On Generalization of Some Coincidence and Common Fixed Point Theorems for Three Self Mappings in Cone Metric Spaces

A.K.Dubey∗[1], Rohit Verma [2], R.P.Dubey [3]

[1] ∗*Department of Mathematics, Bhilai Institute of Technology,*

Bhilai House, Durg, Chhattisgarh 491001, INDIA

[2] *Department of Mathematics, Shri Shankaracharya Institute of Engg. & Tech.,*

Durg, Chhattisgarh 490020, INDIA

[3] *Department of Mathematics, Dr. C. V. Raman University,*

Bilaspur, Chhattisgarh , INDIA

Abstract: The purpose of this paper is to establish some coincidence and common fixed point theorems for three self mappings in cone metric spaces. The results presented in the paper generalize and extend several well-known results in the literature.

Key Words: Cone metric spaces common fixed point, weakly compatible mappings.

1. Introduction

Huang and Zhang [1] recently introduced the concept of cone metric space and established some fixed point theorems for contractive type mappings in a normal cone metric space. Subsequently, some other authors [7,8,9] studied the existence of points of coincidence, and common fixed points of mappings satisfying a contractive type condition in cone metric spaces. Afterwards, Rezapour and Hamlbarani [2] studied fixed point theorems of contractive type mappings by omitting the assumption of normality in cone metric spaces.

Recently, Stojan Redenovic [10] has obtained coincidence point results for two mappings in cone metric spaces which satisfies new contractive conditions. The same concept was further extended by Rangamma et al.[11] and Malhotra et al. [3] and proved coincidence point results and common fixed point results for three self mappings.

In this paper we improve and generalize the results of Rangamma et al.[11] and Malhotra et al. [3] with a new type of contractive condition.

The following definitions and results will be needed in the sequel.

Definition 1.1 [1]. Let E be a real Banach space and P be a subset of E. The set P is called a cone if,

i) P is closed, non-empty and $P \neq \{0_E\}$, here 0_E is the zero vector of E;
ii) $a, b \in \mathbb{R},\ a, b \geq 0,\ x, y \in P$ *implies* $ax + by \in P$;
iii) $x \in P$ and $-x \in P \Rightarrow x = 0_E$.

Given a cone $P \subset E$, we can define a partial ordering $\preccurlyeq$ with respect to P by $x \preccurlyeq y$ if and only if $y - x \in P$. We write $x \prec y$ to indicate that $x \preccurlyeq y$ but $x \neq y$, while $x \ll y$ if and only if for $y - x \in int\ P$, where $int\ P$ denotes the interior of P.

Let P be a cone in a real Banach space E, then P is called normal, if there exist a constant $K > 0$ such that for all $x, y, \in E$,

$$0_E \preccurlyeq x \preccurlyeq y \ implies \ \|x\| \leq K\|y\|.$$

The least positive number K satisfying the above inequality is called the normal constant of P.

Definition 1.2[1]. Let X be a non-empty set, E be a real Banach space. Suppose that the mapping $d: X \times X \to E$ satisfies:

(i) $0_E \preccurlyeq d(x, y)$ for all $x, y \in X$ and $d(x, y) = 0_E$ if and only if $x = y$,
(ii) $d(x, y) = d(y, x)$ for all $x, y \in X$,
(iii) $d(x, y) \preccurlyeq d(x, z) + d(z, y)$ for all $x, y, z \in X$.

Then d is called a cone metric on X, and (X, d) is called a cone metric space.

The concept of cone metric space is more general than that of a metric space, because each metric space is a cone metric space with $E = R$ and $P = [0, +\infty)$.

