Fixed points for α-Ψ Contractive Mapping in 2-Metric spaces

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***Abstract.* In this paper, we introduce the notion of α-Ψ contractive type mappings in 2-metric spaces and establish fixed point theorems for these mappings.**
***Keywords*: 2-metric space, α-Ψ contractive maps, α-admissible.**

I. Introduction.

The concept of 2-metric space has been investigated by Gahler [1] to generalize the concept of metric i.e., distance function. Iseki [2] set out the tradition of proving fixed point theorems for various contractive conditions in 2-metric spaces. The study was further enhanced by Rhoades [5], Iseki [2], Sharma [6, 7, 8], Khan [3] and Ashraf [4].Recently, Samet et.al. [9]. Introduced the notion of α-ψ contractive mappings and α-admissible mapping in metric spaces as follows:

Definition 1.1[1]. Let X denotes a set of nonempty set and d : X×X×X→ R be a map satisfying the following conditions:

For every pair of distinct points a, b ∈ X, there exists a point c ∈X such that d(a, b, c) ≠ 0.

(i) d(a, b, c) = 0, only if at least two of three points are same.

(ii) The symmetry: d(a, b, c) = d(a, c, b) = d(b, c, a) = d(b, a, c) = d(c, a, b) = d(c, b, a) for all a, b, c ϵ X.

(iii) The rectangular inequality: d(a, b, c) ≤ d(a, b, d) + d(b, c, d) + d(c, a, d) for all a, b, c, d ϵ X.

Then d is called 2-metric on X and (X, d) is called a 2-metric.

Definition 1.2[1]. A sequence $\{x_n\}$ is said to be Cauchy sequence in 2-metric space, if for each a ϵ X, $\lim_{m,n\to\infty} d(x_n, x, a) = 0$.

Definition 1.3[1]. A sequence $\{x_n\}$ in 2-metric space is said to be convergent to an element x ϵX , if for each a ϵ X.

Definition 1.4[1]. A complete 2-metric space is one in which every Cauchy sequence in X is convergent to an element of X,

$\lim_{n\to\infty} d(x_n, x, a) = 0$.

Definition 1.5[9]. Let T:X→X and α:X×X→[0,∞). We say that T is α-admissible if x,y∈X, α(x, y) ≥ 1 implies α(Tx, Ty) ≥ 1.

Denote with ψ the family of non decreasing function Ψ:[0,+∞)→ [0,+∞) such that $\sum_{n=1}^{\infty} \Psi^n < +\infty$ for each t>0,where Ψ^n is the nth iterate of Ψ.

Lemma 1.1. For every function Ψ:[0,+∞)→[0,+∞) the following holds:

If Ψ is non decreasing, then for each t>0,$\lim_{t\to\infty} \Psi^n(t)=0$ implies Ψ(t)<t.

Now we shall introduce the notion of α-Ψ contractive type mapping in 2-metric spaces and establish fixed point theorems for these mappings.

Definition 1.6[9]. Let (X, d) be a metric space and T be a self map on X. T is said to be α-Ψ contractive mapping if there exist two functions α:X×X→[0,∞) and Ψ∈ψ such that α(x, y)d(Tx, Ty)≤Ψ(d(x, y)) for all x, y ∈X.

Definition 1.7. Let (X, d) be a 2-metric space and T:X→X be given mapping. We say that T is an α-Ψ contractive mapping if there exists two functions α:X×X×X→[0,∞) and Ψ∈ψ such that

α(x,y,a)d(Tx,Ty,a) ≤ Ψ(d(x,y,a)) for all x,y,a∈X. (1.1)

Remark 1. If T:X→X satisfies the Banach contraction principle in the setting of 2-metric spaces, then T is an α-Ψ contractive mapping, where α(x,y,a)=1 for all x,y,a∈X and Ψ(t)=kt and some k∈[0,1).

Definition 1.8. Let T:X→X and α:X×X×X→[0,∞).We say that T is α-admissible if x,y,a∈X,

α(x,y,a)≥1 implies α(Tx,Ty,a)≥1.

