A Mathematical Model for Blood flow and Cross Sectional Area of an Artery

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Abstract:

The purpose of this work is to study the effect of blood flow and cross sectional area in artery. The cross sectional area plays an important part in order for the blood to flow smoothly through the blood vessels. A small change in the value for the cross sectional area may affect the amount of blood flow rate through the arteries which also may affect the blood pressure. This paper deals with the study of blood flow which was derived from Navier-Stokes equations. A system of non linear partial differential equations for blood flow and cross sectional area of the artery was obtained. The governing equations are solved numerically by using finite difference method.

Key words: Newtonian fluid, finite difference technique, arterial blood flow.

AMS Subject Classification (1991): 760z05, 92c35

1. Introduction:

Blood is a complex fluid consisting of particulate solids suspended in a non-Newtonian fluid. The particulate solids are red cells (RBCs), white blood cells (WBCs) and platelets. The fluid is plasma, which it self is a complex mixture of proteins and other intergradient in an aqueous base. Blood flow is a study of measuring the blood pressure and finding the flow through blood vessels. This work is important for human health.

There are several researchers, who examined the blood flow in the arteries and veins. The blood flow is significantly already and fluid dynamical factors play an important role. Past studies indicated that one of the reasons having hypertension is when the blood vessels become narrow. This work will focus on the diastolic hypertension.

A lot of work is available, but Belardinelli and Cavalcanti (1991) discussed a new non-linear two-dimensional model of blood motion in tapered and elastic vessels. Jung et. al. (2004) gave an idea on the axi-symmetric flows of non-Newtonian fluids in symmetric stenosed artery. In this work, the hemodynamics behavior of the blood flow is influenced by the presence of the arterial stenosis, again Belardinelli and Cavalcanti (1992) investigated about the theoretical analysis of pressure pulse propagation in arterial vessels. In this study, the model is employed to study the propagation along an arterial vessel of a pressure pulse produced by a single flow pulse applied at the proximal vessel extremity, although Takuji and Guimaraes (1998**)** observed that the effect of non-Newtonian property of blood on flow through a stenosed tube.

Kumar and Kumar (2006) studied on numerical study of the axi-symmetric blood flow in a constricted rigid tube. This is the study of the effect of behavior of blood through a constricted rigid tube with an axi-symmetric stenosis, while Sankar and Hemalatha (2007) discussed also a non-Newtonian fluid flow model for blood flow through a catheterized artery-steady flow, again Kumar and Kumar (2009) observed that a mathematical model for Newtonian and non-Newtonian flow through tapered tubes. The most recent work in this field, Sahu et. al. (2010) worked on the study of arterial blood flow in stenosed vessel using non-Newtonian couple stress fluid model. Singh et. al. (2010) observed that the blood flow through an artery having radially non-symmetric mild stenosis.

Blood behaves as a non-Newtonian fluid but in this model, blood is assumed to be a Newtonian fluid. Even though this will make the problem much simpler, it still is valid since blood in large vessel acting almost like a Newtonian fluid.

2. Mathematical formulation:

Let us assume a cylindrical co-ordinate system (r, z, t) having the z-axis along the axis of arterial segment, while r and t are taken along the radial and circumferential direction. The arterial vessel is assumed to be a rectilinear, deformable, thick shell of isotropic, incompressible material with a circular section and without longitudinal movements, while the blood is considered as an incompressible Newtonian fluid and flow is axially symmetric.

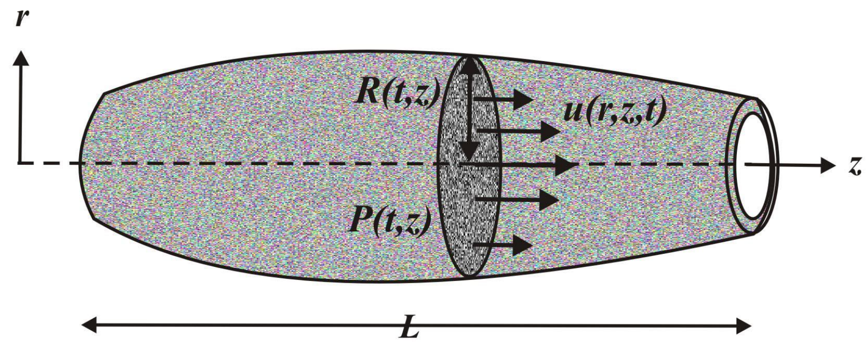

Figure 1: An arterial segment of a visco-elastic artery with length L

The model approach is to use the two-dimensional Navier -stokes equation and continuity equation for a Newtonian and incompressible fluid is given by:

