$m(\alpha)-$Series of The Generalized $\alpha-$Difference Equation to Polynomial Factorial

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In this paper, the authors extend the theory and $m-$series of the generalized difference equation to $m(\alpha)-$series of its $\alpha-$difference equation. We also investigate the complete and summation solutions of certain type of α-difference equation to polynomial factorial with geometric function in the field of finite difference methods.

Key words: Generalized α-difference equation, summation solution, complete solution, generalized polynomial factorial.
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1. Introduction

In 1984, Jerzy Popenda [1] introduced a particular type of difference operator Δ_α defined on $u(k)$ as $\Delta_\alpha u(k) = u(k+1) - \alpha u(k)$. In 1989 Miller and Rose [8] introduced the discrete analogue of the Riemann-Liouville fractional derivative and proved some properties of the fractional difference operator. The general fractional h-difference Riemann-Liouville operator and its inverse $\Delta_h^{-\nu} f(t)$ were mentioned in [3,9]. As application of $\Delta_h^{-\nu}$, by taking $\nu = m$ (positive integer) and $h = \ell$, the sum of m^{th} partial sums on n^{th} powers of arithmetic, arithmetic-geometric progressions and products of n consecutive terms of arithmetic progression have been derived using $\Delta_\ell^{-m} u(k)$ [5].

In 2011, M.Maria Susai Manuel, et.al, [6] have extended the definition of Δ_α to $\Delta_{\alpha(\ell)}$ defined as $\Delta_{\alpha(\ell)} u(k) = u(k+\ell) - \alpha u(k)$ for the real valued function $u(k)$ and $\ell \in (0,\infty)$ is fixed. In [7], the authors have used the generalized α-difference equation;

$$v(k+\ell) - \alpha v(k) = u(k), k \in [0,\infty), \ell \in (0,\infty) \qquad (1)$$

and obtained a summation solution of the above equation in the form

$$v(k) = \Delta_{\alpha(\ell)}^{-1} u(k)\,|_j^k = \sum_{r=1}^{[\frac{k}{\ell}]} \alpha^{r-1} u(k - r\ell), \quad j = k - [\frac{k}{\ell}]\ell. \qquad (2)$$

There are two types of solutions for the equation (1): one is summation another one is closed form solution. If we are able to find a closed form solution of equation (1) which is coinciding with the summation solution of that equation, then we can obtain formula for finding the values of several finite series. In this paper, we extend the theory of generalized m^{th} order difference equation developed in [11] to generalized m^{th} order α-difference equation.

In [12], the authors have defined the $m-$series of $u(k)$. Here we define corressponding

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$m(\alpha)-$series as below

Let $\ell > 0$ and $u(k)$ be real valued function on $[0,\infty)$ and $u(k)=0$ for all $k\in(-\infty,0)$. Then, for $m\in N(1)$, the $m(\alpha)-$series of $u(k)$ with respect to ℓ is defined as below:

$$1(\alpha)-series: u_{1(\alpha(\ell))}(k) = u(k-\ell)+\alpha u(k-2\ell)+\cdots+\alpha^{\left\lceil \frac{k}{\ell}\right\rceil-1} u\left(k-\left\lceil\frac{k}{\ell}\right\rceil\ell\right),$$

$$2(\alpha)-series: u_{2(\alpha(\ell))}(k) = u_{1\alpha(\ell)}(k-\ell)+\alpha u_{1\alpha(\ell)}(k-2\ell)+\cdots+\alpha^{\left\lceil \frac{k}{\ell}\right\rceil-1} u_{1\alpha(\ell)}\left(k-\left\lceil\frac{k}{\ell}\right\rceil\ell\right)$$

and in general $m(\alpha)-$series:

$$u_{m(\alpha(\ell))}(k) = u_{(m-1)\alpha(\ell)}(k-\ell)+\alpha u_{(m-1)\alpha(\ell)}(k-2\ell)+\cdots+\alpha^{\left\lceil \frac{k}{\ell}\right\rceil-1} u_{(m-1)\alpha(\ell)}\left(k-\left\lceil\frac{k}{\ell}\right\rceil\ell\right).$$

Also, we find that the $m(\alpha)-$series of $u(k)$ is the summation solution of the m^{th} order $\alpha-$difference equation

$$\Delta^m_{\alpha(\ell)} v(k) = u(k),\ k\in[0,\infty),\ \ell>0. \tag{3}$$

where $\Delta^m_{\alpha(\ell)}u(k) = \Delta_{\alpha(\ell)}(\Delta^{m-1}_{\alpha(\ell)}u(k))$.

Hence in this paper, we obtain $m(\alpha)-$series to $u(k)$ w.r.to ℓ by equating summation and closed form solution of equation (3).

