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Sequences of Special Dio - Triples

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Abstract -This paper concerns with the study of constructing sequences of special Dio-Triples $(\mathbf{a},\mathbf{b},\mathbf{c})$ such that the product of any two elements of the set added with their sum and increased by a non-zero integer or a polynomial with integer coefficients is a perfect square.

Keywords: Diophantine Triples, Special Dio–Triple, Integer coefficients

I. INTRODUCTION

The problem of constructing the sets with property that product of any two of its distinct elements is one less than a square has a very long history and such sets have been studied by Diophantus [1]. A set of m positive integers $\{a_1, a_2, a_3, \dots a_m\}$ is said to have the property $D(n)$ $n \in Z - \{0\}$ if $(a_i * a_j) + n$ is a perfect square for all $1 \le i \le j \le m$ and such a set is called a Diophantine m - tuple with property $D(n)$

Many mathematicians considered the construction of different formulations of Diophantine Triples with the property $D(n)$ for any arbitrary integer n and also, for any linear polynomials in n. In this context, one may refer [2 – 19] for an extensive review of various problems on Diophantine Triples. This paper aims at constructing sequences of Special Dio-Triples where the product of any two members of the triple with the addition of the same members and the addition with a non-zero integer or a polynomial with integer coefficients satisfies the required property

II. METHOD OF ANALYSIS

Sequence I

An attempt is made to form a sequence of special Dio-Triples $(a,b,c),(b,c,d),(c,d,e),\dots$ with the property $D(2^{3n}+5.2^{2n}+2^n+2)$

Case I

Let $a=2^n$ and $b=2^{3n}+2$

Let c be any non-zero integer.

Consider $ac+a+c+2^{3n}+5.2^{2n}+2^n+2=p^2$ (1)

which yields $(2^n+1)c+2^{3n}+5.2^{2n}+2.2^n+2=p^2$

$bc+b+c+2^{3n}+5.2^{2n}+2^n+2=q^2$ (2)

gives $(2^{3n}+3)c+2.2^{3n}+5.2^{2n}+2^n+4=q^2$

Using some algebra,

$$(2^{3n}+3)p^2-(2^n+1)q^2=2^{6n}+5.2^{5n}-2.2^{3n}+9.2^{2n}+2^n+2 \quad (3)$$

Using the linear transformations

$$\left.\begin{aligned} p&=X+(2^n+1)T\\ q&=X+(2^{3n}+3)T \end{aligned}\right\}$$

and $T=1$, we have $X=2^{2n}+2^n+2$ and $p=2^{2n}+2.2^n+3$

From (1), $c=2^{3n}+2.2^{2n}+3.2^n+7$

Hence (a,b,c) is the Special Dio-Triple with the property $D(2^{3n}+5.2^{2n}+2^n+2)$

Case II

Let $b=2^{3n}+2$ and $c=2^{3n}+2.2^{2n}+3.2^n+7$

Let d be any non-zero integer.

Consider
$$\begin{aligned} bd+b+d+2^{3n}+5.2^{2n}+2^n+2&=\beta^2\\ cd+c+d+2^{3n}+5.2^{2n}+2^n+2&=\gamma^2 \end{aligned} \quad (4)$$

On simplification, we have

$(2^{3n}+3)d+2.2^{3n}+5.2^{2n}+2^n+4=\beta^2$

$(2^{3n}+2.2^{2n}+3.2^n+8)d+2.2^{3n}+7.2^{2n}+4.2^n+9=\gamma^2$

Using some algebra,

$$c\beta^2-b\gamma^2=(c-b)[2.2^{3n}+5.2^{2n}+2^n+4]-b(2.2^{2n}+3.2^n+5)$$

Using the linear transformations

$$\left.\begin{aligned} \beta&=X+bT\\ \gamma&=X+cT \end{aligned}\right\}$$

and $T=1$, we have $X=2^{3n}+2^{2n}+2^n+5$ and $\beta=2^{3n}+2^{2n}+2^n+8$

From (4), $d=4.2^{3n}+4.2^{2n}+5.2^n+20$

Thus (b,c,d) form a Special Dio-Triple with the property $D(2^{3n}+5.2^{2n}+2^n+2)$

Case III

Let $c=2^{3n}+2.2^{2n}+3.2^n+7$ and $d=4.2^{3n}+4.2^{2n}+5.2^n+20$

Let e be any non-zero integer.