$$\rho\left(\frac{\partial u}{\partial t}+w\frac{\partial u}{\partial r}+u\frac{\partial u}{\partial z}\right)=-\frac{\partial P}{\partial z}+\mu\left(\frac{\partial^2 u}{\partial r^2}+\frac{1}{r}\frac{\partial u}{\partial r}+\frac{\partial^2 u}{\partial z^2}\right) \tag{2.1}$$

$$\rho\left(\frac{\partial w}{\partial t}+w\frac{\partial w}{\partial r}+u\frac{\partial w}{\partial z}\right)=-\frac{\partial P}{\partial z}+\mu\left(\frac{\partial^2 w}{\partial r^2}+\frac{1}{r}\frac{\partial w}{\partial r}+\frac{\partial^2 w}{\partial z^2}-\frac{w}{r^2}\right) \tag{2.2}$$

and equation of continuity is:

$$\frac{1}{r}+\frac{\partial}{\partial r}(rw)+\frac{\partial u}{\partial z}=0 \tag{2.3}$$

where, u is the velocity components in radial direction, w is the velocity components in axial direction, P is the pressure, ρ =density, μ = viscosity of fluid.

For convenience, we define a new variable which is the radial co-ordinate η **:**

$$\eta=\frac{r}{R(z,t)} \tag{2.4}$$

where, $R(z,t)$ denote the inner radius of the vessels and η be the arterial wall viscosity.

The pressure P is assumed to be uniform within the cross section. So that P is independent of the radial co-ordinate η. i.e. $P = P(z,t)$

The above equation (2.1), (2.2) and (2.3) can be written in the new co-ordinate (η, z, t) is:

$$\rho\left[\frac{\partial u}{\partial t}+\frac{1}{R}\left\{\eta\left(u\frac{\partial R}{\partial z}+\frac{\partial R}{\partial t}\right)-w\right\}\frac{\partial u}{\partial \eta}++u\frac{\partial u}{\partial z}\right]=-\frac{\partial P}{\partial z}+\frac{\mu}{R^2}\left[\frac{\partial^2 u}{\partial \eta^2}+\frac{1}{\eta}\frac{\partial u}{\partial \eta}\right] \tag{2.5}$$

$$\rho\left[\frac{\partial w}{\partial t}+\frac{1}{R}\left\{\eta\left(\mu\frac{\partial R}{\partial z}+\frac{\partial R}{\partial t}\right)-w\right\}\frac{\partial w}{\partial \eta}+u\frac{\partial w}{\partial z}\right]=\frac{\mu}{R^2}\left[\frac{\partial^2 w}{\partial \eta^2}+\frac{1}{\eta}\frac{\partial w}{\partial z}-\frac{w}{\eta^2}\right] \tag{2.6}$$

and equation of continuity is:

$$\frac{1}{R}\frac{\partial w}{\partial \eta}+\frac{w}{\eta R}+\frac{\partial u}{\partial z}-\frac{\eta}{R}\left(\frac{\partial R}{\partial z}\right)\left(\frac{\partial u}{\partial \eta}\right)=0 \tag{2.7}$$

where, it can be assumed

$$\frac{\partial^2 u}{\partial z^2}\le 1, \qquad \frac{\partial^2 w}{\partial z^2}\le 1, \qquad \frac{\partial p}{\partial r}\le 1$$

and using simple algebra to change the variable such as:

$$\frac{\partial u(r,z,t)}{\partial t}=\frac{\partial u(\eta,t)}{\partial t}\left(\frac{\partial \eta}{\partial t}\right)+\frac{\partial u(\eta,t)}{\partial t}\left(\frac{\partial t}{\partial t}\right)=\frac{\partial u(\eta,t)}{\partial t}\left(\frac{\partial \eta}{\partial t}+1\right)=\frac{\partial u(\eta,t)}{\partial t}\left[\frac{-\eta}{R}\left(\frac{\partial R}{\partial t}\right)+1\right]$$

