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An Upper Bound of an Achromatic Index of K_{14} is 43

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***Abstract*- Achromatic Indices $A(K_n)$ of complete graphs on n vertices are not known in general. It is known that $A(K_{14}) \leq 44$.In this paper we have tightened the upper bound of $A(K_{14})$ to 43.**

***Key Words*- Achromatic index, colouring of graphs, complete edge colouring, proper edge colouring.**

I. INTRODUCTION

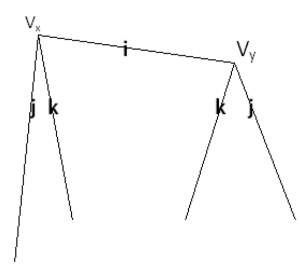
A k-edge colouring of a simple graph G is assigning k colours to the edges of G so that no two adjacent edges receive same colours. If for each pair t_i & t_j of colours there exist adjacent edges with this colours then the colouring is said to be complete.[1] Let G be a simple graph. The achromatic index ψ^{l} (G) of a simple graph G is the maximum number of colours used in the edge colouring of G such that the colouring is complete. All though ψ^{l} (G) is known for some graphs but in general it is not known for arbitrary simple graphs. For Complete graph G of order n , ψ^{l} (G) is denoted by $A(K_n)$. It is known that $39 \leq A(K_{14}) \leq 44$ In this paper we are going to prove that $A(K_{14}) \neq 44$

II. PROOF

Suppose $A(K_{14})=44$
Let C be the optimal colouring of $A(K_{14})$ with 44 colours.
"Any colour in the optimal colouring C has atleast 2 edges of that colour in C " we denote the above argument by *
(If not, Suppose some colour i has only one edge in C then the number of distinct colours represented at the extremities of this edge are at the most 12+12=24
Any other colour apart from above 25 colours (24 + colour i) cann't be present in C as the adjacencies with colour i are exhausted hence C consist of at the most 25 colours
which contradicts to the choice of C. Hence *)
Now $|E(K_{14})|={}^{14}c_2=91$
C can have the following possible structures.

Case i) forty three colours with each colour with twice edges in C & one colour (say t) with five edges.
Or
Case ii)forty two colours with each colour with twice edges in C, one colour (say c1) with three edges & one colour (say c2) with four edges.
Or
Caseiii) forty one colours with two edges of each colour, three colours (say t_1,t_2,t_3) with three edges each.
[Note: Any other combination fails to give the combination $A(K_{14})=44$&| $E(K_{14})|={}^{14}c_2=91$& each colour appearing at least twice in C]

Before discussing the above cases, we will prove that if C contains the following structure for some colour i having exactly two edges of the colour i in C then $A(K_{14}) \leq 43$
Suppose colour i has exactly two edges in C & for some colours j, k the following position is appearing in K_{14}

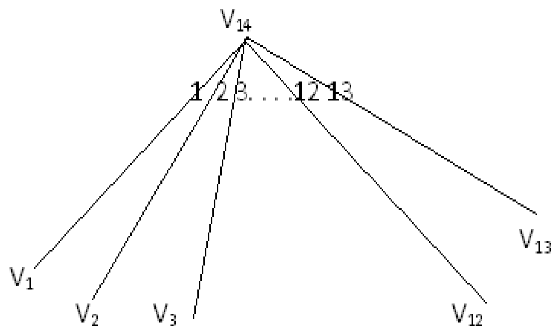


(Numbers on edges represent colours throughout this paper.)

The number of distinct colours at the extremities of the edges coloured i apart from the colours j,k are at the most 10+10+10+10=40
Any other colour apart from above 43 colours
(40+ colours i,j,k)
cann't be present in C as the adjacencies with colour i are exhausted hence C consist of at the most 43 colours which contradicts to the choice of C.
The above position we will denote by "PP".

Now consider the **case i**

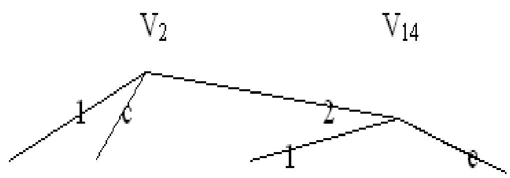
Apart from 10 extremities of colour t, the any of the remaining 4 vertices has 13 colours incident with it & these 13 colours (WLG say 1,2,…..13) have exactly two edges in C (i.e exactly two edges are colored 1,exactly two edges are coloured 2,……,exactly two edges are coloured 13)
WLG we arrive at the following situation.



