Common Fixed Points Theorems of a Countable Family of λ-Demi Contractive Mappings in Banach Spaces

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Abstract: The main purpose of this paper is to study an iterative Mann-type schemes to find a common fixed point of a countable family of λ-demicontractive mapping and L-Lipschitzain mappings in Banach space. We introduce a Mann type iteration procedure then, we generalized the recent result of Daruni Boonchari Satit Saejung [(1), 2010].

Key Words: Demicontractive Mapping, Banach Space, Iteration, Countable, Fixed Point and L-Lipschitzain.

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1. Introduction:

Let E be a real Banach space and E^* be its dual space. The normalized duality mapping: E $\rightarrow 2^{E^*}$ defined by $Jx = \{f \in E^*: \langle x, y\rangle = \|x^2\| = \|f^2\|\}$; for all x$\in E$ where $<.,.>$ denotes the duality paring between E and E^*.

In 1974, Ishikawa [2] introduced an iteration method for finding a fixed point of a Lipschitzain pseudocontractive mapping as follows.

Theorem 1.1 [2]. Let C be a nonempty compact convex subset of a Hilbert space H, $T: C \rightarrow C$ be a Lipschitz pseudocontractive mapping. For a fixed $x_0 \in$ C, define a sequence $\{x_n\}$ by.

$$x_{n+1} = \propto_n x_n + (1-\propto_n)Ty_n; \; y_n = \beta_n x_n + (1-\beta_n)\, Tx_n, \qquad (1.1)$$

Where $\{\propto_n\}$ and $\{\beta_n\}$ are real sequences in [0, 1] satisfying the following conditions;

(i) $\lim_{n\to\infty}\beta_n = 0$, (ii) $\sum_{n=1}^{\infty} \propto_{n\beta_n=\infty}$, (iii) $0 \le \propto_n \le \beta_n < 1$.

Then $\{x_n\}$ converges strongly to a fixed point of T.

It is natural to ask a question of whether or not the simple Mann iteration defined by $x_1 \in$ C and $x_{n+1} = \propto_n x_n + (1 - \propto_n)Tx_n$ can be used to obtain the same conclusion as theorem above. Recently, this question was resolved in the negative by Chidume and Matangadura [3]. They constructed an example of a Lipschitzian pseudo contractive mapping defined on a compact convex subset of the Hilbert space R^2 for which no Mann sequence converges

2. Preliminaries

In the sequel we shall make use of the following lemmas

Lemma 2.1 A mapping T with domain $D(T)$ and range $R(T)$ in Banach space is called

(i) Pseudo contractive [4], if for all $x, y \in D(T)$, there exists $j(x-y) \in J(x-y)$ such that

$$\langle Tx - Ty, j(x-y)\rangle \le \| x - y \|^2 \quad (1.1)$$

Equivalently, for all $x, y \in D(T)$ and for all $s > 0$,

$$\| x - y \| \le \| x - y + s[I - T)x - (I - T)y] \| \quad (1.2)$$

(ii) $\lambda -$Strictly pseudo contractive (in the terminology of Browder and Petryshyn) [4] for all $x, y \in D(T)$, there exists $j(x-y) \in J(x-y)$ such that

$$\langle Tx - Ty, j(x-y)\rangle \le \| x - y \|^2 - \lambda \, (x - Tx) - (y - Ty) \|^2 \quad (1.3)$$

(iii) Strongly pseudo-contractive if there exists $\lambda \in (0,1)$ for all $x, y \in D(T)$,there exists $j(x-y) \in J(x-y)$ such that $\langle Tx - Ty, j(x-y)\rangle \le \lambda \| x - y \|^2$

(iv) L- Lipschitzian if there exists L>0 such that for all $x, y \in D(T)$,

$$\| Tx - Ty \| \le L \| x - y \|$$

In 1974, Ishikawa [2] introduced an iteration method for finding a fixed point of a Lipschitz - pseudo contractive mapping as follows:

Lemma 2.2 (Xu, [5]) Let $q > 1$ and X be a real Banach space. Then the following statement is equivalent:-

X is q-uniformly smooth and for all $x, y \in X, j_q(x) \in J_q(x)$,the following inequality holds:

$$||x + y||^q \geq ||x||^q + q\langle y, j_q(x+y)\rangle + ||y||^q.$$

In the sequel we shall make use of the following lemmas.