Consider
$$\begin{aligned} ce+c+e+2^{3n}+5.2^{2n}+2^n+2&=\delta^2\\ de+d+e+2^{3n}+5.2^{2n}+2^n+2&=\theta^2 \end{aligned} \quad (5)$$

On simplification, we have

$(2^{3n}+2.2^{2n}+3.2^n+8)e+2.2^{3n}+7.2^{2n}+4.2^n+9=\delta^2$

$(4.2^{3n}+4.2^{2n}+5.2^n+21)e+5.2^{3n}+9.2^{2n}+6.2^n+22=\theta^2$

Using some algebra,

$$c\delta^2 - d\theta^2 = 3.2^{6n} + 17.2^{5n} + 15.2^{4n} + 28.2^{3n} + 69.2^{2n} + 15.2^n + 13$$

Using the linear transformations

$$\left.\begin{aligned}\delta &= X + cT\\ \theta &= X + dT\end{aligned}\right\}$$

and $T = 1$, we have $X = 2.2^{3n} + 3.2^{2n} + 4.2^n + 13$ and

$$\delta = 3.2^{3n} + 5.2^{2n} + 7.2^n + 21$$

From (5), $e = 9.2^{3n} + 12.2^{2n} + 16.2^n + 54$

Thus (c, d, e) form a Special Dio – triple with the property $D(2^{3n} + 5.2^{2n} + 2^n + 2)$

From all the above cases, $(a, b, c), (b, c, d), (c, d, e)$ ... will form a sequence of Special Dio-Triples.

Some Numerical examples are tabulated

n	(a, b, c)	(b, c, d)	(c, d, e)	$D(2^{3n} + 5.2^{2n} + 2^n + 2)$
0	(1,3,13)	(3,13,33)	(13,33,91)	$D(9)$
1	(2,10,29)	(10,29,78)	(29,78,206)	$D(32)$
2	(4,66,115)	(66,115,360)	(115,360,886)	$D(150)$
3	(8,514,671)	(514,671,2364)	(671,2364,5558)	$D(842)$

Sequence II

Deriving another sequence of special Dio-Triples (a, b, c), $(b, c, d), (c, d, e), \ldots$ with the property $D(4.5^n + 29)$

Case I

Let $a = 5^n + 2$ and $b = 5^n + 6$

Let c be any non-zero integer.

Consider $ac + a + c + 4.5^n + 29 = p^2$

which yields $(5^n + 3)c + 5.5^n + 31 = p^2$

$bc + b + c + 4.5^n + 29 = q^2$

gives $(5^n + 7)c + 5.5^n + 35 = q^2$

Using some algebra,

$$(5^n + 7)p^2 - (5^n + 3)q^2 = 16.5^n + 112 \quad (6)$$

Using the linear transformations

$$\left.\begin{aligned}p &= X + (5^n + 3)T\\ q &= X + (5^n + 7)T\end{aligned}\right\}$$

in (6), we have

$$X^2 = (5^{2n} + 10.5^n + 21)T^2 + 4.5^n + 28 \quad (7)$$

Let $T_0 = 1$ and $X_0 = (5^n + 7)$ be the initial solution of (7)

yielding $p = 2.5^n + 10$, $q = 2.5^n + 14$ and

$$c = 4.5^n + 23$$

Hence (a, b, c) is the Special Dio-Triple with the property $D(4.5^n + 29)$

Case II

Let $b = 5^n + 6$ and $c = 4.5^n + 23$

Let d be any non-zero integer.

Consider

$$\begin{aligned}bd + b + d + 4.5^n + 29 &= \beta^2 \quad (8)\\ cd + c + d + 4.5^n + 29 &= \gamma^2\end{aligned}$$

Using some algebra,

$$(c + 1)\beta^2 - (b + 1)\gamma^2 = (c - b)[4.5^n + 28]$$

Using the linear transformations

$$\left.\begin{aligned}\beta &= X + (b + 1)T\\ \gamma &= X + (c + 1)T\end{aligned}\right\}$$

and $T = 1$, we have $X = 2.5^n + 14$ and

$$\beta = 3.5^n + 21$$

From (8), $d = 9.5^n + 58$

Thus (b, c, d) form a Special Dio – triple with the property $D(4.5^n + 29)$

Case III

Let $c = 4.5^n + 23$ and $d = 9.5^n + 58$

Let e be any non-zero integer.