Example Let X=[0,∞) define T:X→X and α:X×X×X→[0,∞) by

$$Tx=\begin{cases} 3\ln x & \text{if } x \neq 0 \\ e & \text{otherwise} \end{cases} \quad \text{and}$$

$$\alpha(x,y,z)=\begin{cases} e & if\ x\geq y\geq z \\ 0 & otherwise \end{cases}$$

Then T is α-admissible.

II. Main Results

Now we prove our main results for α-Ψ contractive type mapping in 2-metric spaces.

Theorem 2.1. Let (X,d) be a complete 2-metric space and T:X→X be an α-Ψ contractive mapping satisfying the following conditions:

(2.1) T is α-admissible,

(2.2) there exists, x_0∈X such that α(x_0,Tx_0,a)≥1 for all a∈X,

(2.3) T is continuous.

Then T has a fixed point i.e., there exists u∈X such that Tu=u.

Proof. Let x_0∈X such that α(x_0,Tx_0,a) ≥1 for all a∈X. Define the sequence $\{x_n\}$ in X by Tx_n=x_{n+1} for all n∈N. In particular, if x_n=x_{n+1} for some n∈N then u=x_n is a fixed point for T. Assume that x_n≠x_{n+1} for all n∈N. Since T is α-admissible, we have

α(x_0,x_1,a) = α(x_0,Tx_0,a)≥1 implies

α(Tx_0,Tx_1,a)= α(x_1,x_2,a)≥1.

By induction, we get

$$\alpha(x_n,x_{n+1},a) \geq 1 \text{ for all } n\in N. \qquad (2.4)$$

Using (1.1) and (2.4), we get

$$d(x_n,x_{n+1},a) = d(Tx_{n-1},Tx_n,a)$$
$$\leq \alpha(x_{n-1},x_n,a)d(Tx_{n-1},Tx_n,a)$$
$$\leq \Psi(d(x_{n-1},x_n,a).$$

By induction, we have

d(x_n,x_{n+1},a) ≤Ψ^n(d(x_0,x_1,a) for all n∈N.

Now

$$d(x_n,x_m,a) \leq d(x_n,x_m,x_{n+1}) + d(x_n,x_{n+1},a)+d(x_{n+1},x_m,a)$$
$$\leq \Psi^n(d(x_0,x_1,a)+\Psi^n(d(x_0,x_1,a) + d(x_{n+1},x_m,a) + d(x_n,x_m,a)$$
$$\leq \Psi^n(d(x_0,x_1,a)+\Psi^n(d(x_0,x_1,a) + d(x_{n+1},x_m,x_{n+2})+d(x_{n+1},x_{n+2},a) + d(x_{n+2},x_m,a).$$

$$d(x_n,x_m,a)\leq (\Psi^n + \Psi^{n+1})d(x_0,x_1,x_m) + (\Psi^n + \Psi^{n+1})d(x_0,x_1,a)+ d(x_{n+2},x_m,a)$$

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$$\leq(\Psi^n + \Psi^{n+1}+\dots+\Psi^{m-2})d(x_0,x_1,x_m)+ (\Psi^n + \Psi^{n+1}+\dots+\Psi^{m-2})d(x_0,x_1,a) +d(x_{m-2},x_m,a)$$

$$\leq (\Psi^n + \Psi^{n+1}+\dots+\Psi^{m-2})\, d(x_0,x_1,x_m) +(\Psi^n + \Psi^{n+1}+\dots+\Psi^{m-2} + \Psi^{m-1})\, d(x_0,x_1,a).$$
$$\leq\frac{\Psi^n}{1-\Psi}\, d(x_0,x_1,x_m)+\frac{\Psi^n}{1-\Psi}\, d(x_0,x_1,a). \qquad (2.5)$$

Again, we have

$$d(x_0,x_1,x_m) \leq d(x_0,x_1,x_{m-1})+ d(x_0,x_{m-1},x_m) + d(x_{m-1},x_1,x_m)$$
$$\leq d(x_0,x_1,x_{m-1})+\Psi^{m-1}\, d(x_0,x_1,x_0) +\Psi^{m-2}\, d(x_1,x_2,x_1)$$

= d(x_0,x_1,x_{m-1}).

i.e., d(x_0,x_1,x_m) ≤ d(x_0,x_1,x_{m-1})
≤ d(x_0,x_1,x_{m-2}).