The boundary conditions are:

$$u=0, \quad at\ \eta=1 \qquad \qquad w=0, \quad at\ \eta=0$$

$$\frac{\partial u}{\partial t}=0, \quad at\ \eta=0 \qquad \text{and} \qquad w=\frac{\partial R}{\partial t}, \quad at\ \eta=1 \tag{2.8}$$

Now let we are assuming following expression for the solution purpose of the problem

$$u(\eta,z,t)=\sum_{K=1}^{N} q_K\left(\eta^{2K}-1\right) \tag{2.9}$$

The velocity profile in the radial direction is also expressed as:

$$w(\eta,z,t)=\left(\frac{\partial R}{\partial z}\right)(\eta\, w)+\eta\left(\frac{\partial R}{\partial t}\right)\left[1-\left(\frac{1}{N}\right)\sum_{K=1}^{N}\frac{1}{K}\left(\eta^{2K}-1\right)\right] \tag{2.10}$$

according to Belardinelli and Cavalcanti (1991), for simplicity, we choose $N=1$, the equation (2.9) and (2.10) becomes:

$$u(\eta,z,t)=\left(\eta^2-1\right)q(z,t) \tag{2.11}$$

$$w(\eta,z,t)=\left(\frac{\partial R}{\partial z}\right)(\eta\, w)+\eta\left(2-\eta^2\right)\left(\frac{\partial R}{\partial t}\right) \tag{2.12}$$

By plugging equation (2.11) and (2.12) into equation (2.5) and (2.7), the dynamic equations are:

$$\frac{\partial q}{\partial t} - 4\left(\frac{q}{R}\right)\left(\frac{\partial R}{\partial t}\right) - 2\frac{q^2}{R}\left(\frac{\partial R}{\partial z}\right) + 4\nu\left(\frac{q}{R^2}\right) + \frac{1}{\rho}\left(\frac{\partial P}{\partial z}\right) = 0 \tag{2.13}$$

and

$$\frac{\partial R}{\partial t} + \left(\frac{R}{4}\right)\left(\frac{\partial q}{\partial z}\right) + \left(\frac{q}{2R}\right)\left(\frac{\partial R}{\partial z}\right) = 0 \tag{2.14}$$

The cross-section area S (z, t) and the blood flow rate Q can be defined as:

$$S = \pi R^2, \quad and \quad Q = \iint_s u\, d\eta = \frac{1}{2}\pi\, q\, R^2$$

Then equation (2.13) and (2.14) can be written in terms of Q and S as:

$$\frac{\partial Q}{\partial t} - 3\left(\frac{Q}{S}\right)\left(\frac{\partial S}{\partial t}\right) - 2\left(\frac{Q}{S}\right)^2.\left(\frac{\partial S}{\partial z}\right) + 4\pi\left(\frac{\nu}{S}\right).Q + \frac{S}{2\rho}\frac{\partial p}{\partial z} = 0 \tag{2.15}$$

$$\frac{\partial S}{\partial t} - \frac{\partial Q}{\partial z} = 0 \tag{2.16}$$

The equation (2.15) and (2.16) are non linear partial differential equation. Now we obtained the solution for the cross sectional area of the artery and corresponding blood flow rate by solving equation (2.15) and (2.16). These equations are discretized and transformed in to a state equation by using a finite difference method.

