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**On the Verhult-Logistic Brownian Motion Model for Pricing Option on
Multiple Assets**

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Abstract

We develop a stochastic process that model the adjustment of the market price of a traded security involving multiple assets base on information affecting demand and supply of an asset.This is based on the Walrasian price adjustment assumption ,that change in price is due to excess demand.When supply and demand curve are linearised about the equilibrium point,the process becomes a logistics form of Brownian motion.

Keywords: Excess demand,Walrasian price adjustment process,logistic Brownian motion,Black Scholes Merton *PDE*,Multiple Asset
MSC 60H15, 60H30

1 Introduction

This paper build upon the Walrasian-Samuelson model of [5],where the deterministic and the stochastic case of Walrasian -Samuelson price adjustment model was studied for security involving one asset.

Here ,we develop the model for pricing option involving multiple assets based on the Walrasian -samuelson model.

This focus on the possibility of non-linear drifting as market adjust to new value of asset.The basic driver of such price adjustment is the excess demand over supply at the trading point[6],[8],[9].

We use a linearised version of this driving force to drive an Ito process that model price adjustment in non-steady markets.It is a logistic type process with diffusive Wiener variation referred to as Verhulst -logistic Brownian motion[5]

2 Model

In different works over the year,the Black-Sholes-Merton model for pricing derivatives has extensively been employed.This model which is built upon the assumption of geometric Brownian motion process play a leading role in the valuation of financial derivatives [3]

The equation

$$ds(t) = \mu s(t)d(t) + \sigma s(t)dW(t) \quad (1)$$

which is a stochastic differential equation that measure the dynamics of the price process, where

$s(t)$ is the price of the underlying asset at time t ,

μ is the growth rate of the asset,

σ is a volatility of the asset,and

W is a standard Brownian motion of the asset.

The option price is being determined by observable variables of the asset ; the current price $s(t)$ of the underlying asset, the strike price k , the expiration date T of the contract and the interest rate r together with the volatility σ of the underlying asset.

The study of stability of price equilibrium has been carried out over the years with Walrasian model [1],[6].

We considered the core principle of the standard Walrasian model, that the asset price are directly driven by the excess demand for asset.

We expect cross asset effect in a multi-security market. The dynamic adjustment of price in the market may be expressed in continuous-time Walrasian -samuelson form

$$\frac{1}{p_i(t)} \frac{dp_i(t)}{dt} = \begin{cases} KED(P_i(t)) \\ 0 \end{cases} \quad \text{if } P_i(t) = 0 \quad i = 1, 2, \dots, n \quad (2)$$

where t represent continuous time and $K > 0$ is a positive market adjustment coefficient and $ED(P_i(t)) = Q_D(P_i(t)) - Q_S(P_i(t))$

is the excess demand taken as a continuous function of $P_i(t)$.

Equation (2) can be express as

$$\frac{1}{p_i(t)} \frac{dp_i(t)}{dt} = K[Q_D(P_i(t)) - Q_S(P_i(t))] \quad [4], [9] \quad (3)$$

2.1 Deterministic Price Adjustment

To make the price adjustment more computational, we considered the supply and demand function as a fixed function of instantaneous prices $P_i(t)$. At equilibrium asset price point P_i^* , demand $Q_D(P_i^*)$ is equal to supply $Q_S(P_i^*)$. On the assumption of fixed supply and demand curves, P_i^* is constant $\forall i$.

It is expected that if $P_i(t)$ is not at equilibrium, then excess demand for the securities will increase the prices and excess supply will lower the prices.

Thus the sign of the rate of change of price $P_i(t)$ will depend on the sign of the excess demand

If we linearise $Q_D(P_i(t))$ and $Q_S(P_i(t))$ about the constant equilibrium price P_i^* , the deterministic model of price adjustment becomes

$$\frac{1}{P_i(t)} \frac{dP_i(t)}{dt} = K(\alpha + \beta)(P_i^* - P_i(t)) \quad (4)$$

where $Q_D(P_i(t)) = \alpha(P_i^* - P_i(t))$

$Q_S(P_i(t)) = -\beta(P_i^* - P_i(t))$, α and β are demand and supply sensitivities respectively.

If $r = K(\alpha + \beta)$ then (4) become

$$\frac{dP_i(t)}{dt} = r(P_i(t))(P_i^* - P_i(t)) \quad i = 1, 2..n \quad (5)$$

This is a n-dimensional deterministic logistic first order ordinary differential equation in $P_i(t)$. The logistic equation was first investigated by Pierre-Francois Verhulst (1838) as an improvement on the Malthusian model of population dynamics, hence it is also known as Verhulst- logistic differential equation and it has been applied in

several areas.[4] The solution of equation (5) is given by

$$P(t) = \frac{P^*P(0)}{P(0) + (P^* - P(0))e^{-rP^*(t)}}$$

where $P(0)$ is a parameter interpreted as the initial price of the asset. From the above equation, it can be observed that as $t \rightarrow \infty$ the term $P(t) \rightarrow \frac{P^*P(0)}{P(0)} = P^*$. The asset price thus settles into a constant level called a steady state or equilibrium at which no further changes will occur.

