Some Extensions of the Eneström-Kakeya Theorem

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***Abstract*— The Eneström-Kakeya Theorem states that if $p(z)=\sum_{j=0}^{n} a_j z^j$ is a polynomial satisfying $0<a_0\le a_1\le ...\le a_n$, then $p(z)$ does not vanish in $|z|>1$. In this paper we present related results by considering polynomials with complex coefficients and by putting restrictions on the arguments and moduli of the coefficients.**

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I. Introduction

The following result is well known in the theory of the distribution of zeros of polynomials.

Theorem A (Eneström-Kakeya). If

$$a_n \ge a_{n-1} \ge a_{n-2} \ge ... \ge a_1 \ge a_0 > 0,$$

then for $|z|>1$, $\sum_{j=0}^{n} a_j z^j \ne 0$.

In the literature there already exists ([2], [3], [4]) some extensions of the Eneström-Kakeya Theorem. Govil and Rahman [3, Theorem 2, 4] generalized Theorem A to polynomials with complex coefficients by considering the moduli of the coefficients to be monotonically increasing. More precisely they proved the following:

Theorem B. Let $p(z)=\sum_{j=0}^{n} a_j z^j$ be a polynomial with complex coefficients such that

$$|\arg a_j - \beta| \le \alpha \le \frac{\pi}{2}, \quad j=0,1,...,n,$$

for some real β, and

$$|a_n| \ge |a_{n-1}| \ge ... \ge |a_1| \ge |a_0|,$$

then $p(z)$ has all its zeros on or inside the circle

$$|z| = \cos\alpha + \sin\alpha + \frac{2\sin\alpha}{|a_n|}\sum_{j=0}^{n-1}|a_j|.$$

In this paper we significantly weaken the condition of montonicity on the moduli of the coefficients and obtain the following results:

Theorem 1. If $p(z)=\sum_{j=0}^{n} a_j z^j$ is a polynomial of degree n with complex coefficients, such that for some $k\ge 1$,

$$k|a_n| \ge |a_{n-1}| \ge ... \ge |a_1| \ge |a_0|,$$

and for some real β, $|\arg a_j - \beta| \le \alpha \le \frac{\pi}{2}$, $j=0,1,...,n$, then all the zeros of $p(z)$ lie in

$$|z+k-1|\le \frac{1}{|a_n|}\left[k|a_n|(\cos\alpha+\sin\alpha)-|a_0|(\cos\alpha+\sin\alpha-1)+2\sin\alpha\sum_{j=0}^{n-1}|a_j|\right]$$

For $k=1$, Theorem 1 improves upon Theorem B.

For $\alpha=\beta=0$ our result reduces to a result due to Aziz and Zargar [1] and in addition to the above condition if $k=1$ then it reduces to a result due to Joyal, Labelle and Rahman [4].

If we apply Theorem 1 to the polynomial $p(tz)$, we get the following result.

Corollary 1. If $p(z)=\sum_{j=0}^{n}a_jz^j$ is a polynomial of degree n with complex coefficients such that for some $k\ge 1$,

$$kt^n|a_n|\ge t^{n-1}|a_{n-1}|\ge ...\ge t|a_1|\ge|a_0|,\ t>0,$$

and for some real β, $|\arg a_j-\beta|\le\alpha\le\frac{\pi}{2}$, $j=0,1,...,n$ then all the zeros of $p(z)$ lie in

$$|z+kt-t|\le\left[kt(\cos\alpha+\sin\alpha)-\frac{|a_0|}{t^{n-1}}(\cos\alpha+\sin\alpha-1)+2\sin\alpha\sum_{j=0}^{n-1}\frac{|a_j|}{t^{n-j-1}}\right]$$

Theorem 2. Let $p(z)=\sum_{j=0}^{n}a_jz^j$ be a polynomial of degree n with complex coefficients such that

$$|a_n|\ge|a_{n-1}|\ge...\ge|a_r|\le|a_{r-1}|\le...\le|a_1|\le|a_0|$$

$0\le r\le n-1$, and for some real β, $|\arg a_j-\beta|\le\alpha\le\frac{\pi}{2}$, $j=0,1,...,n$. Then all the zeros of $p(z)$ lie in

$$\left|z+\frac{a_{n-1}}{a_n}-1\right|\le\frac{1}{|a_n|}\left[|a_{n-1}|(\cos\alpha+\sin\alpha)-2|a_r|\cos\alpha+|a_0|(\cos\alpha-\sin\alpha+1)+2\sin\alpha\sum_{j=0}^{n-2}|a_j|\right].$$