We consider segment of the artery with length L is equally divided by N and grids with a step size of $\Delta z = \dfrac{L}{N-1}$. The Mesh point in the finite difference grids are represented by i, where $i = 1, 2, 3, 4 \ldots N \;\; and \;\; N > 2$, by using following finite difference formula for first order.

$$\frac{\partial P_i}{\partial z} = \frac{P_{i+1} - P_i}{\Delta z}, \qquad \frac{\partial Q_i}{\partial z} = \frac{Q_i - Q_{i-1}}{\Delta z}, \qquad \frac{\partial S_i}{\partial z} = \frac{S_{i+1} - S_i}{\Delta z}$$

$$\frac{\partial Q}{\partial t} + 3\left(\frac{Q}{S}\right)\left(\frac{\partial Q}{\partial z}\right) - \left(\frac{Q}{S}\right)^2\left(\frac{\partial S}{\partial z}\right) + \left(\frac{4\pi\nu}{S}\right)Q + \left(\frac{S}{2\rho}\right)\left(\frac{\partial P}{\partial z}\right) = 0$$

Then equation (2.15) and (2.16) can be written as.

$$\frac{\partial Q_i}{\partial t} + 3\left(\frac{Q_i}{S}\right)\left[\frac{Q_i - Q_{i-1}}{\Delta z}\right] - 2\left(\frac{Q_i}{S}\right)^2\left[\frac{S_{i+1} - S_i}{\Delta z}\right] + 4\left(\frac{\pi\nu}{S}\right).Q_i + \frac{S}{2\rho}.\frac{\partial P}{\partial z} = 0 \tag{2.17}$$

$$\frac{\partial S_i}{\partial t} = -\left(\frac{Q_i - Q_{i-1}}{\Delta z}\right) \tag{2.18}$$

We liberalized the dynamic model for the cross-sectional area and corresponding blood flow rate , for the analysis of the nature of the homodynamic behaviors of arteries flow.

The pressure gradient $\frac{\partial P}{\partial z}$ is kept constant and the value is prescribed.

We can simplify the equation (2.17). Then we get:

$$\frac{\partial Q_i}{\partial t} + \beta \frac{\partial P}{\partial z} + \frac{S_i}{2\rho}\frac{\partial P}{\partial z} + (\alpha \upsilon) Q_i = 0 \tag{2.19}$$

where $\alpha = \frac{4\pi\nu}{S_0}$ and $\beta = \frac{S_0}{2\rho}$

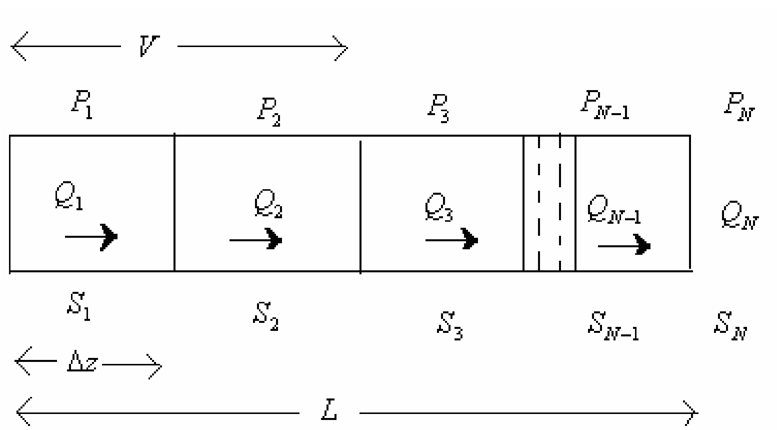

Figure: 2 Discretization of the arterial segment

3. Numerical Approach:

For the simplicity of simulation, the dynamic model in the arterial hemodynamic model was not included. Instead, measured blood pressure signals at section S_1 were used as an input to the arterial model. The definitions of the inputs, state variables and outputs in this simplified model are:

Inputs $u = [P_1 \ P_2]^T$, State variable $y = [Q_1 \ Q_2 .... Q_N S_1 S_2 ...... S_N]^T$