Definition 1.3[1]. Let (X, d) be a cone metric space. Let $\{x_n\}$ be a sequence in X and $x \in X$.

i) If for every $c \in E$ with $0 \ll c$ there is a positive integer $n > n_0$ such that $d(x_n, x) \ll c$ then the sequence $\{x_n\}$ is said to be convergent and converges to x. We denote this by $lim_{n\to\infty} x_n = x$ or $x_n \to x$ as $n \to \infty$.
ii) If for every $c \in E$ with $0 \ll c$ there is a positive integer $n > n_0$, such that, $d(x_n, x_m) \ll c$ for all $n, m > n_0$, then the sequence $\{x_n\}$ is called a Cauchy sequence in X.

(X, d) is called a complete cone metric space, if every Cauchy sequence in X is convergent in X.

Lemma 1.1[1]. Let (X, d) be a cone metric space, P be a normal cone with normal constant K. Let $\{x_n\}$ and $\{y_n\}$ be two sequences in X.

i) $\{x_n\}$ is a Cauchy sequence if and only if $d(x_n, x_m) \to 0_E$ as $n \to \infty$.
ii) If $x_n \to x,\ y_n \to y$ as $n \to \infty$, then $d(x_n, y_n) \to d(x, y)$ as $n \to \infty$.

Remark 1.1[12]. Let P be a cone in a real Banach space E with zero vector 0_E and $a, b, c \in P$, then;

i) If $a \preccurlyeq b$ and $b \ll c$ then $a \ll c$.
ii) If $a \ll b$ and $b \ll c$ then $a \ll c$.
iii) If $0_E \preccurlyeq u \ll c$ for each $c \in int\ P$ then $u = 0_E$.
iv) If $c \in int\ P$ and $a_n \to 0_E$ then there exist $n_0 \in N$ such that, for all $n > n_0$ we have $a_n \ll c$.
v) If $0_E \preccurlyeq a_n \preccurlyeq b_n$ for each n and $a_n \to a, b_n \to b$ then $a \preccurlyeq b$.
vi) If $a \preccurlyeq \lambda a$ where $0 \leq \lambda < 1$ then $a = 0_E$.

Definition 1.4. Let X be a nonempty set and f, g be self maps on X and $x, z \in X$. Then x called coincidence point of pair (f, g) if $fx = gx$, and z is called point of coincidence of pair (f, g) if $fx = gx = z$.

Definition 1.5. Let X be a nonempty set and f, g be self maps on X. Pair (f, g) is called weakly compatible if $f\ and\ g$ commutes at their coincidence point, i.e. $fgx = gfx$, whenever $fx = gx$ for some $x \in X$.

Let E, B be two real Banach spaces, P and C normal cones in E and B respectively. Let " $\preccurlyeq$ " and " $\leq$ " be the partial orderings induced by P and C in E and B respectively. Let $\emptyset: P \to C$ be a function satisfying:

i) If $a, b \in P$ with $a \preccurlyeq b$ then $\emptyset[a] \leq k\emptyset[b]$, for some positive real k;
ii) $\emptyset[a + b] \leq \emptyset[a] + \emptyset[b]$ for all $a, b \in P$;
iii) $\emptyset$ is sequentially continuous i.e. if $a_n, a \in P$ and $lim_{n\to\infty} a_n = a$, then $lim_{n\to\infty}\emptyset[a_n] = \emptyset[a]$;
iv) If $\emptyset[a_n] \to 0_B$ then $a_n \to 0_E$, where 0_E and 0_B are the zero vectors of E and B respectively.

We denote the set of all such functions by $\Phi(P, C)$ i.e. $\emptyset \in \Phi(P, C)$ if $\emptyset$ satisfies all above properties. It is clear that $\emptyset[a] = 0_B$ if and only if $a = 0_E$.

Let (X, d) be a cone metric space with normal cone P and $\emptyset \in \Phi(P, C)$. Since $d(x, y) \preccurlyeq d(x, z) + d(z, y)$ for all $x, y, z \in X$, therefore

$$\emptyset[d(x, y)] \leq k\emptyset[d(x, z)] + k\emptyset[d(z, y)]. \qquad \text{--------(1.1)}$$

Example 1.1[3]. Let E be any real Banach space with normal cone P and normal constant K.Define $\emptyset: P \to P$ by $\emptyset[a] = a$, for all $a \in P$.Then $\emptyset \in \Phi(P, C)$ with $E = B, P = C$ and $k = 1$.