For $r=0$, we obtained the following result which improves the bound obtain in Theorem B, as well as the bound obtained in Theorem 1 (for $k=1$),

Corollary 2. Let $p(z)=\sum_{j=0}^{n}a_jz^j$ be a polynomial of degree n with complex coefficients such that

$$|a_n|\ge|a_{n-1}|\ge...\ge|a_r|\ge|a_0|,$$

and for some real β, $|\arg a_j-\beta|\le\alpha\le\frac{\pi}{2}$, $j=0,1,...,n$. Then all the zeros of $p(z)$ lie in

$$\left|z+\frac{a_{n-1}}{a_n}-1\right|\le\frac{1}{|a_n|}\left[|a_{n-1}|(\cos\alpha+\sin\alpha)-|a_0|(\cos\alpha+\sin\alpha-1)+2\sin\alpha\sum_{j=0}^{n-2}|a_j|\right]$$

By applying Theorem 2 to the polynomial $p(tz)$ we get the following:

Corollary 3. Let $p(z)=\sum_{j=0}^{n}a_jz^j$ be a polynomial of degree n with complex coefficients such that

$$t^n|a_n|\ge t^{n-1}|a_{n-1}|\ge...\ge t^r|a_r|\le t^{r-1}|a_{r-1}|\le...\le t|a_1|\le|a_0|,\ t>0$$

$0\le r\le n-1$, and for some real β, $|\arg a_j-\beta|\le\alpha\le\frac{\pi}{2}$, $j=0,1,...,n$. Then all the zeros of $p(z)$ lie in

$$\left|z+\frac{a_{n-1}}{a_n}-t\right|\le\frac{1}{|a_n|}\left[|a_{n-1}|(\cos\alpha+\sin\alpha)-\frac{2|a_r|}{t^{n-r-1}}\cos\alpha+\frac{|a_0|}{t^{n-1}}(\cos\alpha-\sin\alpha+1)+2\sin\alpha\sum_{j=0}^{n-2}\frac{|a_j|}{t^{n-j-1}}\right]$$

As remarked earlier Corollary 2, gives better result than Theorem B and Theorem 1 for $k=1$, we illustrate this by the following example.

Example.

$$p(z)=z^3+\left(\frac{1}{4}+\frac{i}{4}\right)z^2+\left(\frac{1}{16}+\frac{i}{16}\right)z+\left(\frac{1}{32}+\frac{i}{32}\right).$$

If we take $\alpha=\frac{\pi}{4}$, $\beta=0$ then by Theorem B all the zeros of $p(z)$ are contained in the region $|z|\le 2.1015$, the region containing all the zeros of $p(z)$ due to Theorem 1 (for $k=1$) is $|z|\le 1.99$, while Corollary 2 gives that $p(z)$ has all the zeros in the region $|z-(0.75-0.25i)|\le 0.6692093$.

Next we obtain a result in which real parts of the alternate coefficients of a polynomial are monotonically increasing and not necessarily positive. More precisely, we prove the following:

Theorem 3. Let $p(z)=\sum_{j=0}^{n}a_jz^j$ be a polynomial of degree n with complex coefficients where $a_j=\alpha_{j+i}\beta_j$, $j=0,1,....,n$ and $0\ne\alpha_n\ge\alpha_{n-2}\ge...\ge\alpha_3\ge\alpha_1$ and $\alpha_{n-1}\ge\alpha_{n-3}\ge...\ge\alpha_2\ge\alpha_0$ if n is odd, or

$$0\ne\alpha_n\ge\alpha_{n-2}\ge...\ge\alpha_2\ge\alpha_0 \text{ and } \alpha_{n-1}\ge\alpha_{n-3}\ge...\ge\alpha_3\ge\alpha_1$$

if n is even. Then all the zeros of $p(z)$ lie in

$$\left|z+\frac{a_{n-1}}{a_n}\right|\le\frac{1}{|a_n|}\left\{(|\alpha_0|-\alpha_0)+(|\alpha_1|-\alpha_1)+(\alpha_{n-1}+|\beta_{n-1}|)+(\alpha_n+|\beta_n|)+2\sum_{j=0}^{n-2}|\beta_j|\right\}.$$

If $\beta_j=0$, $j=0,1,....,n$, $\alpha_1>0$ and $\alpha_0>0$, then Theorem 3 reduces to a result due to Aziz Zargar [1].