2. Preliminaries

Before stating and proving our results, we present some notations, basic definitions and preliminary results which will be useful for further subsequent discussions. Let $\ell>0$ be fixed, $k\in[0,\infty)$, $j = k-\left\lceil\frac{k}{\ell}\right\rceil\ell$ where $\left\lceil\frac{k}{\ell}\right\rceil$ denotes the upper integer part of $\frac{k}{\ell}$, $\Delta_{\alpha(\ell)}v(k) = v(k+\ell)-\alpha v(k)$, $\Delta_\ell^{-m}u(k)\|^k_{(m-1)\ell+j} = \Delta_\ell^{-1}\left(\cdots\Delta_\ell^{-1}\left(\Delta_\ell^{-1}u(k)\,|^k_j\right)|^k_{\ell+j}\cdots\right)|^k_{(m-1)\ell+j}$. Throughout this paper, $\alpha\neq 0$ and 1, m is positive integer, $u(k)$ defined on $[0,\infty)$ and $u(k)=0$, $k\in(-\infty,0)$ and $L_{m-1}=\{1,2,\ldots,m-1\}$, $0(L_{m-1}) = \{\phi\}$, ϕ is empty set, $t(L_{m-1})$ = set of all subsets of size t from the set, L_{m-1} $1(L_{m-1}) = \{\{1\},\{2\},\{3\}\cdots,\{m-1\}\}$, $2(L_{m-1}) = \{\{1,2\},\{1,3\},\cdots,\{1,m-1\},\{2,3\},\cdots,\{2,m-1\},\cdots,\{m-2,m-1\}\}$. $t(L_{m-1}) = \{\{1,2,\cdots,m-1\}\}$, $\wp(L_{m-1}) = \bigcup_{t=0}^{m-1} t(L_{m-1})$, power set of L_{m-1}, $\sum_{t=1}^{m-1} f(t) = 0$ for $m\le 1$, and $\prod_{i=2}^{t} f(i) = 1$ for $t\le 1$.

Definition 2.1 *[4] Let $u(k)$, $k\in[0.\infty)$ be a real valued function and $\ell\in(0,\infty)$ be fixed. Then the generalized α difference operator on $v(k)$ is defined as:*

$$\Delta_{\alpha(\ell)}v(k) = v(k+\ell)-\alpha v(k). \tag{4}$$

Lemma 2.2 *[10] If $\Delta_{\alpha(\ell)}v(k) = u(k)$, then $\Delta^{-1}_{\alpha(\ell)}u(k) = v(k)$*

$$\text{and } \Delta^{-1}_{\alpha(\ell)}u(k)\,|^k_j = v(k) - \alpha^{\left\lceil\frac{k}{\ell}\right\rceil} v(j) \tag{5}$$

are solutions of equation (3) when m=1.

Lemma 2.3 *[2] Let* s_q^n *and* S_q^n *are the Stirling numbers of the first and second kinds respectively,* $s_0^0 = S_0^0 = 1$ *and* $s_q^0 = S_q^0 = 0 = s_0^q = S_0^q$ *if* $q \neq 0$, $n \in N(0)$. *Then,*

$$k_\ell^{(n)} = k(k-\ell)\cdots(k-(n-1)\ell) \text{ and } k^{(n)} = k(k-1)\cdots(k-(n-1)), \tag{6}$$

$$k_\ell^{(n)} = \sum_{q=0}^{n} s_q^n \ell^{n-q} k^q,\ k^n = \sum_{q=0}^{n} S_q^n \ell^{n-q} k_\ell^{(q)}, \tag{7}$$

$$\Delta_\ell k_\ell^{(n)} = (n\ell) k_\ell^{(n-1)} \text{ and } \Delta_\ell^{-\nu} k_\ell^{(q)} = \frac{k_\ell^{(q+\nu)}}{(q+\nu)^{(\nu)} \ell^\nu}. \tag{8}$$

3. Main Results

In this section we obtain solution and $m(\alpha)-$ series of $\alpha-$ difference equation to polynomial factorials and geometric function.

Lemma 3.1 *Let* $k \in [\ell, \infty)$, $a^\ell \neq \alpha$ *and* $j = k - \left\lceil \frac{k}{\ell} \right\rceil \ell$. *Then*

$$\Delta_{\alpha(\ell)}^{-1} a^k \big|_j^k = \frac{a^k}{(a^\ell - \alpha)} - \alpha^{\left\lceil \frac{k}{\ell} \right\rceil} \frac{a^j}{(a^\ell - \alpha)} \tag{9}$$

is a solution of $\alpha-$difference equation (3) when $u(k) = a^k$ and $m = 1$.

Proof. The proof follows from (4) and (5).

Theorem 3.2 *Let* $0 < \ell < k$ *and m is a positive integer. Then,*

$$\Delta_{\alpha(\ell)}^{-1}\left(\left\lceil \frac{k}{\ell} \right\rceil^{(m-1)} \alpha^{\left\lceil \frac{k}{\ell} \right\rceil - (m-1)} \right) = \frac{\left(\left\lceil \frac{k}{\ell} \right\rceil \right)^{(m)}}{m} \alpha^{\left\lceil \frac{k}{\ell} \right\rceil - m} \tag{10}$$

is a solution of $\alpha-$difference equation (3) when $u(k) = \left(\left\lceil \frac{k}{\ell} \right\rceil \right)^{(m-1)} \alpha^{\left\lceil \frac{k}{\ell} \right\rceil - (m-1)}$.

