COEFFICIENT ESTIMATES FOR SAKAGUCHI TYPE FUNCTIONS

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Abstract: Let $S^n_{\lambda,\mu}(\alpha, s, t)$ be the class of normalized analytic functions defined in the open unit disk satisfying

$$\Re\left(\frac{(s-t)(D^n_{\lambda,\mu}f(z))'}{D^n_{\lambda,\mu}f(sz)-D^n_{\lambda,\mu}f(tz)}\right) > \alpha, |t| \leq 1, s \neq t$$

for some $\alpha(0 \leq \alpha < 1)$ and $D^n_{\lambda,\mu}$ is a linear multiplier differential operator defined by the author in [8]. The object of the present paper is to discuss some properties of functions $f(z)$ belonging to the classes $S^n_{\lambda,\mu}(\alpha, s, t)$ and $T^n_{\lambda,\mu}(\alpha, s, t)$ where $f(z) \in T^n_{\lambda,\mu}(\alpha, s, t)$ if and only if $zf(z) \in S^n_{\lambda,\mu}(\alpha, s, t)$.

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1. Introduction

Let A denote the family of functions f of the form

$$f(z) = z + \Sigma_{n=2}^{\infty} a_n z^n \tag{1.1}$$

which are analytic in the open unit disc $U = \{z : |z| < 1\}$. For $f(z)$ belongs to A, the multiplier differential operator $D^n_{\lambda,\mu} f(z)$ was defined by the authors in [8] as follows $D^n_{\lambda,\mu} f(z) = f(z)$

$D^1_{\lambda,\mu} = D_{\lambda,\mu} f(z) = \lambda^2(f(z))'' + (\lambda - \mu)(f(z))' + (1 - \lambda + \mu)f(z)$

$D^2_{\lambda,\mu} f(z) = D_{\lambda,\mu}(D^1_{\lambda,\mu} f(z))$

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$D^n_{\lambda,\mu} f(z) = D_{\lambda,\mu}(D^{n-1}_{\lambda,\mu} f(z\psi))$

where $\lambda \geq \mu \geq 0$ and $n \in N_0 = NU0$.

If f is given by (1.1)then from the definition of the operator $D^n_{\lambda,\mu} f(z)$ it is easy to see that

$$D^n_{\lambda,\mu} f(z) = z + \Sigma_{n=2}^{\infty}[1 + (\lambda\mu k + \lambda - \mu)(n-1)]^m a_n z^n \tag{1.2}$$

It should be remarked that the $D^n_{\lambda,\mu}$ is a generalization of many other linear operators considered earlier by different authors. In particular, for $f \in \mathbf{A}$ we have the following

$D^n_{1,0} f(z) \equiv D^n f(z)$ the operator investigated by Salagean.

$D^n_{\lambda,\mu} f(z) \equiv D^n_{\lambda} f(z)$ the operator studied by Al-Oboudi.

$D^n_{\lambda,\mu}$ the operator firstly considered for $0 \leq \mu \leq \lambda \leq 1$, by Raducanu and Orhan.

A function $f(z) \in \mathbf{A}$ is said to be in the class $S^n_{\lambda,\mu}(\alpha, s, t)$ if it satisfies

$$\Re\left(\frac{(s-t)(D^n_{\lambda,\mu} f(z))'}{D^n_{\lambda,\mu} f(sz) - D^n_{\lambda,\mu} f(tz)}\right) > \alpha, |t| \leq 1, s \neq t \tag{1.3}$$

for all $z \in \mathbf{U}$ and some $\alpha(0 \leq \alpha \leq 1)$. We also denote by $T^n_{\lambda,\mu}(\alpha, s, t)$ the subclass of $\mathbf{A}$ consisting of all functions $f(z)$ such that $zf^{'}(z) \in S^n_{\lambda,\mu}(\alpha, s, t)$. The class $S^n_{\lambda,\mu}(\alpha, s, t)$ was introduced and studied by Owa et. al. [8] and by taking t=-1, the class $S^n_{\lambda,\mu}(\alpha, s, t) \equiv S_s(\lambda)$ was introduced by Sakaguchi [8] and is called Sakaguchi function of order α, where as $S_s(0) \equiv S_s$ is the class of starlike functions with respect to symmetrical points in $\mathbf{U}$. Also we note that $S^n_{\lambda,\mu}(\alpha, 1, 0) \equiv S^*(\alpha)$ and $T^n_{\lambda,\mu}(\alpha, 1, 0) \equiv C^*(\alpha)$ which are respectively the familiar classes of starlike functions as order $\alpha(0 \leq \alpha < 1)$ and convex functions of order $\alpha(0 \leq \alpha < 1)$.