Some pair of vertices must be joined to each other coloured as 1.W.L.G. V_2V_3 be coloured as 1.

Now neither of the colours from 2 to 13 can be incident with V_2 or V_3 because if 2 is incident with V_2 or 3 is incident with V_3 will contradict to the fact C is proper edge colouring of K_{14}

If colour c ($3\leq c\leq 13$) is incident with V_2 then "PP" appears as shown below.



Similarly if colour c (either c=2 or $4\leq c\leq 13$) is incident with V_3 then"PP" appears. Similarly we can argue for the remaining vertices,

The vertices from V_1 to V_{13}can be joined pair wise by edges & can accommodate at the most 6 colours from the colours 1,2,…13 leaving no choice for the second edges of the remaining 7 colours. Therefore case i is discarded.

Caseii) Colours c1 & c2 together have 7 edges. If all of them are disjoint then it contradicts to the fact that C is complete colouring of K_{14}.hence ∋ a vetex at which c1& c2 are adjacent to each other once so ∋ a vetex at which neither c1 nor c2 are incident with it & hence v has 13 colours incident with it & these 13 colours (WLG say 1,2,…..13) have exactly two edges in C. So the case becomes similar to case i

hence we discard case ii .

Case iii) If ∋ a vetex as in the case i then we arrive at contradiction. Hence ∌a vertex as in the case i

If at every vertex at least two of t1, t2, t3 are incident then

Let H be the subgraph of K_{14} having edges coloured by t_1, t_2, t_3 therefore $|E(H)|=9$ & $\deg_H V\geq 2$ for all vertices $V\in V(K_{14})$

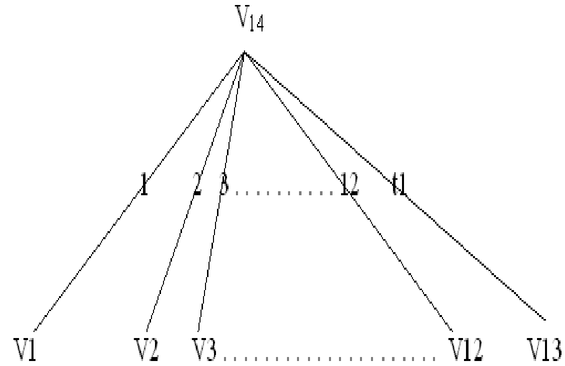
Now $2|E(H)|=18$

$\therefore 18=\sum \deg_H V_i$ for i=1 to 14

But $\sum \deg_H V_i \geq 2\text{x}14=28$

$\therefore 18\geq 28$ which is a contradiction.

∴ ∋ a vertex (wlg say V_{14}) such that exactly one of the colours t_1 or t_2 or t_3 (wlg say t_1) is incident with it. In general we arrive at the following position.



Now we will think of second edges of the colours 1 to 12.

As discuss in the case i, vertices V_1 ,V_2,…..V_{12} are pair wise joined & can be coloured with at most 6 colours from the colours 1 to 12 hence at least one of the colours from the colours 1 to 12 is incident with V_{13}

Suppose q edges are of the type $V_{13}V_j$ where $1\leq j\leq 12$ & are coloured with some of q colours from 1,2,…,12.

∴ the remaining 12- q vertices can accommodate at the most (12-q)/2 colours from the remaining 12-q colours

(at most (12-q)/2 comes from the discussion of case i)

$\therefore q + (12-q)/2 = 12$

$\therefore q=12$

∴V_1V_{13}, V_2V_{13}, …………,$V_{11}V_{13}$, $V_{12}V_{13}$ are coloured using the colours 1,2,….12.

WLG $V_{12}V_{13}$ is coloured with1.

∴The number of distinct colours adjacent to colour 1 in C are at the most

13(namely 1,2,…12,t_1) + 12 (namely V_1V_2,V_1V_3,…,V_1V_{12}) + 10 (namely $V_{12}V_2V_{12}V_3$,………..$V_{12}V_{13}$) = 35 colours

Any other colour apart from above 36 colours (35+ colour 1) Can't be present in C as the adjacencies with colour 1 are exhausted hence C consist of at the most 36 colours which contradicts to the choice of C.

III. CONCLUSION

∌ an complete edge colouring of K_{14} with 44 colours.

$\therefore A(K_{14})\leq 43$

REFERENCES

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