Proof. The proof of (10) follows from (4) and Lemma 2.2 for $m = 1, 2, 3, \cdots$.

Theorem 3.3 *Let* $u(k)$ *and* $v(k)$ *be two real valued functions. Then,*

$$\Delta_{\alpha(\ell)}^{-1}[u(k)v(k)] = u(k)\Delta_{\alpha(\ell)}^{-1} v(k) - \Delta_{\alpha(\ell)}^{-1}[\Delta_{\alpha(\ell)}^{-1} v(k+\ell)\Delta_\ell u(k)]. \tag{11}$$

Proof. From (4), we find that

$$\Delta_{\alpha(\ell)}[u(k)w(k)] = u(k)\Delta_{\alpha(\ell)} w(k) + w(k+\ell)\Delta_\ell u(k). \tag{12}$$

Taking $\Delta_{\alpha(\ell)} w(k) = v(k)$ and $w(k) = \Delta_{\alpha(\ell)}^{-1} v(k)$ in equation (12), we obtain

$$\Delta^{-1}_{\alpha(\ell)}[u(k)v(k)] = u(k)\Delta^{-1}_{\alpha(\ell)}v(k) - \Delta^{-1}_{\alpha(\ell)}[\Delta^{-1}_{\alpha(\ell)}v(k+\ell)\Delta_{\ell}u(k)]. \tag{13}$$

The following theorem gives the summation solution and $m(\alpha)-$ series of the equation (3).

Theorem 3.4 *($\alpha-$summation formula) For $k\in[m\ell,\infty)$,*

$$\Delta^{-m}_{\alpha(\ell)}u(k)\,\|^{k}_{(m-1)\ell+j} = \sum_{r=m}^{\left[\frac{k}{\ell}\right]}\frac{(r-1)^{(m-1)}}{(m-1)!}\alpha^{r-m}u(k-r\ell), \tag{14}$$

is a summation solution of equation (3).

Proof. Taking $\Delta^{-1}_{\alpha(\ell)}$ on (2) and applying (2) for $\Delta^{-1}_{\alpha(\ell)}u(k-r\ell)$, we get

$$\Delta^{-1}_{\alpha(\ell)}\left(\Delta^{-1}_{\alpha(\ell)}u(k)\,|^{k}_{j}\right)|^{k}_{\ell+j} = \sum_{r=1}^{\left[\frac{k}{\ell}\right]}\alpha^{r-1}\Delta^{-1}_{\alpha(\ell)}u(k-r\ell),$$

$$\Delta^{-2}_{\alpha(\ell)}u(k)\,\|^{k}_{\ell+j} = \sum_{r=2}^{\left[\frac{k}{\ell}\right]}(r-1)\alpha^{r-2}u(k-r\ell). \tag{15}$$

Expanding (15) and operating $\Delta^{-1}_{\alpha(\ell)}$ on both sides, we obtain

$$\Delta^{-1}_{\alpha(\ell)}\left(\Delta^{-2}_{\alpha(\ell)}u(k)\right)\|^{k}_{2\ell+j} = \sum_{r=3}^{\left[\frac{k}{\ell}\right]}\frac{(r-1)^{(2)}}{2!}\alpha^{r-3}u(k-r\ell),$$

$$\Delta^{-3}_{\alpha(\ell)}u(k)\,\|^{k}_{2\ell+j} = \sum_{r=3}^{\left[\frac{k}{\ell}\right]}\frac{(r-1)^{(3-1)}}{(3-1)!}\alpha^{r-3}u(k-r\ell). \tag{16}$$

The proof follows by continuing this process m times.

The following theorem gives the complete solution of the equation (3).

Remark 3.5 *In (14) as well as in $m(\alpha)-$ series one can replace $\left\lceil\frac{k}{\ell}\right\rceil$ by $\left\lfloor\frac{k}{\ell}\right\rfloor$ and hence* $j = k-\left[\frac{k}{\ell}\right]\ell$.

Theorem 3.6 *(Complete solution) If $\alpha\neq a^{\ell}$, $k\in[\ell,\infty)$, then.*

$$\Delta^{-m}_{\alpha(\ell)}u(k)\,\|^{k}_{(m-1)\ell+j} = F_{m(\alpha)}(k) - \alpha^{\left\lceil\frac{k}{\ell}\right\rceil-(m-1)}F_{m(\alpha)}((m-1)\ell+j), \tag{17}$$

where $F_{m(\alpha)}(k) = \Delta^{-m}_{\alpha(\ell)}u(k) + \sum_{t=1}^{m-1}\sum_{\{m_t\}\in t(L_{m-1})}(-1)^t\frac{\left(\left\lceil\frac{k}{\ell}\right\rceil\right)^{(m-m_t)}}{(m-m_t)!}\alpha^{\left\lceil\frac{k}{\ell}\right\rceil-(m-1)}$

$$\Delta_{\alpha(\ell)}^{-m_1}u((m_1-1)\ell+j)\prod_{i=2}^{t}\frac{\left\lceil\frac{(m_i-1)\ell+j}{\ell}\right\rceil^{(m_i-m_{i-1})}}{(m_i-m_{i-1})!}\alpha^{\left\lceil\frac{(m_i-1)\ell+j}{\ell}\right\rceil-(m_i-1)} \tag{18}$$

is a complete solution of equation (3).