2. $S^n_{\lambda,\mu}(\alpha, s, t)$ AND $T^n_{\lambda,\mu}(\alpha, s, t)$

Theorem 2.1. *If $f(z) \in \mathbf{A}$ satisfies*

$$\sum_{k=2}^{\infty}\{|k - u_k(s,t)| + (1-\alpha)|u_k(s,t)|\}|a_k| \leq 1 - \alpha \tag{2.1}$$

$$\left(u_k(s,t) = \sum_{j=2}^{\infty} s^{k-j}t^{j-1}\right)$$

for some $\alpha(0 \leq \alpha < 1)$ then $f(z) \in S^n_{\lambda,\mu}(\alpha, s, t)$ where $\mathbf{A}^n_k = [1 + (\lambda\mu k + \lambda - \mu)(k-1)]^n$.

Proof. To prove Theorem 2.1, we show that if $f(z)$ satisfies (2.1) then

$$\left|\frac{(s-t)z(D^n_{\lambda,\mu}f(z))'}{D^n_{\lambda,\mu}f(sz) - D^n_{\lambda,\mu}f(tz)} - 1\right| < 1 - \alpha$$

Evidently, since

$$\left|\frac{(s-t)z(D^n_{\lambda,\mu}f(z))'}{D^n_{\lambda,\mu}f(sz) - D^n_{\lambda,\mu}f(tz)} - 1\right| = \frac{z + \sum_{k=2}^{\infty} kA^n_k z^k}{z + \sum_{k=2}^{\infty} kA^n_k u_n z^k} - 1$$

$$= \frac{\sum_{k=2}^{\infty}(k-u_k)A^n_k a_k z^{k-1}}{1 + \sum_{k=2}^{\infty} A^n_k u_k a_k z^{k-1}}$$

we see that,

$$\left|\frac{(s-t)z(D^n_{\lambda,\mu}f(z))'}{D^n_{\lambda,\mu}f(sz) - D^n_{\lambda,\mu}f(tz)} - 1\right| \leq \frac{\sum_{k=2}^{\infty} A^n_k|k-u_k||a_k|}{1 - \sum_{k=2}^{\infty} A^n_k|u_k|||a_k|}$$

therefore, if $f(z)$ satisfies (2.1), then we have

$$\left|\frac{(s-t)z(D^n_{\lambda,\mu}f(z)')}{D^n_{\lambda,\mu}f(sz) - D^n_{\lambda,\mu}f(tz)} - 1\right| < 1 - \alpha.$$

This completes the proof of Theorem 2.1. □

Theorem 2.2. *If $f(z) \in \mathbf{A}$ satisfies*

$\sum_{k=2}^{\infty}\{|k - u_k(s,t)| + (1-\alpha)|u_k(s,t)|\}|a_k| \leq 1-\alpha$

where

$$\left(u_k(s,t) = \sum_{j=2}^{\infty} s^{k-j}t^{j-1}\right) \tag{2.2}$$

for some $\alpha(0 \leq \alpha < 1)$

then $f(z) \in T^n_{\lambda,\mu}(\alpha, s, t)$, *where* $\mathbf{A}^n_k = [1 + (\lambda\mu k + \lambda - \mu)(k-1)]^n$.

Proof. Noting that $f \in T^n_{\lambda,\mu}(\alpha, s, t)$ if and only if $sf' \in S^n_{\lambda,\mu}(\alpha, s, t)$, we can prove Theorem 2.2.