Finally, we present the following result.

Theorem 4. Let $p(z)=\sum_{j=0}^{n}a_jz^j$ be a polynomial of degree n with complex coefficients. If $\operatorname{Re}a_i=\alpha_j$ and $\operatorname{Im}a_j=\beta_j$, $j=0,1,...,n$, and either

$$0\ne\beta_n\ge\beta_{n-2}\ge...\ge\beta_3\ge\beta_1 \text{ and } \beta_{n-1}\ge\beta_{n-3}\ge...\ge\beta_2\ge\beta_0 \text{ if } n \text{ is odd,}$$

or

$$0\ne\beta_n\ge\beta_{n-2}\ge...\ge\beta_2\ge\beta_0 \text{ and } \beta_{n-1}\ge\beta_{n-3}\ge...\ge\beta_3\ge\beta_1$$

if n is even. Then all the zeros of $p(z)$ lie in

$$\left|z+\frac{a_{n-1}}{a_n}\right|\le\frac{1}{|a_n|}\left\{(|\beta_0|-\beta_0)+(|\beta_1|-\beta_1)+(|\alpha_{n-1}|+\beta_{n-1})+(|\alpha_n|+\beta_n)+2\sum_{j=0}^{n-2}|\alpha_j|\right\}.$$

II. LEMMAS

Lemma. If $p(z)=\sum_{j=0}^{n}a_jz^j$ is a polynomial of degree n such that for some real β, $|\arg a_j-\beta|\le\alpha\le\frac{\pi}{2}$, $j=0,1,...,n$, then for some $t>0$

$$|ta_j-a_{j-1}|\le|t|a_j|-|a_{j-1}||\cos\alpha+(t|a_j|+|a_{j-1}|)\sin\alpha.$$

This lemma follows from inequality (6) in [3].

III. PROCESS OF THE THEOREMS

Proof of Theorem 1. Consider

$$F(z)=(1-z)p(z)=-a_nz^{n+1}+(a_n-a_{n-1})z^n+...+(a_1-a_0)z+a_0$$

Then for $|z|>1$, we have

$$
\begin{aligned}
|F(z)| &= |-a_n z^{n+1} + a_n z^n - Ka_n z^n + (Ka_n - a_{n-1})z^n + (a_{n-1} - a_{n-2})z^{n-1} + \ldots + (a_1 - a_0)z + a_0| \\
&\geq |a_n||z|^n \ |z+K-1| - |z|^n \left\{ |ka_n - a_{n-1}| + \frac{|a_{n-1} - a_{n-2}|}{|z|} + \ldots + \frac{|a_1 - a_0|}{|z|^{n-1}} + \frac{|a_0|}{|z|^n} \right\} \\
&> |a_n| \ |z|^n \ |z+k-1| - |z|^n \{ |ka_n - a_{n-1}| + |a_{n-1} - a_{n-2}| + \ldots + |a_1 - a_0| + |a_0| \}
\end{aligned}
$$

By applying Lemma to the above inequality, we get

$$
\begin{aligned}
|F(z)| \geq |z|^n [|a_n||z+K-1| - \{&(K|a_n| - |a_{n-1}|)\cos\alpha + (K|a_n| + |a_{n-1}|)\sin\alpha + (|a_{n-1}| - |a_{n-2}|)\cos\alpha \\
&+ (|a_{n-1}| + |a_{n-2}|)\sin\alpha + \ldots + (|a_1| - |a_0|)\cos\alpha + (|a_1| + |a_0|)\sin\alpha + |a_0|\} \\
&= |z|^n \left[|a_n||z+K-1| - \left\{ K|a_n|(\cos\alpha + \sin\alpha) - |a_0|(\cos\alpha + \sin\alpha - 1) + 2\sin\alpha \sum_{j=0}^{n-1} |a_j| \right\} \right].
\end{aligned}
$$

This shows that if $|z|>1$, then $|F(z)|>0$ if

$$
|z+k-1| > \frac{1}{|a_n|} \left[k|a_n|(\cos\alpha + \sin\alpha) - |a_0|(\cos\alpha + \sin\alpha - 1) + 2\sin\alpha \sum_{j=0}^{n-1} |a_j| \right]
$$