Proof. Applying the limit j to k on $\Delta_{\alpha(\ell)}^{-1}u(k)$ and using (5), we write

$$\Delta_{\alpha(\ell)}^{-1}u(k)|_j^k=\Delta_{\alpha(\ell)}^{-1}u(k)-\alpha^{\left\lceil\frac{k}{\ell}\right\rceil}\Delta_{\alpha(\ell)}^{-1}u(j), \tag{19}$$

$$=F_{1(\alpha)}(k)-\alpha^{\left\lceil\frac{k}{\ell}\right\rceil-(1-1)}F_{1(\alpha)}((1-1)\ell+j),$$

where $F_{1(\alpha)}(k)=\Delta_{\alpha(\ell)}^{-1}u(k)$, $\Delta_{\alpha(\ell)}^{-1}u(j)$ is a constant and $\Delta_{\alpha(\ell)}^{-1}u(k)$ is a function of k. Taking $\Delta_{\alpha(\ell)}^{-1}$ on both sides of (19) and applying the limit $\ell+j$ to k, we obtain

$$\Delta_{\alpha(\ell)}^{-2}u(k)\|_{\ell+j}^k=\Delta_{\alpha(\ell)}^{-1}[\Delta_{\alpha(\ell)}^{-1}u(k)-\alpha^{\left\lceil\frac{k}{\ell}\right\rceil}\Delta_{\alpha(\ell)}^{-1}u(j)], \tag{20}$$

$$=F_{2(\alpha)}(k)-\alpha^{\left\lceil\frac{k}{\ell}\right\rceil-(2-1)}F_{2(\alpha)}((2-1)\ell+j),$$

where $F_{2(\alpha)}(k)$ is obtained from equation (18) by putting $m=2$.

Taking $\Delta_{\alpha(\ell)}^{-1}$ on (20), applying the limit $2\ell+j$ to k and using (10), we get

$$\Delta_{\alpha(\ell)}^{-3}u(k)\|_{2\ell+j}^k=\Delta_{\alpha(\ell)}^{-1}\left[\Delta_{\alpha(\ell)}^{-2}u(k)-\left\lceil\frac{k}{\ell}\right\rceil\alpha^{\left\lceil\frac{k}{\ell}\right\rceil-1}\Delta_{\alpha(\ell)}^{-1}u(j)-\alpha^{\left\lceil\frac{k}{\ell}\right\rceil-1}\Delta_{\alpha(\ell)}^{-2}u(\ell+j)\right.$$

$$\left.+\alpha^{\left\lceil\frac{k}{\ell}\right\rceil-1}\left\lceil\frac{\ell+j}{\ell}\right\rceil\alpha^{\left\lceil\frac{\ell+j}{\ell}\right\rceil-1}\Delta_{\alpha(\ell)}^{-1}u(j)\right],$$

$$=\Delta_{\alpha(\ell)}^{-3}u(k)-\frac{\left(\left\lceil\frac{k}{\ell}\right\rceil\right)^{(2)}}{2!}\alpha^{\left\lceil\frac{k}{\ell}\right\rceil-2}\Delta_{\alpha(\ell)}^{-1}u(j)-\frac{\left(\left\lceil\frac{k}{\ell}\right\rceil\right)^{(1)}}{1!}\alpha^{\left\lceil\frac{k}{\ell}\right\rceil-2}\Delta_{\alpha(\ell)}^{-2}u(\ell+j)+$$

$$\left\lceil\frac{k}{\ell}\right\rceil\alpha^{\left\lceil\frac{k}{\ell}\right\rceil-2}\left\lceil\frac{\ell+j}{\ell}\right\rceil\alpha^{\left\lceil\frac{\ell+j}{\ell}\right\rceil-1}\Delta_{\alpha(\ell)}^{-1}u(j)-\alpha^{\left\lceil\frac{k}{\ell}\right\rceil-2}[\Delta_{\alpha(\ell)}^{-3}u(2\ell+j)$$

$$-\frac{\left(\left\lceil\frac{2\ell+j}{\ell}\right\rceil\right)^{(2)}}{2!}\alpha^{\left\lceil\frac{2\ell+j}{\ell}\right\rceil-2}\Delta_{\alpha(\ell)}^{-1}u(j)-\frac{\left(\left\lceil\frac{2\ell+j}{\ell}\right\rceil\right)^{(1)}}{1!}\alpha^{\left\lceil\frac{2\ell+j}{\ell}\right\rceil-2}\Delta_{\alpha(\ell)}^{-2}u(\ell+j)+$$