2. Main Results

Theorem 2.1. Let (X, d) be a cone metric space, and P a normal cone with normal constant K. Suppose f, g, h be self maps of X satisfy the condition

$$\emptyset[d(fx, gy)] \leq a_1\emptyset[d(hx, fx)] + a_2\emptyset[d(hy, gy)]$$
$$+a_3\emptyset[d(hy, fx)] + a_4\emptyset[d(hx, gy)]$$
$$+a_5\emptyset[d(hx, hy)] \text{ for all } x, y \in X \qquad \text{----------(2.1)}$$

where $\emptyset \in \Phi(P, C)$ and $a_i \geq 0 (i = 1,2,3,4,5)$ with $a_1 + a_2 + a_3 + a_4 + a_5 < 1$. If $f(X) \cup g(X) \subset h(X)$ and $h(X)$ is a complete subspace of X, then the maps f, g and h have a unique point of coincidence in X. Moreover, if (f, h) and (g, h) are weakly compatible pairs then f, g and h have a unique common fixed point.

Proof. Suppose x_o be any arbitrary point of X. Since $f(X) \cup g(X) \subset h(X)$, starting with x_o we define a sequence $\{y_n\}$ such that

$$y_{2n} = fx_{2n} = hx_{2n+1} \; and$$

$$y_{2n+1} = gx_{2n+1} = hx_{2n+2},$$

for all $n \geq 0$. We shall prove that $\{y_n\}$ is a Cauchy sequence in X.

If $y_n = y_{n+1}$ for some n e.g. if $y_{2n} = y_{2n+1}$, then from (2.1) we obtain

$$\emptyset[d(y_{2n+2}, y_{2n+1})] = \emptyset[d(fx_{2n+2}, gx_{2n+1})]$$
$$\leq a_1\emptyset[d(hx_{2n+2}, fx_{2n+2})] + a_2\emptyset[d(hx_{2n+1}, gx_{2n+1})]$$
$$+a_3\emptyset[d(hx_{2n+1}, fx_{2n+2})] + a_4\emptyset[d(hx_{2n+2}, gx_{2n+1})]$$
$$+a_5\emptyset[d(hx_{2n+2}, hx_{2n+1})]$$
$$= a_1\emptyset[d(y_{2n+1}, y_{2n+2})] + a_2\emptyset[d(y_{2n}, y_{2n+1})]$$

$+a_3 \emptyset[d(y_{2n}, y_{2n+2})] + a_4 \emptyset[d(y_{2n+1}, y_{2n+1})]$

$+a_5 \emptyset[d(y_{2n+1}, y_{2n})]$

Since $y_{2n} = y_{2n+1}$, it follows from above inequality that,

$\emptyset[d(y_{2n+2}, y_{2n+1})] \leq a_1 \emptyset[d(y_{2n+1}, y_{2n+2})] + a_3 \emptyset[d(y_{2n}, y_{2n+2})]$

$\emptyset[d(y_{2n+2}, y_{2n+1})] \leq (a_1 + a_3) \emptyset[d(y_{2n+1}, y_{2n+2})]$

As $a_1 + a_3 < 1$ and from (vi) of remark 1.1, we obtain

$\emptyset[d(y_{2n+2}, y_{2n+1})] = 0_B$ also $\emptyset \in \Phi(P, C)$ therefore we have

$d(y_{2n+2}, y_{2n+1}) = 0_E$ i.e. $y_{2n+2} = y_{2n+1}$.

Similarly we obtain that

$y_{2n} = y_{2n+1} = y_{2n+2} = ------ = \vartheta$ (say)

Therefore $\{y_n\}$ is a Cauchy sequence.