Hence all the zeros of $F(z)$ with $|z|>1$ lie in

$$
|z+k-1| \leq \left[k|a_n|(\cos\alpha + \sin\alpha) - |a_0|(\cos\alpha + \sin\alpha - 1) + 2\sin\alpha \sum_{j=0}^{n-1} |a_j| \right] \tag{3.1}
$$

But those zeros of $F(z)$ with $|z| \leq 1$ already satisfy inequality (3.1). Since all the zeros of $p(z)$ are also the zeros of $F(z)$, therefore, it follows that all the zeros of $p(z)$ lie in the circle defined by (3.1) and this completes the proof of Theorem 1.

Proof of Theorem 2. Consider

$$
\begin{aligned}
F(z) &= (1-z)p(z) \\
&= -a_n z^{n+1} + (a_n - a_{n-1})z^n + \ldots + (a_1 - a_0)z + a_0
\end{aligned}
$$

Let $|z|>1$. Then for $0 \leq r \leq n-1$, we have

$$
\begin{aligned}
|F(z)| &\geq |a_n z^{n+1} - (a_n - a_{n-1})z^n| - |(a_{n-1} - a_{n-2})z^{n-1} + \ldots + (a_1 - a_0)z^n + a_0| \\
&\geq |z^n| \ |a_n z + a_{n-1} - a_n| - \{ |a_{n-1} - a_{n-2}||z|^{n-1} + |a_{n-2} - a_{n-3}||z|^{n-2} + \ldots |a_{r+1} - a_r||z|^{r+1} \\
&\quad + |a_r - a_{r-1}||z|^r + |a_{r-1} - a_{r-2}||z|^{r-1} + \ldots + |a_1 - a_0||z| + |a_0| \} \\
&= |z|^n \Bigg[|a_n z + a_{n-1} - a_n| - \Bigg\{ \frac{|a_{n-1} - a_{n-2}|}{|z|} + \frac{|a_{n-2} - a_{n-3}|}{|z|^2} + \frac{|a_{r+1} - a_r|}{|z|^{n-r-1}} + \frac{|a_r - a_{r-1}|}{|z|^{n-r}} + \frac{|a_{r-1} - a_{r-2}|}{|z|^{n-r+1}} + \ldots \\
&\quad + \frac{|a_2 - a_1|}{|z|^{n-2}} + \frac{|a_1 - a_0|}{|z|^{n-1}} \frac{|a_0|}{|z|^n} \Bigg\} \Bigg]
\end{aligned}
$$

By applying lemma to the above inequality, we get

$$
\begin{aligned}
|F(z)| \geq |z|^n [&|a_n z + a_{n-1} - a_n| \{ (|a_{n-1}| - |a_{n-2}|)\cos\alpha + (|a_{n-1}| + |a_{n-2}|)\sin\alpha + \ldots \\
&+ (|a_{r+1}| - |a_r|)\cos\alpha + (|a_{r+1}| + |a_r|)\sin\alpha + (|a_{r-1}| - |a_r|)\cos\alpha + (|a_r| + |a_{r-1}|)\sin\alpha \\
&+ (|a_{r-2}| - |a_{r-1}|)\cos\alpha + (|a_{r-1}| + |a_{r-2}|)\sin\alpha + \ldots + (|a_0| - |a_1|)\cos\alpha + (|a_1| + |a_0|)\sin\alpha + |a_0| \}] \\
&= |z|^n \left[|a_n z + a_{n-1} - a_n| - \left\{ |a_{n-1}|(\cos\alpha + \sin\alpha) - 2|a_r|\cos\alpha + |a_0|(\cos\alpha - \sin\alpha + 1) + 2\sum_{j=0}^{n-2} |a_j| \sin\alpha \right\} \right] \\
&> 0
\end{aligned}
$$

if

$$
\left| z + \frac{a_{n-1}}{a_n} - 1 \right| > \frac{1}{|a_n|} \left[|a_{n-1}|(\cos\alpha + \sin\alpha) - 2|a_r|\cos\alpha + |a_0|(\cos\alpha - \sin\alpha + 1) + 2\sum_{j=0}^{n-2} |a_j| \sin\alpha \right]
$$