Consider

$$\begin{aligned}ce + c + e + 4.5^n + 29 &= \delta^2 \quad (9)\\ de + d + e + 4.5^n + 29 &= \theta^2\end{aligned}$$

On simplification, we have

$$(4.5^n + 24)e + 8.5^n + 52 = \delta^2$$
$$(9.5^n + 59)e + 13.5^n + 87 = \theta^2$$

Using some algebra,

$$(9.5^n + 59)\delta^2 - (4.5^n + 24)\theta^2 = 20.5^{2n} + 280.5^n + 980$$

Using the linear transformation

$$\left.\begin{aligned}\delta &= X + (4.5^n + 24)T\\ \theta &= X + (9.5^n + 59)T\end{aligned}\right\}$$

and $T = 1$, we have $X = 6.5^n + 38$ and

$$\delta = 10.5^n + 62$$

From (9), $e = 25.5^n + 158$

Thus (c, d, e) form a Special Dio–Triple with the property $D(4.5^n + 29)$

Case IV

Let $d = 9.5^n + 58$ and $e = 25.5^n + 158$

Let f be any non-zero integer.

Consider

$$\begin{aligned}df + d + f + 4.5^n + 29 &= \alpha^2 \quad (10)\\ ef + e + f + 4.5^n + 29 &= \beta^2\end{aligned}$$

On simplification, we have

$$(9.5^n + 59)f + 13.5^n + 87 = \alpha^2$$
$$(25.5^n + 159)f + 29.5^n + 187 = \beta^2$$

Using some algebra,

$$(25.5^n + 159)\alpha^2 - (9.5^n + 59)\beta^2$$
$$64.5^{2n} + 848.5^n + 2800$$

Using the linear transformations

$$\left.\begin{aligned}\alpha &= X + (9.5^n + 59)T\\ \beta &= X + (25.5^n + 159)T\end{aligned}\right\}$$

and $T = 1$, we have $X = 15.5^n + 97$ and

$$\alpha = 24.5^n + 156$$

From (10), $f = 64.5^n + 411$

Thus (d, e, f) form a Dio – triple with the property $D(4.5^n + 29)$

From all the above cases, $(a, b, c), (b, c, d), (c, d, e)$, (d, e, f) ... will form a sequence of Special Dio-Triples.

Some numerical Examples are tabulated.

n	(a,b,c)	(b,c,d)	(c,d,e)	$D(4.5^n+29)$
0	(3,7,27)	(7,27,67)	(27,67,183)	$D(33)$
1	(7,11,43)	(11,43,103)	(43,103,283)	$D(49)$
2	(27,31,123)	(31,123,283)	(123,283,783)	$D(129)$
3	(127,131,523)	(131,523,1183)	(523,1183,3283)	$D(529)$

Sequence III

Forming a special sequence of Dio-Triples $(a,b,c),(b,c,d),(c,d,e),\ldots$ with the property $D(s^2+1)$

Case I

Let $a=r-s$ and $b=r+s$

Note that (a,b) is a Dio-Double with the property $D(s^2+1)$

Let c be any non-zero integer.

Consider $ac+a+c+s^2+1=\alpha^2$ (11)

which yields $(r-s+1)c+r-s+s^2+1=\alpha^2$

$bc+b+c+s^2+1=\beta^2$

gives $(r+s+1)c+r+s+s^2+1=\beta^2$

Using some algebra,

$$(r+s+1)\alpha^2-(r-s+1)\beta^2=2s^3 \quad (12)$$

Using the linear transformations

$$\left.\begin{array}{l}\alpha=X+(r-s+1)T\\ \beta=X+(r+s+1)T\end{array}\right\}$$

in (12), we have $X^2=[(r+1)^2-s^2]T^2+s^2$

When $T=1,\ X=(r+1)$

Hence, $\alpha=(2r-s+2)$

From (11), we have $c=4r+3$

Hence (a,b,c) form a special Dio–Triple with the property $D(s^2+1)$

Case II

Let $b=r+s$ and $c=4r+3$

Let d be any non-zero integer.