We now define

$$S^n_{0,\lambda,\mu}(\alpha, s, t) = \{f \in \mathbf{A} : f satisfies (2.1)\}$$

and

$$S^n_{0,\lambda,\mu}(\alpha, s, t) = \{f \in \mathbf{A} : f satisfies (2.1)\}$$

This completes the proof of Theorem 2.2. □

3. COEFFICIENT INEQUALITIES

Applying Caratheodory function $\mathcal{P}(z)$ defined by

$$\mathcal{P}(z) = 1 + \sum_{k=1}^{\infty} p_k z^k \tag{3.1}$$

in U, we discuss the coefficient inequalities for the functions f in the subclasses $S^n_{\lambda,\mu}(\alpha, s, t)$ and $T^n_{\lambda,\mu}(\alpha, s, t)$.

Theorem 3.1. *If $f(z) \in S^n_{\lambda,\mu}(\alpha, s, t)$, then*

$$|a_k| \leq \frac{\beta}{A^n_k |v_k|} \{1 + \beta \sum_{j=2}^{k-1} \frac{|u_j|}{v_j} + \beta^2 \sum_{j_2>j_1}^{k-1} \sum_{j_1=2}^{k-2} \frac{|u_{j_1} u_{j_2}|}{|v_{j_1} v_{j_2}|} +$$

$$\beta^3 \sum_{j_3>j_2}^{k-1} \sum_{j_2>j_1}^{k-2} \sum_{j_1=2}^{k-3} \frac{|u_{j_1} u_{j_2} u_{j_3}|}{|v_{j_1} v_{j_2} v_{j_3}|} + ... + \beta^{k-2} \prod_{j=2}^{k-1} | \frac{u_j}{v_j} | \Big\}$$

where

$$\beta = 2(1 - \alpha), v_k = k - u_k. \tag{3.2}$$

Proof. We define the function $\mathcal{P}(z)$ by

$$\mathcal{P}(z) = \frac{1}{1-\alpha} \left(\frac{(s-t)(D^n_{\lambda,\mu} f(z))'}{D^n_{\lambda,\mu} f(sz) - D^n_{\lambda,\mu} f(tz)} - \alpha \right) = 1 + \sum_{k-1}^{\infty} p_k z^k$$

(3.3)

for $f(z) \in S^n_{\alpha,s,t}$. Then $\mathcal{P}(z)$ is a Caratheodory function and satisfies

$$|p_k| \leq 2(k \geq 1)$$

since

$$(1-t)z(D^n_{\lambda,\mu}f(z))' = [D^n_{\lambda,\mu}f(sz) - D^n_{\lambda,\mu}f(tz)][\alpha + (1-\alpha)p(z)],$$

we have

$$z + \sum_{k=2}^{\infty} kA^n_k a_k z^k = \left(z + \sum_{k=2}^{\infty} kA^n_k u_k a_k z^k\right)\left(1 + (1-\alpha)\sum_{k=1}^{\infty} p_k z^k\right)$$

Where

$$\left(u_k(s,t) = \textstyle\sum_{j=2}^{\infty} s^{k-j}t^{j-1}\right).$$

So we get

(3.4) ,

$$a_k = \frac{1-\alpha}{A^n_k(k-u_k)}(p_1 A^n_{k-1}u_{k-1}a_{k-1} + p_2 A^n_{k-2}u_{k-2}a_{k-2} + ... + p_{k-2}A^n_2 u_2 a_2 + p_{k-1}). \qquad (3.5)$$

□

From Eq. (3.5), we easily have that

$$|a_2| = \left| \tfrac{(1-\alpha)}{A^n_{2|2-u_2|}} p_1 \right| \leq \tfrac{2(1-\alpha)}{A^n_2 |2-u_2|}.$$

$$|a_3| \leq \frac{2(1-\alpha)}{A^n_3 |3-u_3|} (A^n_3 |u_2 a_2| + 1) \leq \frac{2(1-\alpha)}{A^n_3 |3-u_3|} \left(1 + 2(1-\alpha) \frac{|u_2|}{|2-u_2|}\right)$$

and

$$|a_4| \leq \frac{2(2-\alpha)}{A^n_4 |4-u_4|} \left\{ 1 + 2(1-\alpha) \left(\frac{|u_2|}{|2-u_2|} + \frac{|u_3|}{|3-u_3|} \right) + 2^2 (1-\alpha)^2 \frac{|u_2 u_3|}{|2-u_2||3-u_3|} \right\}$$

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Thus, using the mathematical induction, we obtain the inequality (3.2).