Now we can be written in the form $\frac{\partial y}{\partial t} = f(y)$ of equation (2.17) and (2.18):

where

$$f(y) = \begin{bmatrix} -\left(\alpha . y(1) + \beta \dfrac{\partial P}{\partial z} + \dfrac{y(N+1)}{2\rho}\dfrac{\partial P}{\partial z}\right) \\ -\left(\alpha . y(2) + \beta \dfrac{\partial P}{\partial z} + \dfrac{y(N+2)}{2\rho}\dfrac{\partial P}{\partial z}\right) \\ -\left(\alpha . y(3) + \beta \dfrac{\partial P}{\partial z} + \dfrac{y(N+3)}{2\rho}\dfrac{\partial P}{\partial z}\right) \\ \vdots \quad \vdots \quad \vdots \\ \vdots \quad \vdots \quad \vdots \\ \vdots \quad \vdots \quad \vdots \\ -\left(\alpha . y(N-1) + \beta \dfrac{\partial P}{\partial z} + \dfrac{y(2N-1)}{2\rho}\dfrac{\partial P}{\partial z}\right) \\ -\left(\alpha . y(N) + \beta \dfrac{\partial P}{\partial z} + \dfrac{y(2N)}{2\rho}\dfrac{\partial P}{\partial z}\right) \\ -\left(\dfrac{y(1) - y(0)}{\Delta z}\right) \\ -\left(\dfrac{y(2) - y(1)}{\Delta z}\right) \\ \vdots \quad \vdots \\ \vdots \quad \vdots \\ -\left(\dfrac{y(N-1) - y(N-2)}{\Delta z}\right) \\ -\left(\dfrac{y(N) - y(N-1)}{\Delta z}\right) \end{bmatrix}$$

The hemodynamic processes were simulated by using MATLAB. The required value in the normal condition can be obtained from past works such as:

Blood viscosity $(\mu) = 0.04 \quad poise$, Blood density $(\rho) = 1.06 \ g/cm^3$,

Radius of artery $(r) = 0.5 \quad mm$, Arterial wall viscosity $(\eta) = 100 \ \ dyns/cm^3$,

Length of artery segment $(L) = 15 \ cm$, Kinematic viscosity $(\nu) = 0.035 cm^2/s$,

Initial value of blood flow $Q \ and \ Q_0 = 1 \ to \ 5.4 \ liter/\min ut$,

Initial value of cross sectional area $S \quad and \quad S_0 = 1.5 to 2.0 cm^3$,

Axial pressure gradient $\dfrac{\partial P}{\partial z} = 100 \ \ to \ \ 40 \ \ mmHg$

4. Result and Discussion:

The main purpose of this work, how the cross sectional area of artery affects the blood flow within the artery. Let we are considering, the value of parameters $\rho = 1.06\ g/cm^3$, $S_0 = 1.5 cm^3$, $Q_0 = 16.7 cm^3/\sec$, $\upsilon = 0.035 cm^2/s$, $L = 15 cm$ and node of system $N = 3$ and also consider arteries in a diastole condition only, the choosen time span is 0.2 seconds.

Figure (3) indicated that the blood flow rate against time. It is observed that the result for Q_1, Q_2 and Q_3 are almost same and the value for blood flow is decreasing from its initial value. Figure (4) reveals that the cross-sectional area of artery against time. We observed that the value of S_1, S_2 and S_3 are constant. This could be due to the absence of visco-elastic effect in the present work.