Continuing in this way,
d(x_0,x_1,x_m)≤d(x_0,x_1,x_1)=0.i.e.,d(x_0,x_1,x_m) =0.
Letting n,m→∞ in (2.5),we have d(x_n,x_m,a)→0.
Therefore, we have $\{x_n\}$ is a Cauchy sequence in a complete 2-metric space, so there exists a point u∈X such that x_n→u. Since T is continuous, Tu=$\lim_{n\to\infty} Tx_n$= $\lim_{n\to\infty} x_{n+1}$=u i.e., Tu=u.
Hence T has a fixed point.
In next theorem we omit the continuity hypothesis of T.

Theorem 2.2.Let (X,d) be a 2-metric space and T:X→X be an α-Ψ contractive mapping satisfying the following conditions:

(2.6) T is α-admissible,

(2.7) there exists,x_0∈X such that α(x_0,Tx_0,a)≥1 for all a∈X,

(2.8) if $\{x_n\}$ is sequence in X such that α(x_n,x_{n+1},a)≥1 for all n and x_n→x as n→∞, then α(x_n, x, a) ≥1 for all n.

Then T has a fixed point i.e., there exists u∈X such that Tu=u.

Proof. From the proof of Theorem 2.1, we know $\{x_n\}$ is a Cauchy sequence in complete 2-metric space (X,d), therefore there exists x^*∈X such that x_n→ x^* as n→∞.
On the other hand, from (2.1) and hypothesis (2.4) we have,

α(x_n, x^*,a) ≥1 for all n∈N (2.9)

Now d(Tx^*,x^*,a)≤d(Tx^*,x_n,a)+d(Tx_n,Tx^*,a)
+d(Tx^*, x^*,Tx_n)
= d(Tx_n, x^*,a)+ d(Tx_n,Tx^*,x^*)
+ d(Tx_n, Tx^*,a)

d(Tx^*,x^*,a) =α(x_n, x^*,a) d(Tx_n, Tx^*,a)+
d(Tx_n, x^*,a)+ d(Tx_n, Tx^*,x^*)

=α(x_n, x^*,a)d(Tx_n, Tx^*,Ta)+ d(Tx_n, x^*,a) + d(Tx_n, Tx^*,x^*)
=Ψ(d(x_n, x^*,a)).

Letting n→∞, we have,
d(Tx^*,x^*,a)=0, implies Tx^*=x^*.

Example 2.1. Let X=R with 2- metric d is defined by d(x,y,a)=$|x-y|$ for all x,y,a∈X.
Define the mapping T:X→X by

$$Tx=\begin{cases} 2x-\frac{3}{2} \; if x > 1, \\ \frac{x}{2} \; if\ 0 \le x \le 1, \\ 0 \quad if x < 0. \end{cases}$$

We observe that Banach contraction principle in setting of 2-metric space cannot be applied in this case, since
d(T1,T2,3)=|T1-T2|=2>1=d(2,1,3).
Now we define α:X×X×X→[0,∞) by

$$\alpha(x,y,a)=\begin{cases} 1 \quad \text{if } x,y,a \in [0,1] \\ 0 \quad \text{otherwise.} \end{cases}$$

