; I_2 ; I_3\}$ identified at four arbitrary points $\{(x_0, I_1) ; (x_1, I_1) ; (x_2, I_2) ; (x_3, I_3)\}$. The new solutions of the differential equation are determined by using the new functions expressed in (Equation. 321) where the coefficients $\left\{R'_{(I_0)} ; R'_{(I_1)} ; R'_{(I_2)} ; R'_{(I_3)}\right\}$ are calculated by using shown expressions in (Equation. 322), (Equation. 323), (Equation. 324) and (Equation. 325).

$$\left\{DS'_{(n+i>n)} = R'_{(I_3)}e^{xS_4} + \left(R'_{(I_2)} - R'_{(I_3)}\right)e^{xS_3} + \left(R'_{(I_1)} - R'_{(I_2)}\right)e^{xS_2} + \left(R'_{(I_0)} - \frac{K}{A_0} - R'_{(I_1)}\right)e^{xS_1} + \frac{K}{A_0}; where\ S_k \in \{S_a;\ S_b;\ S_c\ ;\ S_d\}\right\} \quad (321)$$

$$R'_{(I_0)} = \frac{I_0 - \frac{K}{A_0}(1-e^{S_1x_0}) - R'_{(I_1)}(e^{S_2x_0}-e^{S_1x_0}) - R'_{(I_2)}(e^{S_3x_0}-e^{S_2x_0}) - R'_{(I_3)}(e^{S_4x_0}-e^{S_3x_0})}{e^{S_1x_0}} \quad (322)$$

$$R'_{(I_1)} = \frac{I_1 - \frac{K}{A_0}(1-e^{S_1x_1}) - R'_{(I_0)}e^{S_1x_1} - R'_{(I_2)}(e^{S_3x_1}-e^{S_2x_1}) - R'_{(I_3)}(e^{S_4x_1}-e^{S_3x_1})}{e^{S_2x_1}-e^{S_1x_1}} \quad (323)$$

$$R'_{(I_2)} = \frac{I_2 - \frac{K}{A_0}(1-e^{S_1x_2}) - R'_{(I_0)}e^{S_1x_2} - R'_{(I_1)}(e^{S_2x_2}-e^{S_1x_2}) - R'_{(I_3)}(e^{S_4x_2}-e^{S_3x_2})}{e^{S_3x_2}-e^{S_2x_2}} \quad (324)$$

$$R'_{(I_3)} = \frac{I_3 - \frac{K}{A_0}(1-e^{S_1x_3}) - R'_{(I_0)}e^{S_1x_3} - R'_{(I_1)}(e^{S_2x_3}-e^{S_1x_3}) - R'_{(I_2)}(e^{S_3x_3}-e^{S_2x_3})}{e^{S_4x_3}-e^{S_3x_3}} \quad (325)$$

11.5. Fifth proposed Theorem for nth Order Differential Equations

This subsection presents the fifth developed theorem to solve nth order differential equations that are expressed according to the form: $\left\{\left(\sum_{i=0}^{i=N} A_iH^{(i)}(x)\right) = K\right\}$ where $A_N \neq 0$. This Theorem is identifying new additional solutions for nth order differential equations by combining the use of T different roots to express the new solutions, which can allow interconnecting T arbitrary points.

Theorem 24

Supposing having the nth order differential equation $\left\{\left(\sum_{i=0}^{i=N} A_iH^{(i)}(x)\right) = K\right\}$ which is characterized by the nth degree polynomial equation $\left\{\left(\sum_{i=0}^{i=N} A_iX^i\right) = 0\right\}$ where $A_n \neq 0$ and all coefficient are from the group of numbers $\mathbb{R}$.

Supposing the group of *n* roots of the nth degree polynomial equation $\left\{\left(\sum_{i=0}^{i=N} A_iX^i\right) = 0\right\}$ which to be expressed as $\{S'_{(1)}\ ;\ S'_{(2)};\ldots;\ S'_{(n)}\}$, whereas the group of *n* solutions for the corresponding nth order differential equation $\left\{\left(\sum_{i=0}^{i=N} A_iH^{(i)}(x)\right) = K\right\}$ is to be expressed as $\{DS'_{(1)}\ ;\ DS'_{(2)};\ldots;\ DS'_{(n)}\}$ where each solution of the differential equation is calculated by using the identified roots as follows: $DS'_{(i)} = \left(I_0 - \frac{K}{A_0}\right)e^{S'_ix} + \frac{K}{A_0}$.

