On the Parameters of 2- Class Hadamard Association Schemes

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Abstract- We have all possible parameter of 2-class association schemes having the property that suitable (1, -1)- linear combinations of their association matrices yield the blocks of a Hadamard matrix (H-matrix) of certain classical form of Paley and Williamson. Some 2-class association schemes with the above parameters are identified. The known Hadamard Coherent Configurations or 2-Class Association Schemes (CC's or 2-AS's) listed in Table 2 do not yield H-matrices of new order. However we have forwarded new methods of constructing H-matrices of the forms II and III. The developed technique gives several easy constructions of H-matrix from any 2-AS, whose parameters are given.

Keywords- Parameter of Hadamard Matrix, Coherent Configuration, Association scheme.

I. INTRODUCTION

An (n x n) matrix H with entries +1and -1 is called a **Hadamard matrix** (or H-matrix), if $\{H H^T = n I_n\}$. If $(n > 2)$ and H-matrix of order n exists, then $(n = 4t)$, where t is an integer. It is conjectured that H-matrix of order 4t exists for every $(t \geq 1)$. It remains unsettled in spite of various methods of constructions forwarded by different authors. For a brief surveys see Hall (1967), Hedayat et.al. (1978). However the conjecture is supported by the fact that for every order 4t $(t > 3)$ investigated there are several in-equivalent H-matrices of order 4t reported by Seberry (2001). We recall following definitions from Alejandro et al. (2003) and Raghavarao (1988).

1.1 Coherent configuration (CC):

Let $X = \{1, 2, \ldots, n\}$ and $P = \{R_0, R_1, \ldots, R_t\}$ be a set of binary relations on X satisfying the following four relations:

(a) P is a partition of X^2.

(b) there is a subset P_0 of P which is a partition of the diagonal, $D = \{(\alpha, \alpha): \alpha \in X\}$.

(c) for any relation $R_i \in P$, its converse R_i^T(or R_i^{-1}) $\in P$.

(d) for $0 \leq i, j, k \leq t$, there exists an integer p_{ij}^k such that $(\alpha, \beta) \in R_k$ implies the order of the set $\{ \gamma : (\alpha, \gamma) \in R_i$ and $(\gamma, \beta) \in R_j\}$ is p_{ij}^k which is independent of the choice of $(\alpha, \beta) \in R_k$. p_{ij}^k are called intersection numbers or parameters of the CC.

Let $A_i = [a_{jk}]$ be the (0, 1)-matrix, is called adjacency matrix of the relation R_i, defined as:

$$a_{jk} = \begin{cases} 1, \text{if } (j,k) \in R_i \\ 0, \text{otherwise} \end{cases}$$

Clearly $A=\{A_0, A_1, ..A_t\}$ satisfies

$(c_1)\ A_0 + A_1 + ... + A_t = J_n$ (all 1 matrix)

(c_2) there is a subset of the set A, with sum I_n

$(c_3)\ A_i A_j = \sum_{k=o}^{t} p_{ij}^{k} A_k$ ... (1)

A is called basis algebra or coherent algebra of the CC as the matrices belonging to A form a basis of an associative algebra over the field of complex numbers. A CC is faithfully represented by basis matrices A_i of its basis algebra. A CC P=$\{R_0, R_1, . . ., R_t\}$ is called a t-class Association Scheme (t-AS) if it contains an identity relation R_0 (or A_0=I) and its relations R_i are symmetric (or basis matrices A_i of its coherent algebra are symmetric). Basis matrices of an Association Scheme (AS) are called association matrices. A 2-class association scheme is equivalent to a strongly regular graph.

If p_{ij}^{k} are parameters of a 2-class Association Schemes, $p_{ii}^{0} = n_i$ is called the number of i^{th} associates of a point and n_1, n_2 and p_{ij}^{k} satisfy,

$p_{ij}^{k} = p_{ji}^{k}$, $p_{io}^{j} = \delta_{ij}$, $p_{ij}^{0} = 0$; when $i \neq j$,

$n_1 + n_2 = v-1$ and $p_{j1}^{i} + p_{j2}^{i} = n^{j} - \delta_{ij}$., i,j=1,2 . . . (2)

1.2 H-matrices from AS or CC:

A project of the first author is to obtain all ASs and CCs defined by minimum number of relations leading to the construction of an H-matrix of given form. It was motivated by the fact that the construction of H-matrices of Paley uses a family of CC's and that of Williamson uses a family of AS's. Today we have several ASs and CCs used in Statistics and Coding theory but a few which are suitable for the construction of H-matrices. In view of the significance of such schemes, we forward the following definition:

Hadamard CC or AS: A CC or AS $\{A = (A_i)\}$ will be called Hadamard (or H-CC or H-AS related to an H-matrix of a given form) if suitable (1, -1)-linear combination (or combinations) of (0,1)-basis matrices A_i of its coherent algebra yields a Hadamard matrix (or blocks of the Hadamard matrix).

The present paper is a part of the above project confined to Hadamard 2-AS's. A result in this direction is due to Singh, et al. (2002) who forwarded a method of constructing H-matrices from underlying 2-ASs of a partial geometry. Here we obtain the parameters of four families of Hadamard 2-ASs. Finally we have identified and tabulated 2-ASs included in the families.

II. PARAMETERS OF HADAMARD 2-ASS

We consider Hadamard matrices of the following form and order, whose dependence on the association matrices of Hadamard 2-ASs or basis matrices of H-CC's (which are trivial extension of 2-ASs) are shown below.