2.2 Logistic Price Adjustment Model For Multiple Assets

Here, we shall consider the random uncertainty in supply and demand by random change $\delta\alpha, \delta\beta$ in the respective sensitivities. Let $Q_D(P_i(t))$ and $Q_S(P_i(t))$ represent the effect of demand and supply respectively on the price of the asset at time t and suppose that both curves steepen in response to random observed trades, cumulatively, they exercise a Weiner diffusion process, then (5) becomes

$$\frac{dP_i(t)}{P_i(t)dt} = K(\alpha + \beta)(P^* - P(t)) + k(\delta\alpha + \delta\beta)(P^* - P(t)) \quad (6)$$

$$\frac{dP_i(t)}{P_i(t)(P^* - P(t))} = (K(\alpha + \beta) + k(\delta\alpha + \delta\beta))dt \quad (7)$$

Let $\mu = K(\alpha + \beta)$, the logistic growth parameter, and

Let $\sigma dz = k(\delta\alpha + \delta\beta)$, the noise process, then (7) becomes

$$\frac{dP_i(t)}{P_i(t)(P^* - P(t))} = \mu_i dt + \sum_{j=1}^n \sigma_{i_j} dz_j \quad (8)$$

This describes an Ito Process evolving SDE

$$dP_i(t) = \mu_i P_i(t)(P_i^* - P_i(t))dt + \sum_{j=1}^n \sigma_{i_j} P_i(t)(P_i^* - P_i(t))dZ_i \quad i = 1, 2, \dots, n \quad (9)$$

(7) is referred to as the Logistic price Adjustment model or Verhulst price adjustment model

The solution of the 1 dimensional case of (7) by the Ito lemma can be obtained as

$$\ln\left(\frac{P(t)}{|P^* - P(t)|}\right) = \ln\left(\frac{P_0}{|P^* - P(0)|}\right) + \mu P^*(t - t_0) + \sigma P^* Z(t)$$

$$P(t) = \frac{P^*P(0)}{P(0) + (P^* - P(0))e^{-(\mu P^*(t-t_0) + \sigma P^* Z(t))}}$$

This is a price dynamics for Brownian motion of asset prices $P(t)$.

If $V \in C^{1,2}([0, T], \Omega)$, Let $V(S, t)$ be the value of an option depending on the asset price S and time t , then by Ito lemma, we have [7]

$$dV(S(t), t) = \frac{\partial V}{\partial t} dt + \sum_{i=1}^n \frac{\partial V}{\partial s_i} ds_i + \frac{1}{2} \sum_{i,j=1}^n \frac{\partial^2 V}{\partial s_i \partial s_j} ds_i ds_j \quad (10)$$

For logistic Brownian motion ,we have

$$ds = \mu s(s^* - s)dt + \sigma s(s^* - s)dz$$

and

$$ds^2 = \mu s(s^* - s)dt + \sigma^2 s^2(s^* - s)^2 dt$$

for 1 dimensional case.

We extend this to the n-dimensional case for both Geometric Brownian motion and the logistic type with more emphasis on the logistic Brownian motion

$$ds_i(t) = \mu_i s_i(t)dt + \sum_{i,j=1}^n \sigma_{ij} s_i dz_j(t)$$

for the geometric Brownian motion,and

$$ds_i(t) = \mu_i s_i(s_i^* - s_i)dt + \sum_{i,j=1}^n \sigma_{ij} s_i(s_i^* - s_i)dz_j(t)$$

for the logistic Brownian motion

$$ds^2 = ds_i ds_j$$

$$ds_i ds_j = s_i(s_i^* - s_i)[\mu_i dt + \sum_{k=1}^n \sigma_{ik} dz_k(t)]s_j(s_j^* - s_j)[\mu_j dt + \sum_{k=1}^n \sigma_{jk} dz_k(t)]$$

$$ds_i ds_j = s_i(s_i^* - s_i)s_j(s_j^* - s_j)\sigma_{ik}\sigma_{jk}dt$$

$i, j = 1, 2 - - - - n$

Substituting this in (10),we have

$$dV(S(t), t) = \frac{\partial V}{\partial t} dt + \sum_{i=1}^n \frac{\partial V}{\partial s_i} [(\mu_i s_i(s_i^* - s_i)dt + \sum_{i,j=1}^n \sigma_{ij} s_i dz_j(t))$$

$$+ \frac{1}{2} \sum_{i,j=1}^n \frac{\partial^2 V}{\partial s_i \partial s_j} (\sum_{k=1}^n s_i(s_i^* - s_i)s_j(s_j^* - s_j)\sigma_{ik}\sigma_{jk}dt)$$

