Ky Fan's Best Approximation Theorem in Banach Space

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Abstract— **The aim of this paper is to prove a fixed point theorem for condensing mapping using a well-known result of Ky Fan in Banach space.**

Keywords— Fixed point theorem, Condensing mapping, Ky Fan's best Approximation theorem.

I. INTRODUCTION

Several interesting fixed point theorems have been proved by using Ky Fan's best approximation theorem. Most of the fixed point theorems are given for self maps that are for a function with domain and range are the same. In case a function does not have the same domain and range then we need a boundary condition to guarantee the existence of fixed point. The fundamental idea of applying fixed point results to produce theorem in analysis is introduced by Poincare [20] and was developed further in the work of Brikhoff and Kellog [17] and then Schauder [18]. Systematic applications of the Banach principle to various existence theorems in analysis were initiated by Caccioppoli [19]. The Banach contraction principle states that "A contraction mapping on a complete metric space has a unique fixed point". It has become vigorous tool for studying nonlinear functional differential equation in Banach space as well as for both the theoretical and computational aspect in mathematical sciences. The field of approximation theory has become so vast that it intersects with other branch of analysis and plays an important role in application in engineering and applied sciences. In 1969 Ky Fan's [5] establish the theorem known as Ky Fan's best Approximation theorem which has been of great importance in nonlinear analysis. In general fixed point theorems and the related technique have been used to prove the result about best approximation. We refer to Beer et al. [1], Carbone et al. [2], Cheney [3], Depascale [4], Fan [6], Furi and Vignoli [7], Lin [9], Park [12], Sehgal et al. [13], Singh and Watson [15], Waters [16]. In this paper the concept of condensing mapping has been used to establish a Ky fan's best approximation theorem which generalized the results of some standard results on Banach space.

II. PRELIMINARIES

Lemma 2.1 *[14]* Suppose that C be a nonempty close convex bounded subset of a Banach space X. If T be a continuous condensing mapping from S into S, then T has a fixed point in C.

Theorem 2.2 *[5]*
Suppose that C be a nonempty compact convex set in a normed linear space X. For any continuous map T from C into X, $\exists$ a point $v \in C$ such that $\|v - Tv\| = \min_{x \in C} \|v - Tv\|$.

Definition 2.3 *[8]* Suppose that X, Y be two normed linear space and let C be a nonempty subset of X, T be a map from C into Y, T is called nonexpansive if for each $x, y \in C$,we have $\|Tx - Ty\| \le \|x - y\|$.

Definition 2.4 *[8]*
Suppose that D be a nonempty bounded subset of a metric space X. We denote by $\beta(D)$ the infimum of the number r such that D can be covered by a finite number of subsets of X of diameter less than or equal to r.

Definition 2.5 *[14]*
Suppose that C be a nonempty subset of X and let T be a map from C into X. If for every nonempty bounded subset D of C with $\beta(D) > 0$, we have $\beta(T(D)) < \beta(D)$ and then T will be known as condensing. If $\exists$ k, $0 \le k \le 1$, such that $\forall$ nonempty bounded subset D of C we have $\beta(T(D)) < k\beta(D)$, then T is known as k-set contractive.

Lemma 2.6[10]
Suppose that T be a mapping of a compact topological space X into itself. Then $\exists$ a nonempty subset $S \subset X$ such that $S = \overline{T(S)}$.

Lemma 2.7 [11]
Suppose that C be a nonempty closed subset of a Banach space X and T a continuous 1-set contraction map of C into C. Suppose that $T(C)$ is bounded and $(\mathrm{I}-\mathrm{T})(\mathrm{C})$ is closed in X . Implies T has a fixed point in C .

III. MAIN RESULTS

Theorem 3.1 Suppose that X be a Banach space and let B be a ball whose radius is r and its centre is 0. Suppose that $T : B \to X$ be a continuous condensing mapping. Then $\exists$ a $y_0 \in B$ Such that $\|y_0 - Ty_0\| = d(Ty_0, B)$.

Proof: Suppose the radial retraction $R : B \to X$ is defined by

$$Rx = \begin{cases} x, if \ \|x\| \le r, \\ \frac{rx}{\|x\|}, if \ \|x\| \ge r. \end{cases}$$

Then R is a continuous 1-set contraction from X onto B. If $g(x) = RTx$. Then g is continuous map from B into B. Also $\forall$ nonvoid bounded subset A of B with $\beta(A) > 0$ we have $\beta(gA) = \beta(RTA) \le \beta(TA) < \beta(A)$. Therefore g is a condensing map and has a fixed point $y_0 = gy_0 = RTy_0$.

We have

$$\|y_0 - Ty_0\| = \|RTy_0 - y_0\|$$

$$= \begin{cases} \|Ty_0 - Ty_0\| = 0, if \ \|Ty_0\| \le r \\ \left\| \frac{rTy_0}{\|Ty_0\|} - Ty_0 \right\| = \|Ty_0 - r\| = \|Ty_0\| - r, if \ \|Ty_0\| \ge r. \end{cases}$$

$\forall \ y \in B$ We have

$$\|Ty_0\| - r \le \|Ty_0\| - \|y\| \le \|y - Ty_0\|$$

Hence $\|y_0 - Ty_0\| = d(Ty_0, B)$

Theorem 3.2 Suppose that $T : B \to X$ be a continuous condensing mapping, where B is a closed ball about the origin in a Banach space X .Then T has a fixed point if it satisfies any one of the below mentioned condition :

(a) $Tx = \beta x$ for some $x \in \partial B$ then $\beta \le 1$.