TABLE 1

No.	Form of the H-matrix	Order of the H-matrix
I	$H = I - A_1 + A_2$, where A_1, A_2 are association matrices of a 2 - AS on v = 4t points.	O (H) = v =4t
II	$H = \begin{bmatrix} 1 & e \\ e^t & I - A_1 + A_2 \end{bmatrix}$, where A_1, A_2 are association matrices of a 2-AS on v = 4t-1 points.	O (H) = v + 1 =4t
III	$H = H \times I_v + K \times (A_1 - A_2)$, where $K = V_n H$, $V_n = I_{h/2} \times \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$, h =order of an H-matrix H and A_1, A_2 are association Matrices of a 2-AS on v = 2n points.	O(H)= 2nh
IV	$H = \begin{bmatrix} A & B \\ -B & A \end{bmatrix}$, where $A = I - A_1 + A_2$, $B = I + A_1 - A_2$, and A_1 and A_2 are association matrices of a 2-AS on 2t points.	O(H)= 4t

Theorem: Let AS(i), i=1, 2,3,4 be four families of the association schemes required by the forms I, II, III, IV respectively in table1 as per Singh et al. (2009). Then for some values of n and m,

1. A 2-AS of AS(1) has parameters

(i) $v=4n^2$ $n_1 = p_{11}^2 = n(2n-1)$, $p_{11}^1 = p_{11}^2 = n(n-1)$.

or (ii) $v=4n^2$ $n_1 = p_{11}^2 = n(2n+1)$,

$p_{11}^1 = p_{11}^2 = n(n+1)$.

2. A 2-AS of AS(2) has the parameters $v= 4n^2-1$, $n_1= 2n^2$, $p_{12}^1 = n^2-1$, $p_{12}^2 = n^2$ and contains the 2-AS of pg (n,2n+1,n) .

3. A 2-AS of AS(3) has parameters $v= (2n-1)^2+1$, $n_1= p_{11}^2 = m(2n-1)$, $p_{12}^1 = p_{12}^2 = n^2-n$, where $n\geq 2$, $\left\lceil \frac{n}{2} \right\rceil \leq m \leq n-1$ and contains the AS of pg(n,2n,n).

4. AS(4) = AS(3).

A common property shared by all the Hadamard 2-ASs belonging to the above four families is that the parameters p_{12}^1 and p_{12}^2 are as equal as possible i.e. $|p_{12}^1 - p_{12}^2| = 0$ or 1.

III. TABLE OF KNOWN HADAMARD 2-ASs

TABLE 2

Hadamard 2-AS (v, p^0_{11} , p^1_{11}, p^2_{11})	Family AS (i) and form of the H-matrix	Source of the Hadamard 2-AS
(1)(i) AS (4n²,n		Infinitely many 2-ASs obtained from Bush type

(2n-1), n (n-1), n (n-1)) Where (a) n=2r, 4r being the order of an H-matrix.	AS(1)	H-matrices of order16r^2, see Bonato, et al. (2001). For 2-ASs of small order #M_2 see Clatworthy (1973),and Hanaki AS16, # 5 and 6. Infinitely many 2-ASs can be obtained from symmetric Bush-type H- matrices of order 4r^4 vide Muzychuk et al. (2006) and lemma 3 of Bonato (2001).
(b) n= r^2, where r is odd.	AS(1)	There are 32548 2-ASs vide Spence (1995); Mckay et al. (2001) and Janko, et. al. (2001).
(c) n=3	AS(1)	Clatworthy's (1973) AS of Misc. PBIBD #M_3, M_{31a} and Hanaki AS16 #4.
(d) n=5	AS(1)	There are 180 2-ASs vide Spence (1995), Mckay et al. (2001), Brouwer (1996) and Spence.
(1) (ii) AS (4n^2,n(2n+1), n(n+1),n(n	AS(1)	2-ASs of Clatworthy's (1952) Nos. pg9 (n=3) and pg13 (n=4). Dual of BIBD with v=2n^2–n, k=n,

+1)) (a) for n=2 (b) for n=3 (c) n=4		λ□=1 for n = 3, . . .,9 vide Hall no.14, 32, 51, 77, 111,145, 174 respectively by Hall (1967). Also see Mathon et al. (1992).
(2) CC derived from AS of pg (n, 2n + 1, n) n= 3, 4, . . . ,9.	AS(2)	2-ASs of Clatworthy's (1952) Nos. pg7(n=3), pg11(n=4) and pg15(n=5).
(3) AS of pg (n,2n, n)	AS(3)	Dual of BIBD (2n^2-2n+1, k=n , λ = 1) with Hall (1967) No.9, 22, 42 for n= 3, 4, 5 respectively. Also see Mathon et al. (1992).

IV. CONCLUSIONS

Hall (1967); Clatworthy (1973); Spence (1995); Brouwer (1996); Spence; Hanaki et al. reported that H-matrices of new order will be obtained when 2-ASs of corresponding parameters are known. The identifications of the following 2-ASs as Hadamard ones appear to be new:

(i) Clatworthy's AS of Misc. PBIBD #M_3, $M_{31a,}$ 2-ASs of Clatworthy's Nos. pg9(n=3) and pg13(n=4),

(ii) Hanaki's AS16 #4, 5 and 6,

(iii) 180 2-ASs with parameters (36, 14, 4, 6) reported by Spence and AS (64, 36, 22, 22) by Brouwer,

(iv) dual of BIBDs with $v = 2n^2 - n$, $k = n$, $\lambda = 1$ for n= 3, . . .,9 vide Hall no.14, 32, 51, 77, 111, 145, 174 respectively,

(v) dual of BIBD ($2n^2$-2n+1, $k = n$, $\lambda = 1$)with Hall No.9, 22, 42 for n= 3, 4, 5 respectively.

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