Suppose $y_n \neq y_{n+1}$ for all n. Then from (2.1) it follows that

$$\begin{aligned}
\emptyset[d(y_{2n}, y_{2n+1})] &= \emptyset[d(fx_{2n}, gx_{2n+1})] \\
&\leq a_1 \emptyset[d(hx_{2n}, fx_{2n})] + a_2 \emptyset[d(hx_{2n+1}, gx_{2n+1})] \\
&\quad + a_3 \emptyset[d(hx_{2n+1}, fx_{2n})] + a_4 \emptyset[d(hx_{2n}, gx_{2n+1})] \\
&\quad + a_5 \emptyset[d(hx_{2n}, hx_{2n+1})] \\
&= a_1 \emptyset[d(y_{2n-1}, y_{2n})] + a_2 \emptyset[d(y_{2n}, y_{2n+1})] \\
&\quad + a_3 \emptyset[d(y_{2n}, y_{2n})] + a_4 \emptyset[d(y_{2n-1}, y_{2n+1})] \\
&\quad + a_5 \emptyset[d(y_{2n-1}, y_{2n})] \\
&= (a_1 + a_4 + a_5) \emptyset[d(y_{2n-1}, y_{2n})] + (a_2 + a_4) \emptyset[d(y_{2n}, y_{2n+1})]
\end{aligned}$$

i.e. $\emptyset[d(y_{2n}, y_{2n+1})] \leq \frac{a_1+a_4+a_5}{1-a_2-a_4} \emptyset[d(y_{2n-1}, y_{2n})]$

$= \lambda \emptyset[d(y_{2n-1}, y_{2n})]$

where $\lambda = \frac{a_1+a_4+a_5}{1-a_2-a_4} < 1 (since\ a_1 + a_2 + a_3 + a_4 + a_5 < 1)$.

Writing $d_n = \emptyset[d(y_n, y_{n+1})]$, we obtain

$$d_{2n} \leq \boldsymbol{\lambda} d_{2n-1} \qquad \text{----------(2.2)}$$

Again

$$\begin{aligned}
\emptyset[d(y_{2n+2}, y_{2n+1})] &= \emptyset[d(fx_{2n+2}, gx_{2n+1})] \\
&\leq a_1 \emptyset[d(hx_{2n+2}, fx_{2n+2})] + a_2 \emptyset[d(hx_{2n+1}, gx_{2n+1})]
\end{aligned}$$

$$+a_3\emptyset[d(hx_{2n+1}, fx_{2n+2})] + a_4\emptyset[d(hx_{2n+2}, gx_{2n+1})]$$

$$+a_5\emptyset[d(hx_{2n+2}, hx_{2n+1})]$$

$$= a_1\emptyset[d(y_{2n+1}, y_{2n+2})] + a_2\emptyset[d(y_{2n}, y_{2n+1})]$$

$$+a_3\emptyset[d(y_{2n}, y_{2n+2})] + a_4\emptyset[d(y_{2n+1}, y_{2n+1})]$$

$$+a_5\emptyset[d(y_{2n+1}, y_{2n})]$$

$$= (a_1 + a_3)\emptyset[d(y_{2n+1}, y_{2n+2})]$$

$$+(a_2 + a_3 + a_5)\emptyset[d(y_{2n}, y_{2n+1})]$$

i.e. $\emptyset[d(y_{2n+2}, y_{2n+1})] \leq \frac{a_2+a_3+a_5}{1-a_1-a_3}\emptyset[d(y_{2n+1}, y_{2n})]$

$$= \mu\emptyset[d(y_{2n+1}, y_{2n})]$$

where $\mu = \frac{a_2+a_3+a_5}{1-a_1-a_3} < 1(since\ a_1 + a_2 + a_3 + a_4 + a_5 < 1)$.