Remark 3.2. *If we write* $\alpha = t = s = n = 0$ *in Theorem 3.1 then we have well known the result*

$$f \ inS^* \Rightarrow |a_k|,$$

where S^* *is the usual class of starlike functions.*

Remark 3.3. *If we write* $\alpha = \frac{1}{2}, t = 0, s = 0, n = 1, \lambda = 1, \mu = 0$ *in Theorem 3.1 then we obtain*

$$|a_k| \leq \tfrac{1}{k}.$$

Remark 3.4. *If we write* $\alpha = 0, t = -1, s = -1, n = 1,$ *in Theorem 3.1 then we obtain*

$$|a_k| \leq \tfrac{1}{A_k}.$$

where $A_k = 1 + (\lambda\mu k + \lambda - \mu)(k-1)$ *and* $\lambda \geq \mu \geq 0$.

Remark 3.5. *If we write* $\lambda = \mu = 1$ *in Remark 3.4 then we obtain*

$$|a_k| \leq \tfrac{1}{k^2-k+1}.$$

Remark 3.6. *Equalities in Theorem 3.1 are attended for* $f(z)$ *given by*

$$\tfrac{(s-t)\left(D^n_{\lambda,\mu}f(z)\right)'}{D^n_{\lambda,\mu}f(sz)-D^n_{\lambda,\mu}f(tz)} = \tfrac{1+(1-2\alpha)z}{1-z}.$$

Theorem 3.7. *If* $f(z) \in T^n_{\lambda,\mu}(\alpha, s, t)$, *then*

$$|a_k| \leq \frac{\beta}{kA^n_k|v_k|}\{1+\beta\sum_{j=2}^{k-1}\frac{||u_j}{v_j} + \beta^2\sum_{j_2>j_1}^{k-1}\sum_{j_1=2}^{k-2}\frac{|u_{j_1}u_{j_2}|}{|v_{j_1}v_{j_2}|}$$

$$+\beta^3\sum_{j_3>j_2}^{k-1}\sum_{j_2>j_1}^{k-2}\sum_{j_1=2}^{k-3}\frac{|u_{j_1}u_{j_2}u_{j_3}|}{|v_{j_1}v_{j_2}v_{j_3}|} + ... + \beta^{k-2}\prod_{j=2}^{k-1}|\frac{u_j}{v_j}|\Big\}$$

where

$$\beta = 2(1-\alpha), v_k = k - u_k. \tag{3.6}$$

4. DISTORTION INEQUALITIES

For the functions $f(z)$ in the classes $S^n_{0,\lambda,\mu}(\alpha, s, t)$ and $T^n_{0,\lambda,\mu}(\alpha, s, t)$, we derive

Theorem 4.1. *If* $f(z) \in S^n_{0,\lambda,\mu}(\alpha, s, t)$, *then*

$$|z| - \sum_{k=2}^{j} |a_k||z|^k - \mathbf{B_j}|z|^{j+1} \leq |z| + \sum_{k=2}^{j} |a_k||z||^k + \mathbf{B_j}|z|^{j+1} \tag{4.1}$$

where

$$\mathbf{B_j} = \frac{1 - \alpha - \sum_{k=2}^{j} A_k^n \{|k - u_k| + (1-\alpha)|u_k|\}|a_k|}{(j + 1 - \alpha|u_{j+1}|)A^n_{j+1}} \quad (j \geq 2). \tag{4.2}$$

Proof. From the inequality (2.1) we know that

$$\sum_{k=+1}^{\infty} A_k^n \{|k - u_k| + (1-\alpha)|u_k|\}|a_k| \leq 1 - \alpha$$
$$- \sum_{k=2}^{j} A_k^n \{|k - u_k| + (1-\alpha)|u_k|\}|a_k|.$$

On the other hand

$$\{|k - u_k| + (1-\alpha)|u_k|\} - \alpha|u_k|,$$

and $k - \alpha|u_k|$ is monotonically increasing with respect to k. Thus we deduce

