Rough Sets on Topological Spaces

Anitha K[1*], Venkatesan P[2]

[1]Department Of Mathematics,S.A.Engineering College,Chennai-77.

[2]Department Of Statistics,National Institute for Research in Tuberculosis,ICMR,Chennai-31.

***Abstract*— Rough Set Theory propsed by Zdzilaw Pawlak as a mathematical tool for approaching imprecise knowledge and it has many applications in Datamining, Machine Learning,Morphology, Control Theory and Medicine etc., The basic concept of Rough Set Theory depends on equivalence relation. In this paper we introduce topologies into the realm of Rough Sets and investigate their properties.Mathematical properties of Rough Set Theory can be constructed from fragments of Algebra, Logical Implications and Topology. We aim at presenting a properties of Rough Set Theory by Topological Methods**

***Keywords*— Approximations, Indiscernibility Function, Equivalence Relation, Topological Spaces,** $\Pi_0 - Rough\ Sets,$ **Rough Metric, Isometric**

I. Introduction

The basic concepts of Rough Set Theory proposed by Zdzislaw Pawlak in earlies of 1980, mainly for the purpose of data and knowledge analysis. The basic assumption of Rough Set Theory [2] is every knowledge in universe depends upon their capability of its classifications. So that equivalence relations are considered to define Rough Sets. The suggested topological properties of Rough Sets widely useful in granular computing [1]. This paper describes the basic topological properties of Rough Sets. Section II presents basic definitions of Rough Sets and Section III explains role on Rough Sets on Topological spaces with basic theorems. The conclusion work appears in section IV.

II. ROUGH SETS BASIC DEFINITIONS AND NOTATIONS

Elementary Knowledge is encoded in Rough Set Theory in the form of a pair (U, R) where U is a Universe of objects and R is an equivalence relation on U. For such an elementary knowledge, we can form elementary granules of knowledge which are nothing but Equivalence Classes of R and they are defined for each object $x \in U$, consider the set $[x]_R$ such that

$$[x]_R = \{ y \in U : (x, y) \in R\} \text{ ---------------- (1)}$$

The set of all equivalence classes fo R is called Quotient Set and it is denoted by

$$U/_R = \{ [x]_R : x \in U\} \text{ ,--------------------- (2)}$$

INDISCERNIBILITY RELATION AND INFORMATION TABLE

Any two objects $x, y \in R$ which satisfy the property that $[x]_R = [y]_R$ then x, y are called Indiscernible and this relation is called Indiscernibility Relation.

In Rough Set Theory knowledge is represented in the form of Information table . Information table is a pair (U, A) where U is a set of objects represented in Rows and A set of Attributes in columns . Each attribute $a \in A$ is a function $a: U \to V_a$ where V_a is the value set of the attribute a. The Indiscernibility of R with respect to a is defined if and only if $a(x) = a(y)$ and denoted with the symbol IND_a . Each set $B \subseteq A$ of attributes induces the classification

$$IND_{B=} \cap \{IND_a : a \in B \} \text{ ---------------- (3)}$$

Which is called $B - Indiscernibility\ Relation.$

LOWER AND UPPER APPROXIMATION

Equivalence relation is the basic property of Rough Sets for constructing Upper and Lower Approximation. Let Y be a Rough Set and $Y \subset U$ can be approximated using only the information contained within R by constructing the Lower and Upper approximation of R. Equivalence classes contained within Y belong to the Lower Approximation $(\underline{R}Y)$. Equivalence classes within X and along its border form the Upper Approximation $(\bar{R}Y)$. They are expressed as

$\underline{R}Y = \{ x \in U : [x]_R \subseteq Y\} - - - - Lower\ Approximation.$ ---------------------------- (4)

$\overline{R}Y = \{x : [x]_R \cap Y \neq \emptyset \} - - - - - Upper\ Approximation$------------------------- (5)

The Subset generated by Lower Approximation is characterized by objects that will definitely form part of a subset where as upper approximation will possibly form a part of a subset [4].

REMARKS :

The Lower and Upper Approximations satisfy the following properties

(i) $\underline{R}(X \cap Y) = \underline{R}X \cap \underline{R}Y$

(ii) $\underline{R}X \subseteq X$

(iii) $\overline{R}(X \cup Y) = \overline{R}X \cup \overline{R}Y$

(iv) $\underline{R}Y \subseteq Y \subseteq \overline{R}Y$

III.ROUGH TOPOLOGY

Topology is a theory of certain set structures which have been motivated by attempts to generalize a geometric reasoning. We consider here Rough Sets that arise in an information system from the point of view of topology. Clopen Topology is a special type of topology which can be constructed from equivalence classes in which every open set is closed[1].