$$\left\lceil\frac{2\ell+j}{\ell}\right\rceil\alpha^{\left\lceil\frac{2\ell+j}{\ell}\right\rceil-2}\left\lceil\frac{\ell+j}{\ell}\right\rceil\alpha^{\left\lceil\frac{\ell+j}{\ell}\right\rceil-1}\Delta^{-1}_{\alpha(\ell)}u(j)\Bigg],$$

which is the same as

$$\Delta^{-3}_{\alpha(\ell)}u(k)\,\|^{k}_{2\ell+j}=\Delta^{-3}_{\alpha(\ell)}u(k)+\sum_{t=1}^{2}\sum_{\substack{\{m_t=1\},\{m_t=2\}\in 1(L_2)\\ \{m_1=1,m_t=2\}\in 2(L_2)}}(-1)^t\frac{\left(\left\lceil\frac{k}{\ell}\right\rceil\right)^{(3-m_t)}}{(3-m_t)!}\alpha^{\left\lceil\frac{k}{\ell}\right\rceil-(3-1)}\times$$

$$\times\Delta^{-m_1}_{\alpha(\ell)}u((m_1-1)\ell+j)\prod_{i=2}^{2}\frac{\left\lceil\frac{(m_i-1)\ell+j}{\ell}\right\rceil}{(m_i-m_{i-1})!}\alpha^{\left\lceil\frac{(m_i-1)\ell+j}{\ell}\right\rceil-(m_i-1)}-\alpha^{\left\lceil\frac{k}{\ell}\right\rceil-2}[\Delta^{-3}_{\alpha(\ell)}u(2\ell+j)$$

$$+\sum_{t=1}^{2}\sum_{\substack{\{m_t=1\},\{m_t=2\}\in 1(L_2)\\ \{m_1=1,m_t=2\}\in 2(L_2)}}(-1)^t\frac{\left(\left\lceil\frac{2\ell+j}{\ell}\right\rceil\right)^{(3-m_t)}}{(3-m_t)!}\alpha^{\left\lceil\frac{2\ell+j}{\ell}\right\rceil-(3-1)}\Delta^{-m_1}_{\alpha(\ell)}u((m_1-1)\ell+j)\times$$

$$\times\prod_{i=2}^{2}\frac{\left\lceil\frac{(m_i-1)\ell+j}{\ell}\right\rceil}{(m_i-m_{i-1})!}\alpha^{\left\lceil\frac{(m_i-1)\ell+j}{\ell}\right\rceil-(m_i-1)}]$$

$$=F_{3(\alpha)}(k)-\alpha^{\left\lceil\frac{k}{\ell}\right\rceil-(3-1)}F_{3(\alpha)}((3-1)\ell+j).$$

In the same way, we find that

$$\Delta^{-4}_{\alpha(\ell)}u(k)\,\|^{k}_{3\ell+j}=\Delta^{-4}_{\alpha(\ell)}u(k)+\sum_{t=1}^{3}\sum_{\{m_t\}\in t(L_3)}(-1)^t\frac{\left(\left\lceil\frac{k}{\ell}\right\rceil\right)^{(4-m_t)}}{(4-m_t)!}\alpha^{\left\lceil\frac{k}{\ell}\right\rceil-(4-1)}\times$$

$$\times\Delta^{-m_1}_{\alpha(\ell)}u((m_1-1)\ell+j)\prod_{i=2}^{3}\frac{\left\lceil\frac{(m_i-1)\ell+j}{\ell}\right\rceil^{(m_i-m_{i-1})}}{(m_i-m_{i-1})!}\alpha^{\left\lceil\frac{(m_i-1)\ell+j}{\ell}\right\rceil-(m_i-1)}-\alpha^{\left\lceil\frac{k}{\ell}\right\rceil-(4-1)}$$

$$[\Delta^{-4}_{\alpha(\ell)}u(3\ell+j)+\sum_{t=1}^{3}\sum_{\{m_t\}\in t(L_3)}(-1)^t\frac{\left(\left\lceil\frac{3\ell+j}{\ell}\right\rceil\right)^{(4-m_t)}}{(4-m_t)!}\alpha^{\left\lceil\frac{3\ell+j}{\ell}\right\rceil-(4-1)}$$

$$\Delta^{-m_1}_{\alpha(\ell)}u((m_1-1)\ell+j)\prod_{i=2}^{3}\frac{\left\lceil\frac{(m_i-1)\ell+j}{\ell}\right\rceil^{(m_i-m_{i-1})}}{(m_i-m_{i-1})!}\alpha^{\left\lceil\frac{(m_i-1)\ell+j}{\ell}\right\rceil-(m_i-1)}]$$

$$= F_{4(\alpha)}(k) - \alpha^{\left\lceil \frac{k}{\ell} \right\rceil - (4-1)} F_{4(\alpha)}((4-1)\ell + j).$$

The proof completes by continuing this process m times.