Therefore $d_{2n+1} \leq \mu d_{2n}$ ---------(2.3)

From (2.2) and (2.3) we get

$$d_{2n} \leq \boldsymbol{\lambda} d_{2n-1} \leq \lambda\mu d_{2n-2} \leq ----\leq \lambda^n\mu^n d_0$$

and

$$d_{2n+1} \leq \mu d_{2n} \leq \lambda\mu d_{2n-1} \leq ----\leq \lambda^n\mu^n d_0.$$

Thus

$$d_{2n} + d_{2n+1} \leq \lambda^n\mu^n(1+\mu)d_0 \quad \text{--------(2.4)}$$

$$d_{2n+1} + d_{2n+2} \leq \lambda^n\mu^{n+1}(1+\lambda)d_0 \quad \text{-------(2.5)}$$

Let $n, m \in N$, then for the sequence $\{y_n\}$ we consider $\emptyset[d(y_n, y_m)]$ in two cases.

If n is even and $m > n$, then using (1.1) and (2.4) we obtain

$$\emptyset[d(y_n, y_m)] \leq k\emptyset[d(y_n, y_{n+1})] + k\emptyset[d(y_{n+1}, y_{n+2})] +$$

$$------- + k\emptyset[d(y_{m-1}, y_m)]$$

$$\leq k[d_n + d_{n+1} + d_{n+2} + d_{n+3} + ----]$$

$$\leq k[\lambda^{\frac{n}{2}}\mu^{\frac{n}{2}}(1+\mu)\,d_0 + \lambda^{\frac{n+2}{2}}\mu^{\frac{n+2}{2}}(1+\mu)d_0 + ---]$$

$$\emptyset[d(y_n, y_m)] \leq \frac{k(\lambda\mu)^{\frac{n}{2}}(1+\mu)}{1-\lambda\mu}d_0.$$

If n is odd and $m > n$, then again using (1.1) and (2.5) we obtain

$$\emptyset[d(y_n, y_m)] \leq k\emptyset[d(y_n, y_{n+1})] + k\emptyset[d(y_{n+1}, y_{n+2})] +$$

$$-------- + k\emptyset[d(y_{m-1}, y_m)]$$

$$\leq k[d_n + d_{n+1} + d_{n+2} + d_{n+3} + ----]$$

$$\leq k[\lambda^{\frac{n-1}{2}}\mu^{\frac{n-1}{2}+1}(1+\lambda)d_0 + \lambda^{\frac{n+1}{2}}\mu^{\frac{n+1}{2}+1}(1+\lambda)d_0 + ---]$$

$$\emptyset[d(y_n, y_m)] \leq \frac{k(\lambda\mu)^{\frac{n-1}{2}}(1+\lambda)}{1-\lambda\mu}d_0.$$

Since $\lambda < 1, \mu < 1$ therefore $\lambda\mu < 1$, so in both the cases $\emptyset[d(y_n, y_m)] \to 0_B$ as $n \to \infty$, and since $\emptyset \in \Phi(P, C)$ we have $d(y_n, y_m) \to 0_E$ as $n \to \infty$. So by lemma 1.1, $\{y_n\} = \{hx_{2n-1}\}$ is a Cauchy sequence.

Since $h(X)$ is complete, there exists $\vartheta \in h(X)$ and $u \in X$ such that $lim_{n\to\infty} y_n = \vartheta$ and $\vartheta = hu$.

We shall show that u is a coincidence point of pairs (f, h) and (g, h) i.e. $fu = gu = hu$. If $fu \neq hu$ then $0_E \prec d(fu, hu)$. Using (2.1) we obtain