Given R_n , where $\Re = \{ R_n = IND_{a_n} ; n = 1,2,3 \dots\}$, the family $\Re$ does induce various topologies on the universe U the topology Π_n is obtained by the family $P_n = \{[x]_{R_n} : x \in U\}$ as an open base. A set $Z \subseteq U$ is Π_0-Exact if $Int_{\Pi_0}(Z) = Cl_{\Pi_0}(Z)$ otherwise it is said to be Π_0-Rough.

REMARK:

Given a Π_0-Rough Set $[X]_r$, where r-is an equivalaence relation on the collection $R\Pi_0$. We say that $X\ r\ Y$ if and only if $Int_{\Pi_0}(X) = Int_{\Pi_0}(Y)\ and\ Cl_{\Pi_0}(X) = Cl_{\Pi_0}(Y)$ and Π_0 – Closed Rough sets can be represented as a pair (Q, T) where $Cl_{\Pi_0}(X), T = {}^{U}/_{Int_{\Pi_0}(X)}$.

Proposition (i) : A pair (Q, T) of Π_0 – Closed Subsets in U satisfies conditions $Q = Cl_{\Pi_0}(X)$, $T = {}^{U}/_{Int_{\Pi_0}(X)}$ with a rough subset $X \subseteq U$ if and only if $U = Q \cup T, Q \cap T \neq \phi$ and $Q \cap T$ does not contain any point x such that the singleton $\{x\}$ is $\Pi_0 - Open$.

METRIC TOPOLOGY ON ROUGH SETS

Let us define the relation R_n for fixed $n = 1,2, \dots$. such that $d_n : U\ X\ U \longrightarrow R^+$, where R^+ is the set of non-negative real numbers by the following combination

$d_n(x,y) = \begin{cases} 1 & when\ [x]_n \neq [y]_n \\ 0 & Otherwise \end{cases}$ -------------------------------- (6)

where $d_n(x,y)$ satisfies $d_n(x,x) = 0, d_n(x,y) = d_n(y,x)$ ----------- (7)

The function $d: U\ X\ U \longrightarrow R^+$ is defined as

$d(x,y) = \sum_n 10^{-n} \ d_n(x,y)$ ------------- (8)

Which satisfies the properties of metric space.

Proposition (ii): The metric topology induced by d coincides with the topology Π_0

Proposition (iii): For any pair K, H of closed sets $d_H(K,H) = 0$, if $cl_n K = cl_n H$.

Proposition (iv): If n is the first among indices j such that $cl_j K \neq cl_j H$ then $d_H(K,H) = \frac{1}{9}.10^{-n+1}$

Proposition (v): For each descending sequence $\{[x_n]_n\}_n$ of equivalence classes $\cap_n [x_n]_n \neq \emptyset$

Definition

The Sequence $(Q_n, \ T_n)_n$ is $D^* - Fundamental$ if and only if for each positive real number ε there is a natural number M such that $D^*\left((Q_i, T_i), (Q_j, T_j)\right) < \varepsilon$ whenever $i, j > M$

Theorem (1):

Each $D^* - Fundamental \ sequence \ (Q_n, T_n)_n$ of Rough Sets converges in the metric D to a Rough Set

Proof : From the definition of $D^* - Fundamental \ D^*\left((Q_i, T_i), (Q_j, T_j)\right) < \varepsilon$ when $\varepsilon = \frac{1}{n}$ and $m_n = m(\frac{1}{n})$. Hence the sequence $(Q_n, T_n)_n$ converges.

Definition

Let $\mathbb{D}$- be the family of all descending sequences $([x_n]_n)_n$ of equivalence classes. Then limit of the sequence is denoted by (Q^*, T^*) and defined by the following sets.

$Q^* = \cup\{\cap x : x \in \mathbb{D}, \forall_n, x_{j_n} \cap Q_{m_n} \neq \emptyset\}$ -------------------------------- (9)

$T^* = \cup\{\cap x : x \in \mathbb{D}, \forall_n, x_{j_n} \cap T_{m_n} \neq \emptyset\}$ ---------------------------------(10)

Theorem (2)

Sets Q^*, T^* are $\Pi_0 - Closed$

Proof

Let $x \notin Q^*$, Where $x = ([x]_n)_n$, then from the above definition $x_{j_n} \cap Q_{m_n} = \emptyset$ for some n

$\Rightarrow [x]_{j_n} \cap Q^* = \emptyset$

$\Rightarrow (Q^*)^c$ is open.