Lemma 2.3 (Lemma 2.2, [6]). Let $\{\sigma_n\}$ and $\{\beta_n\}$ be sequence of nonnegative real numbers satisfying the following inequality:

$$\{\beta_{n+1}\} \leq (1+\sigma_n)\,\beta_n \quad ,n \geq 0$$

If $\sum_{n=1}^{\infty} \sigma_n < \infty$ then $\lim_{n\to\infty}\beta_n$ exists.

Definition 2.4. [7]. Let$\{T_n\}$ be a family of mappings from a subset C of a Banach space E into E with $\cap_{n=1}^{\infty} F(T_n) \neq \emptyset$ We say that$\{T_n\}$ satisfies the AKTT-condition if for each bounded subset B of C,

$$\sum_{n=1}^{\infty} sup_{z\in B} \parallel T_{n+1}z - T_n z \parallel < \infty$$

Remark 2.5. (Lemma 3.2, [7]). Suppose that $\{T_n\}$ satisfies the AKTT-condition. Then, for each y$\in C$, , T_ny converges strongly to a point in C. Moreover, let T be defined by;

$$Ty = \lim_{n\to\infty} T_n y \text{ for all } y \in C,$$

Then for each bounded subset B of C, $\lim_{n\to\infty} sup_{z\in B} \parallel T_z - T_n z \parallel = 0$

3. Main results:

Motivated by [8], we have the following lemma**.**

Lemma 3.1. Let C be a closed convex subset of a Banach space E. Let$\{T_n\}_{n=1}^{\infty}$: C → C be a family of λ -demi contractive and L- Lipschitzian mappings such that $\cap_{n=1}^{\infty} F(T_n) \neq \emptyset$ .define a sequence $\{x_n\}$ by $x_1 \in C$,

$$x_{n+1} = \propto_n x_n + (1 - \propto_n) T_n x_n \quad \text{for all } n \geq 1,$$

Where $\{x_n\} \subset [0,1]$ satisfying $\sum_{n=1}^{\infty} \propto_n = \infty$ and $\sum_{n=1}^{\infty} \propto_n{}^p < \infty$ if $\{T_n\}_{n=1}^{\infty}$satisfies AKTT-condition, then

(1) $\lim_{n\to\infty} \| x_n - p \|$ exits for all $p \in F$ and hence $\{x_n\}$ is bounded;

(2) $\lim_{n\to\infty} \| x_n - T_n x_n \| = 0$

Proof. (1) Let all $p \in F$. then $\| T_n x_n - p \| \le L \| x_n - p \|$. Moreover,

$$\| x_{n+1} - P \| \le \propto_n \| x_n - p \| + (1-\propto_n) \| T_n x_n - p \| \le (\propto_n + (1-\propto_n)L) \| x_n - p \|$$
$$\le (1 + L) \| x_n - p \| \tag{3.1}$$

Consequently,

$$\| x_n - T_n x_n \| \le \| x_n - p \| + \| p - T_n x_n \| \le (1 + L) \| x_n - p \| \tag{3.2}$$

From (3.1), we have, $\| x_{n+1} - T_n x_{n+1} \| \le \| x_{n+1} - p \| + \| p - T_n x_{n+1} \|$

$$\le (1 + L) \| x_{n+1} - p \| \le (1 + L)^2 \| x_n - p \| \tag{3.3}$$

It follows from (3.2) that

$$\| x_{n+1} - x_n \| = (1-\propto_n) \| T_n x_n - x_n \| \le (1-\propto_n)(1 + L) \| x_n - p \| \tag{3.4}$$

Since T_n is a λ -demi contractive mapping, there exists $j\,(x_{n+1} - p) \in J(x_{n+1} - p)$ such that

$$\langle x_{n+1} - T_n x_{n+1}, j(x_{n+1} - p) \rangle \ge \lambda \| (x_{n+1} - T_n x_{n+1}) \|^2,$$