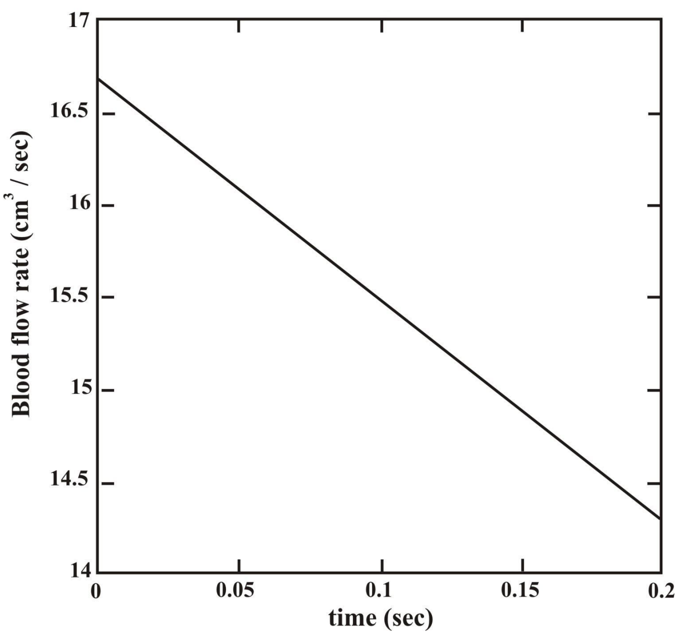

Figure 3: Plot of the blood flow rate against time

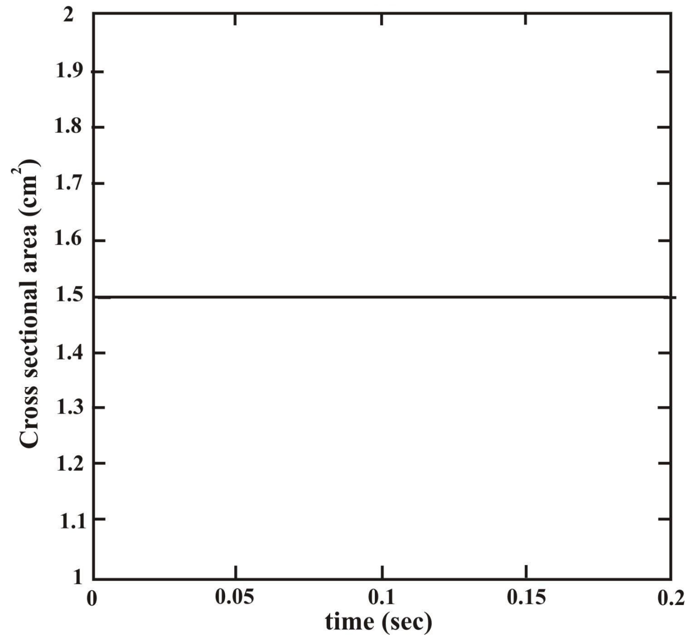

Figure 4: Plot of the cross-sectional area against time

Now, since there is not much difference in the blood flow rate between the sections, we will consider only one section which is S_2 to make the comparison of the different values of the cross sectional area.

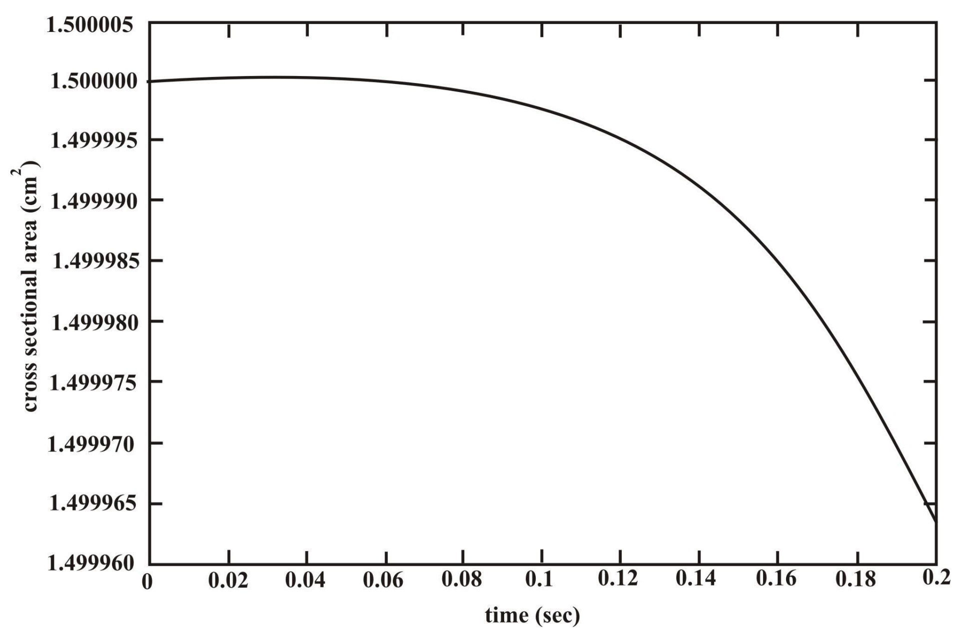

Figure 5: Plot of without changing the value of pressure gradient and the cross-sectional area against time