$i, j = 1, 2 - - - - n$

$$= [\frac{\partial V}{\partial t} + \sum_{i=1}^n \frac{\partial V}{\partial s_i} \mu_i s_i(s_i^* - s_i) + \frac{1}{2} \sum_{i,j=1}^n C_{ij} s_i(s_i^* - s_i)s_j(s_j^* - s_j) \frac{\partial^2 v}{\partial s_i \partial s_j}] dt + \sum_{i,j=1}^n \sigma_{ij} s_i(s_i^* - s_i) dz_j$$

where C_{ij} is a covariance matrix.

With this result, we can obtain the multi-dimensional model for pricing derivative involving multiple assets driven by a logistic Brownian motion

A portfolio Π consisting of one option V and δ_i of the underlying assets s_i is formed by shorting the contingent claim V and long δ_i unit of the underlying asset i.e

$$\begin{aligned}
\Pi(t) &= -V(S_i(t), t) + \sum_{i=1}^n \delta_i s_i \\
d\Pi(t) &= -dV + d\left(\sum_{i=1}^n \delta_i s_i\right) \\
&= -dV + \sum_{i=1}^n \delta_i ds_i \\
&= -\left[\left(\frac{\partial V}{\partial t} + \sum_{i=1}^n \frac{\partial V}{\partial s_i} \mu_i s_i (s_i^* - s_i) + \frac{1}{2} \sum_{i,j=1}^n C_{ij} s_i (s_i^* - s_i) s_j (s_j^* - s_j) \frac{\partial^2 v}{\partial s_i \partial s_j}\right) dt \right. \\
&\quad \left. + \sum_{i,j=1}^n \sigma_{ij} s_i (s_i^* - s_i) dz_j\right] + \sum_{i=1}^n \delta_i (\mu_i s_i (s_i^* - s_i) dt) + \sum_{i,j=1}^n \sigma_{ij} s_i (s_i^* - s_i) dz_j(t)
\end{aligned}$$

This become risk- less when the noise term vanishes.

This is possible if $\delta_i = \frac{\partial V}{\partial s_i} \quad i = 1, 2, \dots, n$

Then,

$$d\Pi(t) = -\left(\frac{\partial V}{\partial t} + \frac{1}{2} \sum_{i,j=1}^n C_{ij} s_i (s_i^* - s_i) s_j (s_j^* - s_j) \frac{\partial^2 v}{\partial s_i \partial s_j}\right) dt$$

As this portfolio is risk-less, it must have the same return as a Bank account, i.e if an investor deposited π_0 in a Bank account, with risk-less interest rate r over a period of time t , then at time t , the investor will have a return of

$$\Pi(t) = \Pi_0 e^{rt}$$

$$d\Pi(t) = r(-V + \sum_{i=1}^n s_i \frac{\partial v}{\partial s_i}) dt$$

$$\implies -\left(\frac{\partial V}{\partial t} + \frac{1}{2} \sum_{i,j=1}^n C_{ij} s_i (s_i^* - s_i) s_j (s_j^* - s_j) \frac{\partial^2 v}{\partial s_i \partial s_j}\right) dt = r(-V + \sum_{i=1}^n s_i \frac{\partial v}{\partial s_i}) dt$$

$$\implies -\frac{\partial V}{\partial t} - \frac{1}{2} \sum_{i,j=1}^n C_{ij} s_i (s_i^* - s_i) s_j (s_j^* - s_j) \frac{\partial^2 v}{\partial s_i \partial s_j} + rV - \sum_{i=1}^n r s_i \frac{\partial v}{\partial s_i} = 0$$

$$\implies \frac{\partial V}{\partial t} + \frac{1}{2} \sum_{i,j=1}^n C_{ij} s_i (s_i^* - s_i) s_j (s_j^* - s_j) \frac{\partial^2 v}{\partial s_i \partial s_j} + \sum_{i=1}^n r s_i \frac{\partial v}{\partial s_i} - rV = 0$$

This is the modify Black-Scholes model associated with the Walrasian price adjustment process driven by the logistic Brownian motion.

3 Conclusion

We have relaxed one of the assumptions of the Black-Scholes (1973) and Merton (1973) and we used the excess demand function to derive a logistic Brownian motion in terms of the market equilibrium S^* and the asset price S at time t to obtain the multiple assets with the no arbitrage argument.

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