(b) $T(B) \subseteq B$.

(c) $T(\partial B) \subseteq B$.

(d) $\|Tx - x\|^2 \ge \|Tx\|^2 - \|x\|^2$ for all $x \in \partial B$.

Proof: By using theorem 3.1 we obtain $y_0 \in B$ such that $RTy_0 = y_0$.Let $\|Ty_0\| \le r$ then $y_0 = RTy_0 = Ty_0$ and y_0 is a fixed point of T. If $\|Ty_0\| > r$ then it contradicts. Actually $\|Ty_0\| > r$ implies that $RTy_0 = \frac{rTy_0}{\|Ty_0\|}$.

Thus, $Ty_0 = \frac{\|Ty_0\|}{r} y_0$ and $\|y_0\| = r$

i.e. $y_0 \in \partial B$.

If $y_0 \in \partial B$ and $y_0 \ne Ty_0$

Then $\beta = \frac{\|Ty_0\|}{r} > 1$, which contradicts condition (a). It is observed that each of condition (b) and (c) implies condition (a).We have to prove that condition (d) implies (a).

Let $Tx = \beta x$ for some $x \in \partial B$.

Then (d) implies that $(\beta - 1)^2 \ge \beta^2 - 1$ or that $\beta \le 1$ which satisfies condition (a).

Theorem 3.3 Suppose that C be a non empty closed convex subset of a Hilbert space H and T be a continuous 1-set contraction map of C into H. Let either $(I - P \circ T)(clco(P \circ T(C)))$ is closed in H or $(I - P \circ T)(C)$ is closed in H where P is the proximity map of H into C. If T(C) is bounded then $\exists$ a point v in C such that

$$\|v - T(v)\| = d(T(v), C)$$

Proof: Since P is nonexpansive in a Hilbert space H, therefore $P \circ T$ is a continuous 1-set contraction map of C into C and also $clco(P \circ T(C))$ into $clco(P \circ T(C))$ .Since T(C) is bounded therefore $P \circ T(C)$ is also bounded. Using lemma 2.7 $\exists$ a point v in C such that $P \circ T(v) = v$.

Hence

$$\|v - T(v)\| = \|P(T(v) - T(v)\| = d(T(v), C).$$

Corollary 3.4 Suppose C be a nonempty closed convex subset of a Hilbert space H, T be a continuous condensing map of C into H. If T(C) is bounded then $\exists$ a point v in C such that

$$\|v - T(v)\| = d(T(v), C).$$

Theorem 3.5 Suppose that X be a Banach space and let S be a closed bounded convex subset of a Banach space X. Let $F : S \to S$ be a condensing mapping. Then T has at least one fixed point.

Proof: y_0 in S we consider the sequence $\{F^n(y_0) : n = 1, 2, 3, ..\}$ and suppose G be its closure. Then G is invariant and compact. Therefore by lemma 2.6 $\exists$ a nonempty subset $D \subset G$ such that $D = \overline{F(D)}$.

Let $f = \{B \subset S : D \subset B$, B closed convex invariant under F$\}$. Let $A = \bigcap\{B : B \in f\}$.

Therefore $A = \overline{co}(F(A))$, the convex closure of F(A). Since $\beta(\overline{co}(F(A))) = \beta(F(A))$ .We get A is compact. Then $\exists$ a point x in A such that $F(x) = x$.

Theorem 3.6 Suppose that B be an open ball about the origin in a Banach space X. If $F : \overline{B} \to X$ be a condensing mapping that satisfies the boundary condition then $f(F)$ the set of fixed points of F in $\overline{B}$ is nonempty.

Proof: Let radial retraction be defined as $R : X \to \overline{B}$. Then R is a 1-set contraction of X onto $\overline{B}$. If the mapping is defined as $F_1(y) = R(F(y)), \forall$ x $\in$ B.

Then F_1 is a continuous mapping of $\overline{B}$ into $\overline{B}$ is also condensing.

Since $F : \overline{B} \to X$ is condensing $R : X \to \overline{B}$ is a 1-set contraction and therefore

$$\beta(F_1(\overline{B}) = \beta(R(F(\overline{B}))) \le \beta(F(\overline{B})) < \beta(\overline{B}) .$$

Hence by using theorem 3.5 F_1 has at least one fixed point x in $\overline{B}$. But then x is also a fixed point of F . If $x \in B$ then $F(x) = x$. Since $\|Fx\| \ge r$ gives

$F(x) = \frac{\|F(x)\|}{r} x$, which is contradiction $\|x\| \le r$. If $x \in B$ and x is not a fixed point of F , then

$k = \frac{\|F(x)\|}{r} > 1$ which opposes the boundary condition.

Thus x is a fixed point of F and hence $f = f$(F) is nonempty set in $\overline{B}$.

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