If there are T different roots $\{S_{p_1};\ S_{p_2};\ldots;\ S_{p_T}\}$ among the group of *n* roots $\{S'_{(1)}\ ;\ S'_{(2)};\ldots;\ S'_{(n)}\}$ of the nth degree polynomial equation, then we can use the nth order differential equation to interconnect T arbitrary values $\{I_0\ ;\ldots;I_{T-1}\}$ identified at T arbitrary points $\{(x_0, I_0)\ ;\ \ldots\ ;\ (x_{T-1}, I_{T-1})\}$. The new solutions of the differential equation are determined by using the new functions expressed in (Equation. 326) where the values of the coefficients $\left\{R'_{(I_0)}\ ;\ R'_{(I_1)}\ ;\ \ldots\ ;\ R'_{(I_{T-1})}\right\}$ are calculated by using shown expressions in (Equation. 327) whereas the used parameters $R_{(x_k)}$, $O_{(I_k,\ x_k)}$, $P_{(x_k)}$ are as shown in (Equation. 328), (Equation. 329) and (Equation. 330).

$$\left\{DS'_{(n+i>n)} = R'_{(I_{(T-1)})}e^{xS_T} + \left(\sum_{k=1}^{k=T-2}\left[\left(R'_{(I_k)} - R'_{(I_{(k+1)})}\right)e^{xS_{(k+1)}}\right]\right) + \left(R'_{(I_0)} - \frac{K}{A_0} - R'_{(I_1)}\right)e^{xS_1} + \frac{K}{A_0}; where\ S_j \in \{S_{p_1};\ S_{p_1};\ldots;\ S_{p_T}\}\right\} \quad (326)$$

$$R'_{(I_k)} = \frac{I_k - \frac{K}{A_0}(1-e^{S_1x_k}) - R_{(x_k)} + O_{(I_k,\ x_k)}}{P_{(x_k)}} \quad (327)$$

$$R_{(x_k)} = R'_{(I_0)}e^{S_1x_k} + \sum_{j=1}^{j=T-1} R'_{(I_j)}\left(e^{S_{j+1}x_k} - e^{S_jx_k}\right) \quad (328)$$

$$O_{(I_k,\ x_k)} = \begin{cases} R'_{(I_0)}e^{S_1x_k}, & k = 0 \\ R'_{(I_k)}(e^{S_{k+1}x_k} - e^{S_kx_k}), & k > 0 \end{cases} \quad (329)$$

$$P_{(x_k)} = \begin{cases} e^{S_1x_k}, & k = 0 \\ (e^{S_{k+1}x_k} - e^{S_kx_k}), & k > 0 \end{cases} \quad (330)$$

11.6. Sixth proposed Theorem for nth Order Differential Equations

This subsection presents the sixth developed theorem to solve nth order differential equations that are expressed according to the form: $\left\{\left(\sum_{i=0}^{i=N} A_i * H^{(i)}(x)\right) = K\right\}$ where $A_N \neq 0$. This Theorem is identifying new additional solutions for nth order differential equations by combining the use of T' different roots $\{T' \in [\![2, T]\!]\}$ to express the new solutions, which can allow interconnecting T' arbitrary points $\{(x_{q_1}, I_0)\ ;\dots;\ (x_{q_{T'}}, I_{T'-1})\}$.

Theorem 25

Supposing having the nth order differential equation $\left\{\left(\sum_{i=0}^{i=N} A_i H^{(i)}(x)\right) = K\right\}$ which is characterized by the nth degree polynomial equation $\left\{\left(\sum_{i=0}^{i=N} A_i X^i\right) = 0\right\}$ where $A_n \neq 0$ and all coefficient are from the group of numbers $\mathbb{R}$.

Supposing the group of n roots of the nth degree polynomial equation $\left\{\left(\sum_{i=0}^{i=N} A_i X^i\right) = 0\right\}$ which to be expressed as $\{S'_{(1)}\ ;\ S'_{(2)};\dots;\ S'_{(n)}\}$, whereas the group of n solutions for the corresponding nth order differential equation $\left\{\left(\sum_{i=0}^{i=N} A_i * H^{(i)}(x)\right) = K\right\}$ is to be expressed as $\{DS'_{(1)}\ ;\ DS'_{(2)};\dots;\ DS'_{(n)}\}$ where each solution of the differential equation is calculated by using the identified roots as follows: $DS'_{(i)} = \left(I_0 - \frac{K}{A_0}\right)e^{S'_i x} + \frac{K}{A_0}$.