Figure (5) shows that without changing the value of the pressure gradient and cross-sectional area of the arteries. It decreases in smaller ranges and also shows that the blood flow rate through the arteries is decreasing as time is increasing. Figure (6) observed that if the value of cross-sectional is small, then the blood flow rate is decreasing, which shows that when the cross-sectional area is decreased then the blood flow rate is increased. This condition occurs when the value of cross sectional area in range between $1.5\ cm^2\ to\ 0.9\ cm^2$.

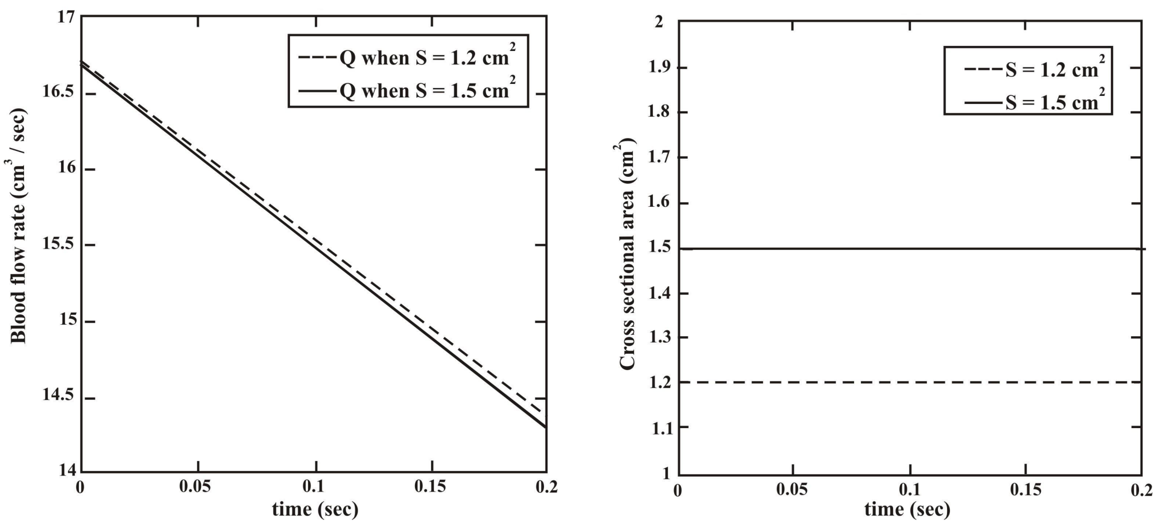

Figure (6) Blood flow rate with different value of cross-sectional area

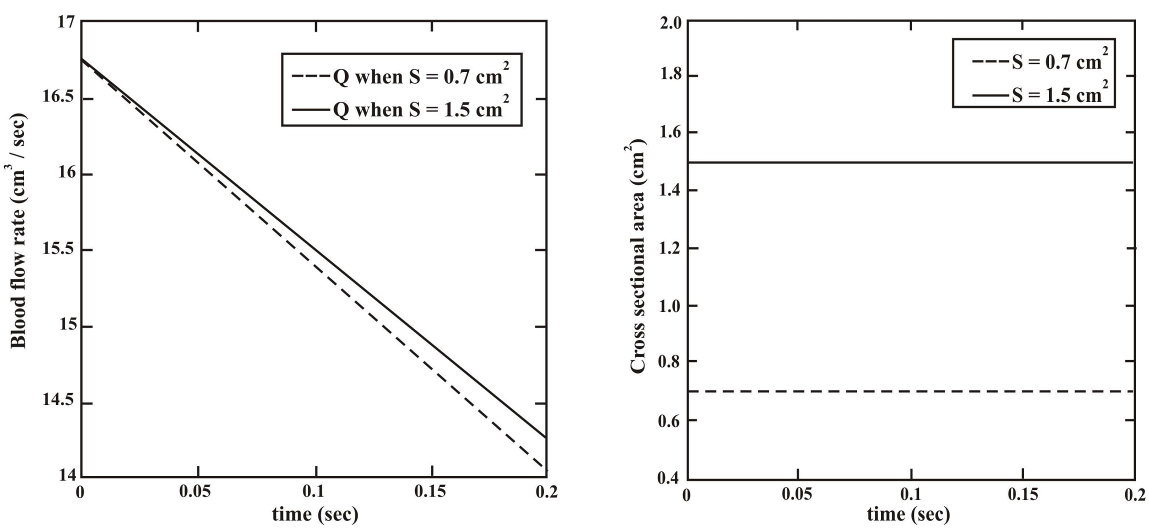

Figure (7): Blood flow rate at normal cross-sectional area and much smaller cross-sectional area

The pressure in artery may increase because of large amount of blood flow through the arteries in a smaller cross sectional area. Thus the blood pressure increases and contributes to high blood pressure.