Theorem 3.7 *($m(\alpha)-$ series formula) The summation-complete relation of equation (3) is given by*

$$\sum_{r=m}^{\left\lceil \frac{k}{\ell} \right\rceil} \frac{(r-1)^{(m-1)}}{(m-1)!} \alpha^{r-m} u(k-r\ell) = F_{m(\alpha)}(k) - \alpha^{\left\lceil \frac{k}{\ell} \right\rceil - (m-1)} F_{m(\alpha)}((m-1)\ell + j), \tag{21}$$

where $F_{m(\alpha)}(k)$ is as given in equation (18).

Proof. The proof follows by equating the summation solution given in Theorem 3.4 and the complete solution given in Theorem 3.6.

Remark 3.8 *In equation (18), $\Delta_{\alpha(\ell)}^{-m} u(k)$ is called the simple solution of the equation (3). $F_{m(\alpha)}(k) - \Delta_{\alpha(\ell)}^{-m} u(k)$ is called balancing factor of the equation (3) and L.H.S of (21) is $\alpha-$ summation solution of the equation (3).*

Corollary 3.9 *$m(\alpha)-$ series formula to $k_\ell^{(n)} a^k$ is*

$$\sum_{r=m}^{[\frac{k}{\ell}]} \frac{(r-1)^{(m-1)}}{(m-1)!} \alpha^{r-m} (k-r\ell)_\ell^{(n)} a^{k-r\ell} = F_{m(\alpha)}^{(n)}(k) - \alpha^{\left\lceil \frac{k}{\ell} \right\rceil - (m-1)} F_{m(\alpha)}^{(n)}((m-1)\ell + j), \tag{22}$$

where $$F_{m(\alpha)}^{(n)}(k) = \sum_{r=1}^{n+1} (-1)^{r-1} \frac{n^{(r-1)}}{(r-1)!} [\frac{(m+r-2)^{(r-1)} \ell^{r-1}}{(a^\ell - \alpha)^{r+m-1}} k_\ell^{(n+1-r)} a^{k+(r-1)\ell} +$$

$$\sum_{t=1}^{m-1} \sum_{\{m_t\} \in t(L_{m-1})} (-1)^t \frac{\left(\left\lceil \frac{k}{\ell} \right\rceil\right)^{(m-m_t)}}{(m-m_t)!} \alpha^{\left\lceil \frac{k}{\ell} \right\rceil - (m-1)} \frac{(m_1 + r - 2)^{(r-1)} \ell^{r-1}}{(a^\ell - \alpha)^{r+m_1-1}}$$

$$\times ((m_1-1)\ell + j)_\ell^{(n-r+1)} a^{((m_1-1)\ell + j + (r-1)\ell)} \prod_{i=2}^{t} \frac{\left\lceil \frac{(m_i-1)\ell + j}{\ell} \right\rceil^{(m_i - m_{i-1})}}{(m_i - m_{i-1})!} \alpha^{\left\lceil \frac{(m_i-1)\ell + j}{\ell} \right\rceil - (m_i - 1)}].$$

Proof. Taking $u(k) = k_\ell^{(n)}$ and $v(k) = a^k$ in (11), we obtain

$$\Delta_{\alpha(\ell)}^{-1}(k_\ell^{(n)} a^k) = k_\ell^{(n)} \frac{a^k}{(a^\ell - \alpha)} - \Delta_{\alpha(\ell)}^{-1} \left[\frac{a^{k+\ell}}{(a^\ell - \alpha)} \left\{ n\ell k_\ell^{(n-1)} \right\} \right].$$

Using (6), (8) and applying (9) for $k_\ell^{(n-1)} a^k, k_\ell^{(n-2)} a^k, \cdots k_\ell^{(1)} a^k$, we arrive

$$\Delta^{-1}_{\alpha(\ell)}(k_\ell^{(n)}a^k)=\sum_{r=1}^{n+1}(-1)^{r-1}\frac{n^{(r-1)}}{(r-1)!}(r-1)^{(r-1)}\frac{\ell^{r-1}}{(a^\ell-\alpha)^r}k_\ell^{(n+1-r)}a^{k+(r-1)\ell}. \tag{23}$$

Taking $\Delta^{-1}_{\alpha(\ell)}u(k)$ on (23) for $m-1$ times, we arrive

$$\Delta^{-m}_{\alpha(\ell)}(k_\ell^{(n)}a^k)=\sum_{r=1}^{n+1}(-1)^{r-1}\frac{n^{(r-1)}}{(r-1)!}\frac{(m+r-2)^{(r-1)}\ell^{r-1}}{(a^\ell-\alpha)^{r+m-1}}k_\ell^{(n+1-r)}a^{k+(r-1)\ell}. \tag{24}$$

The proof follows by taking $u(k)=k_\ell^{(n)}a^k$ in Theorem 3.7.

The following example illustrates corollary 3.9.