If there are T different roots $\{S_{p_1};\ S_{p_2};\dots;\ S_{p_T}\}$ among the group of n roots $\{S'_{(1)}\ ;\ S'_{(2)};\dots;\ S'_{(n)}\}$ of the nth degree polynomial equation, then for each value of $T'\ (2 \leq T' \leq T)$, we can select a specific group of T' different roots $\{S_{q_1};\ S_{q_2};\dots;\ S_{q_{T'}}\}$, then we can use the nth order differential equation to interconnect T' arbitrary values $\{I_0\ ;\dots;I_{T'-1}\}$ identified at T' arbitrary points $\{(x_0, I_0)\ ;\dots;\ (x_{T'-1}, I_{T'-1})\}$. The new solutions of the differential equation are determined by using the new functions expressed in (Equation. 331) where the values of the coefficients $\left\{R'_{(I_0)}\ ;\ R'_{(I_1)}\ ;\dots;\ R'_{(I_{T'-1})}\right\}$ are calculated by using shown expressions in (Equation. 332) whereas the used parameters $R_{(x_L)}$, $O_{(I_L,\ x_L)}$, $P_{(x_L)}$ are as shown in (Equation. 333), (Equation. 334) and (Equation. 335).

$$\left\{DS'_{(n+i>n)} = R'_{(I_{(T'-1)})}e^{xS_{T'}} + \sum_{L=1}^{L=T'-2}\left[\left(R'_{(I_L)} - R'_{(I_{(L+1)})}\right)e^{xS_{(l+1)}}\right] + \left(R'_{(I_0)} - \frac{K}{A_0} - R'_{(I_1)}\right)e^{xS_1} + \frac{K}{A_0};where\ S_k \in \{S_{q_1};\ S_{q_2};\dots;\ S_{q_{T'}}\}\right\} \tag{331}$$

$$R'_{(I_L)} = \frac{I_L - \frac{K}{A_0}(1 - e^{S_1 x_L}) - R_{(x_L)} + O_{(I_L,\ x_L)}}{P_{(x_L)}} \tag{332}$$

$$R_{(x_L)} = R'_{(I_0)}e^{S_1 x_L} + \sum_{j=1}^{T'-1} R'_{(I_j)}\left(e^{S_{j+1}x_L} - e^{S_j x_L}\right) \tag{333}$$

$$O_{(I_L,\ x_L)} = \begin{cases} R'_{(I_0)}e^{S_1 x_L},\ L = 0 \\ R'_{(I_L)}(e^{S_{L+1}x_L} - e^{S_L x_L}),\ L > 0 \end{cases} \tag{334}$$

$$P_{(x_L)} = \begin{cases} e^{S_1 x_L},\ L = 0 \\ (e^{S_{L+1}x_L} - e^{S_L x_L}),\ L > 0 \end{cases} \tag{335}$$

12. Conclusion

This paper presents new engineered methodologies to solve nth order differential equations and nth degree polynomial equations step by step while providing the necessary logic, expressions, conditions, and formulas to solve these equations.

This paper presents the results of deploying the proposed engineered methodologies into solving fourth degree, fifth degree and sixth degree polynomial equations in general forms while presenting the results according to specific theorems and formulas.

This paper also presents the results of deploying the proposed engineered methodologies into solving fourth order, fifth order and sixth order differential equations in general forms while presenting the results of this deployment according to specific theorems and formulas expressing the solutions of these differential equations.

In addition, this paper presents generalized theorems along with specific formulated solutions which we propose to solve nth order differential equations and nth degree polynomial equations in general forms and in complete forms.

Furthermore, this paper presents new theorems expressing new additional solutions for nth order differential equations in order to allow the use of these differential equations and their roots into interconnecting many arbitrary values accorded to specific points, which open the way toward scaling up the use of these differential equations and their solutions in business analytics, data analytics predictive analysis and systems control.

Conflicts of Interest

The author states there is no conflict of interest.

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