Clearly, T is α-Ψ contractive mapping with Ψ(t)=$\frac{t}{2}$ for all t≥0.
Infact, for all x,y,a∈X we have α(x,y,a)d(Tx,Ty, a)≤$\frac{1}{2}$d(x,y,a).
Moreover, there exists x_0∈X such that α(x_0,Tx_0,a)≥1. Infact for $x_0 = 1$ we have α(1,T1,a)=1.
Obviously, T is continuous, so it remain to show that T is α-admissible.
Let x,y,a∈X such that α(x,y,a)≥1 implies x,y,a∈[0,1].
By definition of T and α we have Tx=$\frac{x}{2}$∈[0,1] Ty=$\frac{y}{2}$∈[0,1] Ta=$\frac{a}{2}$∈[0,1] and α(Tx,Ty,a)=1 then T is α-admissible.
Now all hypothesis of Theorem 2.1 are satisfied, consequently T has a fixed point. Note that Theorem 2.2 guarantees only the existence of fixed point but not uniqueness. In this example 0 and $\frac{3}{2}$ are two fixed point of T.
Now we give an example involving a function that is not continuous.

Example 2.2. Let X=R with standard metric d(x,y,a)=$|x-y|$ for all x,y,a∈X.
Define the mapping T:X→X by

$$Tx=\begin{cases} 2x-\frac{3}{2} \; if\ x > 1, \\ \frac{x}{2} \; if\ 0 \le x \le 1, \\ 0 \quad if x < 0. \end{cases}$$

It is clear that T is not continuous at point 1 therefore Banach contraction principle in the Theorem 2.1. is not applicable in this case.
Define the mapping α:X×X×X→[0,∞) by

$$\alpha(x,y,a)=\begin{cases} 1 \quad if x,y,a \in [0,1], \\ 0 \quad otherwise. \end{cases}$$

Clearly T is an is α-Ψ contractive mapping with Ψ(t)=$\frac{t}{4}$ for all t≥0.
In fact, x,y,a ∈X, we have

$$\alpha(x,y,a)d(Tx,Ty,a) \leq \frac{1}{4}d(x,y,a).$$

Moreover, there exists x_0∈X such that α(x_0,Tx_0,a)≥1. Infact, for $x_0 = 1$ we have α(1,T1,a)=1.
Now let x,y,a∈X such that α(x,y,a)≥1 implies x,y,a∈[0,1].
By definition of T and α we have Tx=$\frac{x}{4}$∈[0,1] Ty=$\frac{y}{4}$∈[0,1] Ta=$\frac{a}{4}$∈[0,1] and α(Tx,Ty, a)=1 then T is α-admissible.
Finally let $\{x_n\}$ be sequence in X such that α(x_n,x_{n+1},a)≥1 for all n and x_n→x as n→∞.
Since α(x_n,x_{n+1},a)≥1 for all n and by definition of α we have x_n∈[0,1] for all n and x∈[0,1].
Then α(x_n,x,a)=1.
Therefore, all hypothesis of Theorem 2.3 are satisfied. So T has fixed point. Here 0 and $\frac{3}{2}$ are two fixed point of T.
To assure the uniqueness of fixed point we will consider the following hypothesis:
(H): For all x,y, ∈X there exists z ∈X such that α(x,y,z)≥1, α(y,x,z)≥1.
Theorem 2.3. Adding condition (H) to hypothesis of Theorem 2.2, we obtain uniqueness of fixed point of T.

Proof. Suppose x^* and y^* are two fixed point of T.
From (H) there exists z∈ X such that
α(x^*,y^*,z)≥1, α(y^*,x^*,z)≥1. (2.10)
Since T is α-admissible from (2.10), we have
α(x^*,y^*,T^nz)≥1 and α(y^*,x^*,T^nz)≥1 for all n∈N. (2.11)
From (2.11) and (1.1)

$$d(x^*,y^*,T^n z,)=d(Tx^*,Ty^*,TT^{n-1}z)$$
$$\leq\alpha(x^*,y^*,T^n z)\, d(Tx^*,Ty^*\,TT^{n-1}z)$$
$$\leq\Psi\, d(x^*,y^*,T^n z).$$

This implies $d(x^*,y^*,T^n z,)\leq\Psi^n(d(x^*,y^*,T^n z))$ for all n∈N.
Letting n→∞,we get $x^*=y^*$. (2.12)
Hence T has a unique fixed point.

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