Consider $bd+b+d+s^2+1=p^2$ (13)

which yields $(r+s+1)d+r+s+s^2+1=p^2$

$cd+c+d+s^2+1=q^2$

gives $(4r+4)d+4r+3+s^2+1=q^2$

Using some algebra,

$$(4r+4)p^2-(r+s+1)q^2=3rs^2+3s^2-s^3 \quad (14)$$

Using the linear transformations

$$\left.\begin{array}{l}p=X+(r+s+1)T\\ q=X+(4r+4)T\end{array}\right\}$$

in (14), we have

$$X^2=(4r^2+4rs+8r+4s+4)T^2+s^2$$

When $T=1,\ X=(2r+s+2)$

Hence, $p=3r+2s+3$

From (13), we have $d=9r+3s+8$

Hence (b,c,d) form a special Dio–Triple with the property $D(s^2+1)$

Case III

Let $c=4r+3$ and $d=9r+3s+8$

Let e be any non-zero integer.

Consider $ce+c+e+s^2+1=p^2$ (15)

which yields $(4r+4)e+4r+3+s^2+1=p^2$

$de+d+e+s^2+1=q^2$

gives $(9r+3s+9)e+9r+3s+8+s^2+1=q^2$

Using some algebra,

$$(9r+3s+9)p^2-(4r+4)q^2=s^2(5r+3s+5) \quad (16)$$

Using the linear transformations

$$\left.\begin{array}{l}p=X+(4r+4)T\\ q=X+(9r+3s+9)T\end{array}\right\}$$

in (16), we have $X^2=(9r+3s+9)(4r+4)T^2+s^2$

When $T=1,\ X=6r+s+6$

Hence, $p=10r+s+10$

From (15), we have $e=25r+5s+24$

Hence (c,d,e) form a special Dio–Triple with the property $D(s^2+1)$

Case IV

Let $d=9r+3s+8$ and $e=25r+5s+24$

Let f be any non-zero integer.

Consider $df+d+f+s^2+1=p^2$ (17)

which yields $(9r+3s+9)f+9r+3s+8+s^2+1=p^2$

$ef+e+f+s^2+1=q^2$

gives $(25r+5s+25)f+25r+5s+24+s^2+1=q^2$

Using some algebra,

$$(25r+5s+25)p^2-(9r+3s+9)q^2=s^2(16r+2s+16) \quad (18)$$

Using the linear transformations

$$\left.\begin{array}{l}p=X+(9r+3s+9)T\\ q=X+(25r+5s+25)T\end{array}\right\}$$

in (18), we have

$$X^2=(25r+5s+25)(9r+3s+94)T^2+s^2$$

When $T=1,\ X=15r+4s+15$

Hence, $p=24r+7s+24$

From (17), we have $f=64r+16s+63$

Hence (d,e,f) form a special Dio–Triple with the property $D(s^2+1)$

Case V

Let $e=25r+5s+24$ and $f=64r+16s+63$

Let g be any non-zero integer.

Consider $eg+e+g+s^2+1=p^2$ (19)

which yields

$(25r+5s+25)g+25r+5s+25+s^2=p^2$

$fg+f+g+s^2+1=q^2$

gives

$(64r+16s+64)g+64r+16s+24+s^2+64=q^2$

Using some algebra,

$$(64r+16s+64)p^2-(25r+5s+25)q^2=s^2(39r+11s+39) \quad (20)$$

Using the linear transformations

$$\left.\begin{array}{l}p=X+(25r+5s+25)T\\ q=X+(64r+16s+64)T\end{array}\right\}$$

in (20), we have

$$X^2=(64r+16s+64)(25r+5s+25)T^2+s^2$$

When $T = 1$, $X = 40r + 9s + 40$ and $p = 65r + 14s + 65$
From (19), we have $g = 169r + 39s + 168$
Hence (e, f, g) form a Dio–Triple with the property $D(s^2 + 1)$
From all the above cases, we have derived a special sequence of Dio-Triples of the form $(a, b, c), (b, c, d), (c, d, e), (d, e, f), (e, f, g)$, ... with the property $D(s^2 + 1)$
Numerical examples are tabulated.

r	s	(a, b, c)	(b, c, d)	(c, d, e)	$D(s^2 + 1)$
2	1	(1,3,11)	(3,11,29)	(11,29,79)	$D(2)$
3	2	(1,5,15)	(5,15,41)	(15,41,109)	$D(5)$
4	2	(2,6,19)	(6,19,50)	(19,50,134)	$D(5)$
5	3	(2,8,23)	(8,23,62)	(23,62,164)	$D(10)$

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