# Geodesic and Exponential Maps on a Riemannian Manifold

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***Abstract:-* Geodesic curve plays an important role in the development of Geometry. Our aim is to ensure local diffeomorphism on a Riemannian manifold which are obtained with the help of exponential mapping.**

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## 1. INTRODUCTION

We introduce Riemannian manifold which is a natural development in so far as differential geometry of surfaces are concerned. Given a surface S measuring lengths of tangent vectors is enabled by the inner product <, > of vectors tangent to the surface at a point this is the induced inner product in $R^3$. Then to compute the length of curves in S is to integrate the length of the velocity vector. The inner product not only permits us to measure the length of curves in S but also the area of domains in S, as well as the angle between two curves and all the metric ideas arising in geometry ,more generally these notions lead to define on S certain special curves Geodesics which possess the length minimizing property that is given any two points p & q on a Geodesic sufficiently close, the length of such a curve is less than or equal to the length of any other curve joining p to q such curves in many situations behave like the straight lines of S. These curves play an important role in the development of Geometry.

## 2. PRELIMINARIES

Before entering into our work, we recall the following definitions and some results.

**Definition 2.1(Riemannian Metric) [1]:** Let M be a smooth manifold of dimension n. A Riemannian metric or Riemannian structure on M is a correspondence which associates to each point $p \in M$ an inner product $<, >_p$ i.e. a symmetric, bilinear, positive definite form on the tangent space $T_pM$, which varies differentiably in the following sense:

If x: U --->M is a system of coordinates around p, with x $(x_1,---------,x_n)$ = q(U) and $\frac{\partial}{\partial x}(q) = dx_q(0, \dots 1, \dots 0)$,then $<\frac{\partial}{\partial x_i}(q), \frac{\partial}{\partial x_j}(q)> = g_{ij}(x_1, \dots\dots x_n)$is a differentiable function on U.

Clearly this definition is independent of the choice of coordinate system. Another way to express the differentiability of the Riemannian metric is to say that for any pair of vector fields X and Y which are differentiable in a neighbourhood V of M, is the function <X, Y> differentiable on V. Generally, we omit the index p in $< ,>_p$ and simply write < , >. The function $g_{ij}(= g_{ji})$ is called the local expression of theRiemannian metric in the coordinate systemx : $U \subset Rn --> M$.

A differentiable manifold with a given Riemannian metric will be called a Riemannian manifold.

**Definition 2.2 [1]:** Let M and N be Riemannian manifolds A diffeomorphism f : M -----> N , i.e. a differentiable bijection with a differentiable inverse called an isometry if

$\langle u , v\rangle_p = \langle dfp(u), \ dfp(v)\rangle_{f(p)}$ ---------(1) $\forall p \in M$, $( u,v) \in T_pM$.

**Definition 2.3 [1]:** For f : M -----> N, M,N be the differentiable manifold, f is called a local isometry at $p \in M$ if there is a neighbourhood U of M of p such that f : U -----> f(U),is a diffeomorphism satisfying (1).

It is common to say that a Riemannian manifold M is locally isometric to a Riemannian manifold N if for every p in M there exists a neighbourhood U of p in M and local isometry f : U -----> $f(U) \subset N$.

**Example 2.4:** Let $M = R^n$ with $\frac{\partial}{\partial x_i}$ identified with $e_i = (0, \dots\dots 1, \dots\dots, 0)$ is trivially a Riemannian manifold. The metric is given by $\langle ei, ej\rangle = \delta_{ij}$. $R^n$ is called Euclidean space of dimension n and the Riemannian geometry of this space is metric Euclidean geometry.

**Definition 2.5 (Immersion and Embeddings) [1] :** Let M and N be differentiable manifolds of dimension m and n respectively. A differentiable mapping φ: M ----> N is said to be Immersion if $d\varphi_p : T_pM \longrightarrow T_{\varphi(p)}N$ is injective for all p∈M. If in addition φ is a homeomorphism φ(M) ⊂ N, where φ(M) has the subspace topology induced from N, we say that φ is an embedding. If M ⊂ N and the inclusion η : M ⊂ N is an embedding we say that M is a submanifold of N. It can be seen that if φ : M ----> N is an immersion, then m < n ; the difference n –m is called the co-dimension of the immersion φ.

**Example 2.6:** The curve $\alpha(t) = (t^3-4t, t^2-4)$ is an immersion α : R ---->$R^2$ which has a self-intersection for t = 2 and t = -2. α is therefore not an embedding.

**Example 2.7:** The curve α : R ---->$R^2$ given by $\alpha(t) = (t^3, t^2)$ is a differentiable but is not an immersion. For immersion, in this case in particular, α'(0)≠ 0 which is not so for t = 0.

**Example 2.8:** The curve

$$\alpha(t) = \begin{cases} (0, -(t+2)), & t \in (-3,-1) \\ \text{regular curve}, & t \in (-1, -\frac{1}{x}) \\ \left(-t, -\sin\frac{1}{t}\right), & t \in \left(-\frac{1}{x}, 0\right) \end{cases}$$

is an immersion α : (-3,0) --->$R^2$ without self intersection never the less α is not an embedding.