Example 3.10 *Consider the case when* $m=4$ *and* $n=2$. *In this case,*

$L_3=\{1,2,3\}$, $1(L_3)=\{\{1\},\{2\},\{3\}\}$, $2(L_3)=\{\{1,2\},\{1,3\},\{2,3\}\}$,

$3(L_3)=\{\{1,2,3\}\}$ and (22) becomes

$$\sum_{r=4}^{\lceil \frac{k}{\ell}\rceil}\frac{(r-1)^{(3)}}{3!}\alpha^{r-4}(k-r\ell)_\ell^{(2)}a^{k-r\ell}=F^{(2)}_{4(\alpha)}(k)-\alpha^{\lceil \frac{k}{\ell}\rceil-3}F^{(2)}_{4(\alpha)}((3)\ell+j), \tag{25}$$

$$\textit{where}\quad F^{(2)}_{4(\alpha)}(k)=\sum_{r=1}^{3}(-1)^{r-1}\frac{2^{(r-1)}}{(r-1)!}[\frac{(2+r)^{(r-1)}\ell^{r-1}}{(a^\ell-\alpha)^{r+3}}k_\ell^{(3-r)}a^{k+(r-1)\ell}+$$

$$\sum_{t=1}^{3}\sum_{\{m_t\}\in t(L_3)}(-1)^t\frac{\left(\lceil \frac{k}{\ell}\rceil\right)^{(4-m_t)}}{(4-m_t)!}\alpha^{\lceil \frac{k}{\ell}\rceil-3}\frac{(m_1+r-2)^{(r-1)}\ell^{r-1}}{(a^\ell-1)^{r+m_1-1}}((m_1-1)\ell+j)_\ell^{(3-r)}$$

$$\times a^{((m_1-1)\ell+j+(r-1)\ell)}\prod_{i=2}^{t}\frac{\left\lceil\frac{(m_i-1)\ell+j}{\ell}\right\rceil^{(m_i-m_{i-1})}}{(m_i-m_{i-1})!}\alpha^{\lceil\frac{(m_i-1)\ell+j}{\ell}\rceil-(m_i-1)}].$$

The double summation expression of (25) will be obtained by adding the sums corresponds to $1(L_3)$:

$$\frac{\left(\lceil \frac{k}{\ell}\rceil\right)^{(3)}}{3!}\alpha^{\lceil \frac{k}{\ell}\rceil-3}\frac{(r-1)^{(r-1)}\ell^{r-1}}{(a^\ell-\alpha)^r}(j)_\ell^{(3-r)}a^{j+(r-1)\ell}$$

$$+\frac{\left(\lceil \frac{k}{\ell}\rceil\right)^{(2)}}{2!}\alpha^{\lceil \frac{k}{\ell}\rceil-3}\frac{(r)^{(r-1)}\ell^{r-1}}{(a^\ell-\alpha)^r}(\ell+j)_\ell^{(3-r)}a^{\ell+j+(r-1)\ell}$$

$$+\frac{\left(\lceil \frac{k}{\ell}\rceil\right)^{(1)}}{1!}\alpha^{\lceil \frac{k}{\ell}\rceil-3}\frac{(r+1)^{(r-1)}\ell^{r-1}}{(a^\ell-\alpha)^{r+2}}(2\ell+j)_\ell^{(3-r)}a^{2\ell+j+(r-1)\ell},$$

corresponds to $2(L_3)$:

$$\frac{\left(\left\lceil\frac{k}{\ell}\right\rceil\right)^{(2)}}{2!}\alpha^{\left\lceil\frac{k}{\ell}\right\rceil-3}\frac{(r-1)^{(r-1)}\ell^{r-1}}{(a^{\ell}-\alpha)^{r}}(j)_{\ell}^{(3-r)}a^{j+(r-1)\ell}\left\lceil\frac{\ell+j}{\ell}\right\rceil\alpha^{\left\lceil\frac{\ell+j}{\ell}\right\rceil-1}$$

$$+\frac{\left(\left\lceil\frac{k}{\ell}\right\rceil\right)^{(1)}}{1!}\alpha^{\left\lceil\frac{k}{\ell}\right\rceil-3}\frac{(r-1)^{(r-1)}\ell^{r-1}}{(a^{\ell}-\alpha)^{r}}(j)_{\ell}^{(3-r)}a^{j+(r-1)\ell}\frac{\left(\left\lceil\frac{2\ell+j}{\ell}\right\rceil\right)^{(2)}}{2!}\alpha^{\left\lceil\frac{2\ell+j}{\ell}\right\rceil-2}$$

$$+\frac{\left(\left\lceil\frac{k}{\ell}\right\rceil\right)^{(1)}}{1!}\alpha^{\left\lceil\frac{k}{\ell}\right\rceil-3}\frac{(r)^{(r-1)}\ell^{r-1}}{(a^{\ell}-\alpha)^{r+1}}(\ell+j)_{\ell}^{(3-r)}a^{\ell+j+(r-1)\ell}\left\lceil\frac{2\ell+j}{\ell}\right\rceil\alpha^{\left\lceil\frac{2\ell+j}{\ell}\right\rceil-2},$$