By Lemma 2.2, (3.1) and (3.4), we have

$$\| x_{n+1} - p \|^q = \| (x_n - p) + (1-\propto_n)(T_n x_n - x_n) \|^q$$
$$= \| x_n - p \|^q + q(1-\propto_n)\langle T_n x_n - x_n, j\,(x_{n+1} - p)\rangle + \|(1-\propto_n)(T_n x_n - x_n)\|^q$$
$$= \| x_n - p \|^q + q(1-\propto_n)\langle T_n x_n - T_n x_{n+1}, j(x_{n+1} - p)\rangle + q(1-\propto_n)\langle T_n x_{n+1} - x_{n+1}, j\,(x_{n+1} - p)\rangle + q(1-\propto_n)\langle x_{n+1} - x_n, j\,(x_{n+1} - p)\rangle$$
$$+ (1-\propto_n)\|(T_n x_n - x_n)\|^q$$

$$\le \| x_n - p \|^q + q(1-\propto_n)L \| x_n - x_{n+1} \| \| (x_{n+1} - p) \| - q(1-\propto_n)\lambda \| T_n x_{n+1} - x_{n+1} \|^q + q\,(1-\propto_n^q) \| T_n x_n - x_n \| \| x_{n+1} - p \| + (1-\propto_n)\|(T_n x_n - x_n)\|^q$$

$$\leq\| x_n - p \|^q + q(1-\propto_n^q)L(1+L)^2 \| x_n - p \|^q - q(1-\propto_n)\lambda \| T_n x_{n+1} - x_{n+1} \|^q + q(1-\propto_n^q)(1+L)^2 \| x_n - p \|^q + (1-\propto_n)\|(T_n x_n - x_n)\|^q$$

$$\leq\| x_n - p \|^q + q(1-\propto_n^q)(1+L)^3 \| x_n - p \|^q - q(1-\propto_n)\lambda \| x_{n+1} - T_n x_{n+1} \|^q + \left((1-\propto_n)\mathrm{L}\right) \| x_n - \mathrm{p} \|^q \quad (3.5)$$

Consequently,

$$\| x_{n+1} - p \|^q \leq (1 + q(1-\propto_n^q)(1+L)^3) \| x_n - p \|^q + \left((1-\propto_n)\mathrm{L}\right) \| x_n - \mathrm{p} \|^q$$

From Lemma 2.3 and $\Sigma_{n=1}^{\infty} \propto_n^2 < \infty$, we get that $\lim_{n\to\infty} \| x_n - p \|$ exists and hence $\{x_n\}$ is bounded.

(2) We first show that $\lim_{n\to\infty} \| x_{n+1} - T_n x_{n+1} \| = 0$. Suppose that $\lim_{n\to\infty} \| x_{n+1} - T_n x_{n+1} \| = \delta > 0$. There exists N ∈ ℕsuch that $\| x_{n+1} - T_n x_{n+1} \| \geq \frac{\delta}{2}$ for all n≥ N since $\{x_n\}$ is bounded, put M=$\sup_{n\in\mathbb{N}}\{\| x_n - p \|\}$ from (3.5),

$$\| x_{n+1} - p \|^q \leq \| x_n - p \|^q - q(1-\propto_n)\lambda \| x_{n+1} - T_n x_{n+1} \|^q + q(1-\propto_n^q)(1+L)^3) \| x_n - p \|^q + \| (T_n x_n - \mathrm{p}) \|^q$$

$$\leq \| x_n - p \|^q - (1-\propto_n)\lambda\frac{\delta^2}{2} + q\left(1-\propto_n^q\right)(1+L)^3)M^q + \| T_n x_n - \mathrm{p} \|^q \text{ for all } n \geq N$$

It follows that

$$(1-\propto_n)\lambda\frac{\delta^2}{2} \leq (\| x_n - p \|^q - \| x_{n+1} - p \|^q) + q\left(1-\propto_n^q\right)(1+L)^3)M^2 + (\| T_n x_n - \mathrm{p} \|^q - \| T_{n+1} x_{n+1} - p \|^q)$$

For any $m \geq N$ we have,

$$\lambda\frac{\delta^2}{2}\Sigma_{n=N}^m(1-\propto_n) \leq \Sigma_{n=N}^m(\| x_n - p \|^q - \| x_{n+1} - p \|^q) + q(1+L)^2M^2\Sigma_{n=N}^m(1-\propto_n^q) + \Sigma_{n=N}^m(\| T_n x_n - \mathrm{p} \|^q - \| T_{n+1} x_{n+1} - p \|^q)$$

$$\leq \| x_n - p \|^q + q(1+L)^2M^2\Sigma_{n=N}^m \propto_n^q + \| T_n x_n - \mathrm{p} \|^q$$

Because $\Sigma_{n=1}^{\infty} \propto_n^2 < \infty$ we have $\Sigma_{n=1}^{\infty} \propto_n < \infty$ which is a contradiction.