$(j + 1 - \alpha|u_{j+1}|)A^n_{j+1} \sum_{k=j+1}^{\infty} |a_k| \leq 1 - \alpha - \sum_{k=2}^{j} A_k^n \{|k - u_k| + (1-\alpha)|u_k|\}|a_k|$,

which'implies that

$$\sum_{k=+1}^{\infty} |a_k| \leq \mathbf{B_j} \tag{4.3}$$

Therefore we have the following

$$|f(z)| \leq |z| + \sum_{k=2}^{j} |a_k||z|^k + \mathbf{B_j}|z|^{j+1}$$

and

$$|f(z)| \geq |z| - \sum_{k=2}^{j} |a_k||z|^k - \mathbf{B_j}|z|^{j+1}$$

This completes the proof of the Theorem. □

Theorem 4.2. *If $f(z) \in T^n_{0,\lambda,\mu}(\alpha, s, t)$, then*

$$(4.4) \qquad |z| - \sum_{k=2}^{j} |a_k||z|^k - \mathbf{C_j}|z|^{j+1} \leq |z| + \sum_{k=2}^{j} |a_k||z||^k + \mathbf{C_j}|z|^{j+1}$$

and

$$(4.5) \quad 1 - \sum_{k=2}^{j} k|a_k||z|^{k-1} - \mathbf{D_j}|z|^j \leq |f'(z)| \leq 1 + \sum_{k=2}^{j} k|a_k||z||^{k-1} + \mathbf{D_j}|z|^j$$

where

$$(4.6) \qquad \mathbf{C_j} = \frac{1 - \alpha - \sum_{k=2}^{j} kA_k^n\{|k - u_k| + (1-\alpha)|u_k|\}|a_k|}{(j+1)j + 1 - \alpha|u_{j+1}|A^n_{j+1}} \quad (j \geq 2)$$

$$(4.7) \qquad \mathbf{D_j} = \frac{1 - \alpha - \sum_{k=2}^{j} kA_k^n\{|k - u_k| + (1-\alpha)|u_k|\}|a_k|}{(j + 1 - \alpha|u_{j+1}|)A^n_{j+1}} \quad (j \geq 2)$$

Remark 4.3. *If we choice n=0, s=t=-1, j=2 in Theorems 4.1 and 4.2, then we get the results given by Cho et. al. [8]*

5. RELATION BETWEEN THE CLASSES

By the definitions for the classes $S^n_{o,\lambda,\mu}(\alpha,s,t)$ and $T^n_{0,\lambda,\mu}(\alpha,s,t)$, evidently we have

$$S^n_{o,\lambda,\mu}(\alpha,s,t)\subset S^n_{o,\beta,\mu}(\alpha,s,t) \qquad (0\le\beta\le\alpha<\beta<1)$$

and

$$T^n_{o,\lambda,\mu}(\alpha,s,t)\subset T^n_{o,\beta,\mu}(\alpha,s,t) \qquad (0\le\beta\le\alpha<\beta<1).$$

Theorem 5.1. *If* $f(z)\in T^n_{0,\lambda,\mu}(\alpha,s,t)$*, then* $f(z)\in S^n_{o,\lambda,\mu}(\frac{1+\alpha}{2},s,t)$

Proof. Let $f(z)\in T^n_{0,\lambda,\mu}(\alpha,s,t)$. Then if $f(z)$ satisfies

$$\frac{|k-u_k|+(1-\beta)|u_k|}{1-\beta}\le k\frac{|k-u_k|+(1-\beta)|u_k|}{1-\beta} \tag{5.1}$$

for all $k\ge 2$, then we have that $f(z)\in S^n_{o,\lambda,\mu}(\beta,s,t)$. From 5.1, we have

$$\beta\le 1-\frac{(1-\alpha)|k-u_k|}{k|k-u_k|+(1-\alpha)(k-1)|u_k|}. \tag{5.2}$$

Furthermore, since for all $k\ge 2$

$$\tfrac{|k-u_k|}{k|k-u_k|+(1-\alpha)(k-1)|u_k|}\le\tfrac{1}{k}\le\tfrac{1}{2},$$

we obtain

$$f(z)\in S^n_{o,\lambda,\mu}(\tfrac{1+\alpha}{2},s,t).$$

$\square$

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