As we can see in figure (7), when the cross sectional area is below $0.8\ cm^2$, then the blood flow rate decreases faster than the normal rate. But figure (8) slows that the effect of blood flow rate when the cross sectional area is in the range between $0.1\ cm^2$ *to* $0.8 cm^2$. It is indicated that if cross-sectional area of artery continuous to decrease below $0.8\ cm^2$, then the blood flow rate also decreases.

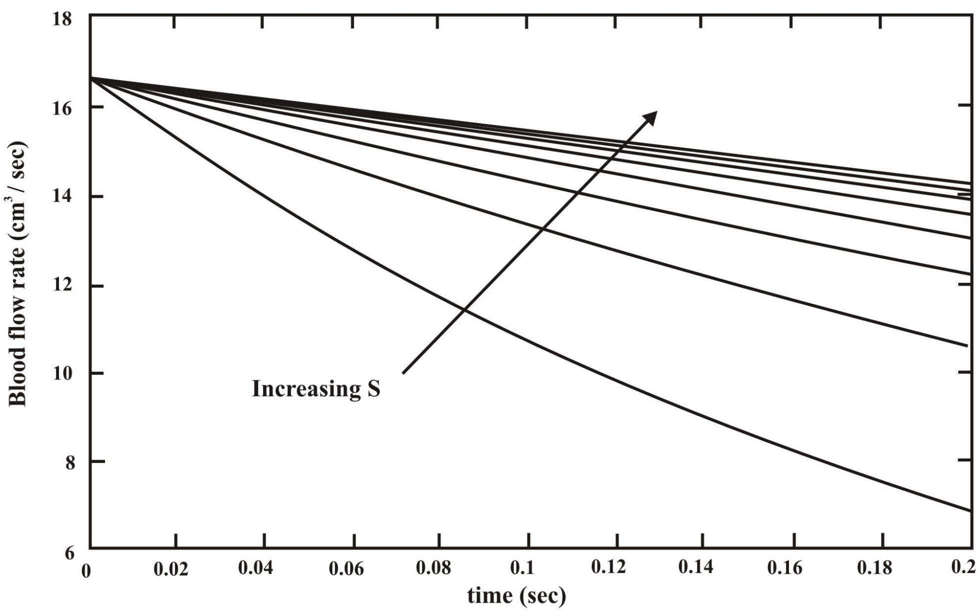

Figure 8: Blood flow rate when the cross-sectional area is in the range between

$0.1\ cm^2$ *to* $0.8\ cm^2$

Conclusion:

In this study**,** we have study the effect of blood flow and cross-sectional area in artery. Even though, the model does not include visco-elastic effect. We observed that a small change in the value for the cross-sectional may affect the amount of blood flow rate through the arteries, which also may affect the blood pressure.

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