Hence $\lim_{n\to\infty} inf \| x_{n+1} - T_n x_{n+1} \| = 0$ (3.6)

Consequently, since $\{x_n\}$ is bounded,

$$\| x_{n+1} - T_{n+1} x_{n+1} \| \leq \| x_{n+1} - T_n x_{n+1} \| + \| T_n x_{n+1} - T_{n+1} x_{n+1} \| + \| T_n x_{n+1} - T_{n+1} x_{n+1} \|$$

$$\leq \| x_{n+1} - T_n x_{n+1} \| + sup_{z\in\{x_n\}} \| T_z - T_n z \|.$$

Using (3.6) and AKTT-condition of $\{T_n\}$, we have

$\lim_{n\to\infty} inf \| x_{n+1} - T_n x_{n+1} \| = 0$

This completes the proof.

Theorem 3.2. Let C be a nonempty closed convex subset of real Banach space E.

Let $\{T_n\}: C \to C$ be a family of λ-demicontractive and L-Lipschitzian mappings with $\bigcap_{n=1}^{\infty} F(T_n) \neq \emptyset$ for a fixed $x_0 \in C$ define a sequence $\{x_n\}$ by

$$x_{n+1} = \propto_n x_n + (1 - \propto_n) T_n x_n$$

Where $\{\propto_n\}$ is a real sequence in $[0,1]$ satisfying $\Sigma_{n=1}^{\infty} \propto_n = \infty$ and $\Sigma_{n=1}^{m} \propto_n^p < \infty$ suppose that $\{T_n\}$ satisfies the AKTT-condition. Let $T: C \to C$ be the mapping defined by $Tz = \lim_{n\to\infty} T_n z$ for all $z \in C$ and suppose that F (T) =F. If there exists a compact subset K of E such that $T_n(c) \subset k$ for all n$\in \mathbb{N}$ then x_n converges strongly to a common fixed point of $\{T_n\}$.

Proof. From Lemma 3.1, we have $\lim_{n\to\infty} inf \| x_{n+1} - T_n x_{n+1} \| = 0$. Since $T_n(c) \subset k$ for all n$\in \mathbb{N}$ there exists a subsequence $\{T_{n_k}\}$ of $\{x_n\}$ such that $\lim_{k\to\infty} \| x_{n_k} - T_{n_k} x_{n_k} \| = 0$ and $T_{n_k} x_{n_k} \to q$ for some $q \in c$. this implies that $x_{n_k} \to q$. From Remark 2.4, we have $T_{n_k} q \to Tq$ and hence,

$$\| q \to Tq \| \leq \| q - x_{n_{k_i}} \| + \| x_{n_{k_i}} - T_{n_{k_i}} q \| + \| T_{n_{k_i}} q - Tq \| \to 0$$

Thus $q \in F(T) = F$.From Lemma 3.1, we have $\lim_{n\to\infty} \| x_n - q \|$ exists and hence $x_n \to q \in F$. If $T_n = T$, then the existence of a compact subset K of E such that T(c) $\subset k$ can be replaced by the demicompactness of the mapping and we immediately obtain the following corollary.

Corollary 3.3. Let C be a nonempty closed convex subset of a real Banach space E and $T: C \to C$ be a λ-demicontractive and L-Lipschitzian mappings with $\bigcap_{n=1}^{\infty} F(T_n) \neq \emptyset$ for fixed $x_1 \in C$,define a sequence $\{x_n\}$ by

$$x_{n+1} = \propto_n x_n + (1 - \propto_n) T x_n,$$

where $\{\propto_n\}$ is a real sequence satisfying $\sum_{n=1}^{\infty} \propto_n = \infty$ *and* $\sum_{n=1}^{\infty} \propto_n^q < \infty$, If T is demicompact, then $\{x_n\}$ converges strongly to a fixed point of T.