In a Riemannian manifold the length of a segment, is defined by $L^b_a(c)=\int_a^b \langle \frac{dc}{dt}, \frac{dc}{dt}\rangle^{1/2} dt$ .

## 3. MAIN RESULTS

**Definition 3.1 (Geodesics map):** A parametrized curve γ : I----> M, is a geodesic at $t_o$∈ I, if $\frac{D}{dt}\left(\frac{dr}{dt}\right) = 0$ at $t_o$.

If γ is a geodesic at t, for all t ∈I we say that γ is a geodesic.

If γ is a geodesic and [a,b]⊂ I is such that γ is restricted to [a, b] in I then γ is called a geodesic segment joining γ(a) and γ(b).

If γ : I----> M is a geodesic then $\frac{d}{dt}\langle \frac{d\gamma}{dt} \frac{d\gamma}{dt}\rangle = 2 \langle \frac{D}{dt}\frac{d\gamma}{dt}, \frac{d\gamma}{dt}\rangle = 0$

This implies the length of the length of the tangent vector $\frac{d\gamma}{dt}$ is constant. If $|\frac{d\gamma}{dt}| = c \neq 0$ means that $\gamma$ is not reduced to a point as a geodesic segment.

Local equation satisfied by a geodesic, $\gamma$ : I----> M be a geodesic and (U,x) be a chart about x in M. $\gamma$ has coordinates in (U, x) at $\gamma(t_o)$ in U, a curve $\gamma$ given by

$\gamma(t) = (x_1(t), \ldots\ldots, x_n(t))$ is a geodesic if and only if it satisfies a second order system of ordinary differential equations$\frac{d^2x_k}{dt^2} + \sum_{i,j} \Gamma_{ij}{}^k \frac{dx_i}{dt}\frac{dx_j}{dt} = 0$, k = 1,…..n,

We do not study the geodesic flows but however, we use geodesics to define an exponential map, which would enable us to obtain local diffeomorphism.

**Definition 3.2 (Exponential map):** Let M be as usual a smooth manifold and let TM denote the tangent bundle associated with M $p \in M$. Let $u \subset TM$ be an open set in TM. Then the map exp : u ----->M given by $\exp(q, v) = \gamma(1,q,v) = v(|v|, q, \frac{v}{|v|})$ , $(q, v) \in u$ is called the exponential map on u.

Clearly exponential is differentiable and in general, we restrict exponential map to a tangent space $T_pM$, in particular in open ball $\exp_q: B_\in(0) \subset T_qM$ -----> M by $\exp_q(v) = \exp(q, v)$,where $B_\in(0)$ is an open ball with centre 0 and radius $\in$.

Geometrically, $\exp_v$ is a point of M obtained by going to the length equal to $|v|$, starting from q, along a geodesic which passes through q with velocity equal to $\frac{v}{|v|}$.

**Proposition 3.3:** Given $q \in M$ there exits an $\in > 0$ such that $\exp_q : B_\in(0) \subset T_qM$ -----> M is a diffeomorphism of $B_\in(0)$ onto an open subset of M.

**Proof:** Let us calculate $d(\exp_q)_0(v) = \frac{d}{dt}\left(\exp_q(tv)\right)|t=0 = \frac{d}{dt}(1, q, tv)|t=0$

$$= \frac{d}{dt}(t, q, v)|t = 0 = v.$$

Hence $d(\exp_q)_0$ is the identity of $T_pM$, and it follows from the inverse function theorem that $\exp_q$ is a local diffeomorphism on a neighbourhood of 0.

Diffeomorphism Via exponential maps:

(i) Let (M, S) be a Riemannian manifold $p \in M$ and $v \in T_pM$, tangent vector at p to M. We give local geometric feature associated with the exponential map.

For each $p \in M$ and $v \in T_pM$ there is a unique speed geodesic such that

$\gamma_v$:[0,∞) -------->M. Starting from p with $\gamma_v(0) = p$ and $\gamma'_v(0) = v$. Given $p \in M$, let $\exp_p : T_pM$----->M denote the exponential map defined by $\exp_p(v) = \gamma_v(1)$

By the uniqueness of geodesics with given initial data, one has

$\gamma_v(t) = \gamma_{tv}(1)$

$\gamma_v(t) = \exp_p(tv)$

Further, observe that $(\exp_p)_* : T_o(T_pM)$---------> $T_pM$ is the natural isomorphism.

(ii) By the inverse mapping theorem ,there exists $\in>0$ such that

$\exp_p|_{B(0,\in)}$------> $\exp_p(B(0,\in)) \subset M$ is a diffeomorphism, where B(v, r) denotes the open ball with $v \subset T_pM$ and radius $r > 0$.

## REFERENCES


[1] DoCarmo M., Differential of curves and Surfaces, Englewood, cliffs, Prentice Hall.

[2] Flanders H., Differential forms with applications to Physical sciences, Academic N.Y, 1963.

[3] Kobayashi and Nomizu K, Foundation of Differential Geometry , Vol I and II, Inter Science (Wiley).

[4] DoCarmo M., Rimennian Geometry, Birkhauser Boston Basel Berlin -1992.

[5] Hu S.T., Differential Manifolds, Academic Press. 1962