This shows that all the zeros of $F(z)$ with $|z|>1$ lie in

$$
\left| z + \frac{a_{n-1}}{a_n} - 1 \right| \leq \frac{1}{|a_n|} \left[|a_{n-1}|(\cos\alpha + \sin\alpha) - 2|a_r|\cos\alpha + |a_0|(\cos\alpha - \sin\alpha + 1) + 2\sum_{j=0}^{n-2} |a_j| \sin\alpha \right] \tag{3.2}
$$

But those zeros of $F(z)$ with $|z|\le 1$ already satisfy (3.2). Hence we conclude that all the zeros of $F(z)$ and therefore those of $p(z)$ lie in the circle defined by (3.2) and this completes the proof of Theorem 2.

Proof of Theorem 3. Consider

$$F(z)=(1-z^2)p(z)$$
$$=-a_n z^{n+2}-a_{n-1}z^{n+1}+(a_n-a_{n-2})z^n+(a_{n-1}-a_{n-3})z^{n-1}+...+(a_3-a_1)z^3+(a_2-a_0)z^2+a_1 z+a_0.$$

Then for $|z|>1$, we have

$$|F(z)|\ge |z|^n\left\{|z|\,|a_n z+a_{n-1}|-\left|(a_n-a_{n-2})+\frac{(a_{n-1}-a_{n-3})}{|z|}+...+(a_3-a_1)\frac{1}{z^{n-3}}+(a_2-a_0)\frac{1}{z^{n-2}}+\frac{a_1}{z^{n-1}}+\frac{a_0}{z^n}\right|\right\}$$
$$\ge |z|^n\left[|z|\,|a_n z+a_{n-1}|-\left\{|a_n-a_{n-2}|+\frac{|a_{n-1}-a_{n-3}|}{|z|}+...+\frac{|a_3-a_1|}{|z|^{n-3}}+\frac{|a_2-a_0|}{|z|^{n-2}}+\frac{|a_1|}{|z|^{n-1}}+\frac{a_0}{|z|^n}\right\}\right]$$
$$>|z|^n\left[|z|\,|a_n z+a_{n-1}|-\{|a_n-a_{n-2}|+|a_{n-1}-a_{n-3}|+...+|a_2-a_0|+|a_1|+|a_0|\}\right]$$
$$>|z|^n\,[|a_n z+a_{n-1}|-\{(\alpha_n-\alpha_{n-2})+(|\beta_n|+|\beta_{n-2}|)+(\alpha_{n-1}-\alpha_{n-3})+(|\beta_{n-1}|+|\beta_{n-3}|)+(\alpha_{n-2}-\alpha_{n-4})$$
$$+(|\beta_{n-2}|+|\beta_{n-4}|)+....+(|\alpha_1|+|\beta_1|)+(|\alpha_0|+|\beta_0|)]$$
$$=|z|^n\left[|a_n+a_{n-1}|-\left\{(|\alpha_0|-\alpha_0)+(|\alpha_1|-\alpha_1)+(\alpha_{n-1}+|\beta_{n-1}|)+(\alpha_n+|\beta_n|)+2\sum_{j=0}^{n-2}|\beta_j|\right\}\right]$$

This show that for $|z|>1$, $|F(z)|>0$, if

$$|a_n z+a_{n-1}|>\left\{(|\alpha_0|-\alpha_0)+(|\alpha_1|-\alpha_1)+(\alpha_{n-1}+|\beta_{n-1}|)+(\alpha_n+|\beta_n|)+2\sum_{j=0}^{n-2}|\beta_j|\right\}.$$

Hence those zeros of $F(z)$ with $|z|>1$ lie in

$$\left|z+\frac{a_{n-1}}{a_n}\right|\le\frac{1}{|a_n|}\left\{(|\alpha_0|-\alpha_0)+(|\alpha_1|-\alpha_1)+(\alpha_{n-1}+|\beta_{n-1}|)+(\alpha_n+|\beta_n|)+2\sum_{j=0}^{n-2}|\beta_j|\right\}. \quad (3.3)$$

Also those zeros of $F(z)$ with $|z|\le 1$ already lie in the circle (3.3). Therefore we conclude that all the zeros of $F(z)$ and hence those of $p(z)$ lie in the circle (3.3). Hence Theorem 3 is proved.

Proof of Theorem 4. The proof of the Theorem 4 follows on the same lines as that of Theorem 3. We omit the details.

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