Recall that T is said to be demicompact if for every bounded sequence $\{x_n\}$ in C such that $\{x_n - Tx_n\}$ converges strongly contains a strongly convergent subsequence.

Remark 3.4. The result of Corollary 3.3 is proved by Chidume et al. [8] only for a strictly pseudo contractive mapping with a fixed point. It is easy to see that such a mapping is demi contractive and Lipschitzian. But the converse is not true as shown in the following example.

Example 3.5. Let T: $[0,3] \to [0,3]$ be defined by;

$$Tx = \begin{cases} 2x - 4 & if\ x \in [2,3] \\ 0 & if\ x \in [0,2) \end{cases}$$

Then T is 1-demicontractive and 2-Lipschitzian but it is not even a pseudocontractive mapping. To see the latter, let $x = 3$ and $y = 2$. Then $\langle Tx - Ty, x - y \rangle = 2 \nleq 1 = |x - y|^2$

Following Bruck's idea [9] and [10, Lemma 3.1], we have the following result.

Theorem 3.6. Let C be a nonempty closed convex subset of a smooth Banach space E.

Let$\{\propto_n\} \subset [0,1]$ satisfying the following conditions:

$(i) \sum_{n=1}^{\infty} \propto_n = \infty$, (ii) $\sum_{n=1}^{\infty} \propto_n^q < \infty$,Let $\{s_k\}_{k=1}^{\infty}$ be a sequence of λ_k-demicontractive and L-Lipschitzian mappings of K into itself with a common fixed point and inf . $\{\lambda_k : k\} \in \mathbb{N};= \lambda > 0$

Let $\{x_n\}$ be a sequence defined by $x_1 \in c$ and

$$x_{n+1} = \propto_n x_n + (1 - \propto_n) \sum_{k=1}^{n} \beta_n{}^k s_k x_n, \quad (n \geq 1), \tag{3.7}$$

where $\{\beta_n{}^k\}$ satisfies conditions (i)–(iii) of Lemma 3.7 If,

(a) there exists a compact and convex subset K of E such that$s_k(c) \subset k$ for all k$\in \mathbb{N}$, or

(b) The mapping T : C→C is defined by $T_n x = \sum_{k=1}^{n} \beta_n{}^k s_k x, \ (x \in c)$, $then$ $Tx = \lim_{n\to\infty} T_n x$ and $F(T) = \bigcap_{n=1}^{N} F(T_n) = \bigcap_{k=1}^{\infty} F(s_k)$. then$\{x_n\}$ converges strongly to a common fixed point of the family $\{s_k\}_{k=1}^{\infty}$

Proof. We write the iteration (3.7) as

$$x_{n+1} = \propto_n x_n + (1 - \propto_n) T_n x_n \qquad (n \geq 1),$$

where T_n is defined by (3.7). It is clear that each mapping T_n is L-Lipschitzian. By Theorem 3.2 , the conclusion follows.

Example 3.7. Let $\{{\beta_n}^k\}$ be defined by;

$${\beta_n}^k = \begin{cases} 2^{-k} & (k < n) \\ 2^{l-k} & (k = n), \end{cases}$$

for all n, k $\in N$ with k ≤ n. In this case, the sequence $\{T_n\}$ of mappings generated by $\{S_k\}$ is defined as follows: For x ∈ C,

$$T_1 x = S_1 x,$$

$$T_2 x = \tfrac{1}{2} S_1 x + \tfrac{1}{2} S_2 x,$$

$$T_3 x = \tfrac{1}{2} S_1 x + \tfrac{1}{4} S_2 x + \tfrac{1}{4} S_3 x,$$

$$T_4 x = \tfrac{1}{2} S_1 x + \tfrac{1}{4} S_2 x + \tfrac{1}{8} S_3 x + \tfrac{1}{8} S_4 x,$$

:

$$T_n x = \tfrac{1}{2} S_1 x + \tfrac{1}{4} S_2 x + \tfrac{1}{8} S_3 x + \tfrac{1}{16} S_4 x + \cdots + \tfrac{1}{2^{n-1}} S_{n-1} x + \tfrac{1}{2^{n-1}} S_n x.$$

Letting $S_N = S_{N+1} = S_{N+2} = \ldots$ yields the following result.

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