and to $3(L_3)$:

$$\frac{\left(\left\lceil\frac{k}{\ell}\right\rceil\right)^{(1)}}{1!}\alpha^{\left\lceil\frac{k}{\ell}\right\rceil-3}\frac{(r-1)^{(r-1)}\ell^{r-1}}{(a^{\ell}-\alpha)^{r}}(j)_{\ell}^{(3-r)}a^{j+(r-1)\ell}\left\lceil\frac{\ell+j}{\ell}\right\rceil\alpha^{\left\lceil\frac{\ell+j}{\ell}\right\rceil-1}\left\lceil\frac{2\ell+j}{\ell}\right\rceil\alpha^{\left\lceil\frac{2\ell+j}{\ell}\right\rceil-2}.$$

In particular when $k=15.2$, $\ell=3.4$, $\alpha=1.2$, $a=2$ and $j=-1.8$ in (25)

$$\sum_{r=4}^{\left\lceil\frac{15.2}{3.4}\right\rceil}\frac{(r-1)^{(3)}}{3!}1.2^{r-4}(15.2-3.4r)_{3.4}^{(2)}\times 2^{15.2-3.4r}=F_{4(\alpha)}^{(2)}(15.2)-1.2^{5-3}F_{4(\alpha)}^{(2)}(8.4)$$

$$F_{4(\alpha)}^{(2)}(15.2)=35.3826516-23.24923775-7.101670544-0.8266197456=4.20512356$$

and

$$F_{4(\alpha)}^{(2)}(8.4)=3.461713602-1.6145304-1.479514697-0.344424894=0.023243611.$$

Theorem 3.11 ***(Special cases on factorial for*** $m=1$***)***

$$(i)\quad\sum_{r=1}^{\left\lceil\frac{k}{\ell}\right\rceil}\alpha^{r-1}(k-r\ell+\ell-\alpha)(k-r\ell)_{\ell}^{\left(\left\lceil\frac{k-r\ell}{\ell}\right\rceil\right)}=k_{\ell}^{\left(\left\lceil\frac{k}{\ell}\right\rceil\right)}-\alpha^{\left\lceil\frac{k}{\ell}\right\rceil}j_{\ell}^{\left(\left\lceil\frac{j}{\ell}\right\rceil\right)}.\qquad(26)$$

$$(ii)\quad\sum_{r=1}^{\left\lceil\frac{k}{\ell}\right\rceil}\alpha^{r-1}\left[(k-r\ell+\ell)^{n+1}-\alpha(k-r\ell)^{n}\right](k-r\ell)_{\ell}^{\left(\left\lceil\frac{k-r\ell}{\ell}\right\rceil\right)}=k^{n}k_{\ell}^{\left(\left\lceil\frac{k}{\ell}\right\rceil\right)}-\alpha^{\left\lceil\frac{k}{\ell}\right\rceil}j_{\ell}^{\left(\left\lceil\frac{j}{\ell}\right\rceil\right)}.\qquad(27)$$

$$(iii)\quad\sum_{r=1}^{\left\lceil\frac{k}{\ell}\right\rceil}\alpha^{r-1}\left[(k-r\ell+(n+1)\ell)(k-r\ell+\ell)-\alpha(k-r\ell+n\ell)\right](k-r\ell)_{\ell}^{\left(\left\lceil\frac{k-r\ell}{\ell}\right\rceil\right)}$$

$$=(k+n\ell)k_{\ell}^{\left(\left\lceil\frac{k}{\ell}\right\rceil\right)}-\alpha^{\left\lceil\frac{k}{\ell}\right\rceil}(j+n\ell)(j)_{\ell}^{\left(\left\lceil\frac{j}{\ell}\right\rceil\right)}.\qquad(28)$$

$$(iv)\quad \sum_{r=1}^{\left\lceil \frac{k}{\ell} \right\rceil} \alpha^{r-1} \left[(k-r\ell-(n-1)\ell)(k-r\ell+\ell) - \alpha(k-r\ell-n\ell) \right] (k-r\ell)_{\ell}^{\left(\left\lceil \frac{k-r\ell}{\ell} \right\rceil\right)}$$

$$= (k-n\ell) k_{\ell}^{\left(\left\lceil \frac{k}{\ell} \right\rceil\right)} - \alpha^{\left\lceil \frac{k}{\ell} \right\rceil} (j-n\ell)(j)_{\ell}^{\left(\left\lceil \frac{j}{\ell} \right\rceil\right)}. \quad (29)$$

Proof. The proof of (26), (27), (28) and (29) follows from (4), (6) and (14).

Conclusion: Here we obtained summation and complete solution of higher order generalized $\alpha-$ difference equation and derived several formula on finite series by equating summation solution and complete solution of $\alpha-$difference equation to polynomial factorial with geometric function.

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