$\Rightarrow Q^*$ is closed. $\Rightarrow \Pi_0 - Closed$.

Theorem(3)

If $(Q_j, T_j)_j$ is a sequence of Π_0 Rough Sets approximately converging to a pair (Q, T) of Π_0Closed Sets then (Q, T) does represent a $\Pi_0 - Rough \ Sets$.

Proof: If the sequence $(Q_j, T_j)_j$ of Π_0 Rough Sets approximately converging then it has the limit sets Q^*,T^* which are closed and have approximation space, then they are $\Pi_0 - Rough\ Sets$.

Theorem (4)

Rough Complement is an isometric function on the space $(R\Pi_0, D)$

Proof

A Map $f: X \to Y$ between metric spaces with metrices d_X and d_Y is called Isometric or Isometric Embedding if for any pair of points x, x', $d_X(x,x') = d_Y(f(x), f(x'))$. ---------- (11)

If an Isometric embedding is a bijection then it is called Isometry. If there is an Isometry between two metric spaces they are called Isometric which is similar to homomorphimsm of topological spaces.

In Π_0 $Rough\ Sets$, $Int_{\Pi_0}(Z) \neq Cl_{\Pi_0}(Z)$ $\Rightarrow$ Its complement is closed.
Then by Proposition (iii): For any pair K, H of closed sets $d_H(K,H) = 0$, if $cl_n K = cl_n H$.
Which satisfies the condition (11) , Rough complement is isometric.

ROUGH ENTROPY MEASURE

Rough Entropy Measure [9] is defined on the basis of uncernity in granulation obtained by the relation on the universe which quantifies information with the underlying elements with limited discernibility between them . The Entropy Measure($H_R^L(X)$) defined on the basis of roughness measure to quantify the incompleteness which is given by the following expression

$$H_R^L(X) = -\frac{1}{2}\left[\gamma(X) + \gamma(X^C)\right] \text{------------------- (12)}$$

Where $\gamma(X) = \rho_R(X) log_{\rho_R}(X)/\beta$ for any set $X \subseteq U$ and $\rho_R(X) = 1 - \frac{\underline{R}X}{\overline{R}X}$, Roughness Measure, β - Base of Logarithmic function.

IV.CONCLUSION

We conclude that construction of topological properties on Rough Sets will help us to find rough measure which will enable to find the missing attribute values. Topological properties of Rough sets gives useful tool for creating the consistency base for Knowledge Discovery in Database (KDD) by rule induction and Clopen Topology plays vital role in digital image processing. Finally Topological Structure of this theory brings various techniques for Data Processing and Bio- mathematics.

REFERENCES

[1]. Abo Khadra , A.A., Kozae .A.M., Lashin . E.F., Medhat . T., (2005), Rough Set Theory for Topological Spaces,International Journal of Approximate Reasoning , 35-43

[2]. Feifei Xu, Duoqian Miao, Yiyu Yao. (1991), Rough Set Approximations in Formal Concept Analysis and Knowledge Spaces, Kluwer academic publishers (London).

[3]. Lukshmi R.A, Geetha P.V, Venkatesan, P., (2013) Rough Set Theory for Attribute Reduction,Indian Journal of Automation and Artificial Intelligence,1(3),2320-4001.

[4]. Milan Vlach,(2008) , Algebraic and Topological Aspects of Rough Set Theory, Fourth International Workshop on Computational Intelligence and Applications, IEEE

[5]. Nguyen Duc Thuan, (2009), Covering Rough Sets From a Topological Point of View , International Journal of Computer Theroy and Engineering , 1(5),1793-8201

[6]. Raghavan,R., Tripathy,B.K.., (2011) On Some Topological Properties of Multigranular Rough Sets, Advances in Applied Science Research,2(3),536-543

[7]. Salma.A.S., (2011), Some Topological Properties of Rough Sets with Tools for Data Mining,IJCSI,1694-0814

[8]. Polkowski.L.,(2002), Rough Sets- Mathematical Foundations, Springer-Verlag Berlin Heidelberg GmbH.

[9]. Sankar.K.Pal., Pradipta Maji, (2012),Rough –Fuzzy Pattern Recognition, IEEE