$$\emptyset[d(fu, y_{2n+1})] = \emptyset[d(fu, gx_{2n+1})]$$

$$\leq a_1\emptyset[d(hu, fu)] + a_2\emptyset[d(hx_{2n+1}, gx_{2n+1})]$$

$$+a_3\emptyset[d(hx_{2n+1}, fu)] + a_4\emptyset[d(hu, gx_{2n+1})]$$

$$+a_5\emptyset[d(hu, hx_{2n+1})]$$

$$= a_1\emptyset[d(hu, fu)] + a_2\emptyset[d(y_{2n}, y_{2n+1})]$$

$$+a_3\emptyset[d(y_{2n}, fu)] + a_4\emptyset[d(hu, y_{2n+1})]$$

$$+a_5\emptyset[d(hu, y_{2n})]$$

$$= (a_1 + a_3)\emptyset[d(hu, fu)] + a_2 d_{2n}.$$

Since $y_{2n} \to hu, d_{2n} \to 0_B, d(fu, y_{2n+1}) \to d(fu, hu)$ as $n \to \infty$ and $\emptyset \in \Phi(P, C)$, therefore letting $n \to \infty$ in above inequality and using remark 1.1 we get

$$\emptyset[d(fu, hu)] \leq (a_1 + a_3)\emptyset[d(fu, hu)]$$

$$< \emptyset[d(fu, hu)] \text{ (since } a_1 + a_3 < 1),$$

a contradiction. Therefore $fu = hu$. Similarly it can be shown that $gu = hu$.

$$fu = gu = hu = \vartheta \quad \text{-------------(2.6)}$$

Thus ϑ is point of coincidence of pairs (f, h) and (g, h). We shall show that it is unique.

Suppose w is another point of coincidence of these pairs i.e. $fz = gz = hz = w$ for some $z \in X$.

Then from (2.1) it follows that

$$\emptyset[d(w, \vartheta)] = \emptyset[d(fz, gu)]$$

$$\leq a_1\emptyset[d(hz,fz)] + a_2\emptyset[d(hu,gu)] + a_3\emptyset[d(hu,fz)]$$

$$+a_4\emptyset[d(hz,gu)] + a_5\emptyset[d(hz,hu)]$$

$$= a_1\emptyset[d(w,w)] + a_2\emptyset[d(\vartheta,\vartheta)] + a_3\emptyset[d(\vartheta,w)]$$

$$+a_4\emptyset[d(w,\vartheta)] + a_5\emptyset[d(w,\vartheta)]$$

$$= (a_3 + a_4 + a_5)\emptyset[d(w,\vartheta)].$$

Since $a_3 + a_4 + a_5 < 1$, by remark 1.1 we obtain

$\emptyset[d(w,\vartheta)] = 0_B$ i.e. $w = \vartheta$. Thus point of coincidence is unique.

If pairs (f,h) and (g,h) are weakly compatible, from (2.6) we have $f\vartheta = fhu = hfu = h\vartheta$ and $g\vartheta = ghu = hgu = h\vartheta$, therefore $f\vartheta = g\vartheta = h\vartheta = p$ (say). This shows that p is another point of coincidence, therefore by uniqueness, we must have $p = \vartheta$ i.e.

$$f\vartheta = g\vartheta = h\vartheta = \vartheta.$$

Thus ϑ is unique common fixed point of self maps f, g and h.

Corollary 2.1.

(i) If $a_3 = a_4 = 0$ in Theorem 2.1, then we have the Theorem 2.1 of [3].

(ii) If $a_1 = a_2 = a_3 = a_4 = 0$, Theorem 2.1 is generalization of Theorem 1 of [1], Theorem 2.1 of [4] and Theorem 2.3 of [5].

(iii) If $a_3 = a_4 = a_5 = 0$, Theorem 2.1 is generalization of Theorem 3 of [1], Theorem 2.3 of [4] and Theorem 2.6 of [2].

(iv) If $a_1 = a_2 = a_5 = 0$, Theorem 2.1 is generalizes Theorem 5 of [1].

Theorem 2.2. Let (X,d) be a cone metric space and P a normal cone with normal constant K. Suppose f,g,h be self maps of X satisfy the condition.

$$\emptyset[d(fx,gy)] \leq a_1\emptyset[d(hx,hy)] + a_2\emptyset[d(fx,hx)]$$

$$+a_3\emptyset[d(gy,hy)] + a_4\emptyset[d(fx,hy) + d(gy,hx)]$$

$$\text{for all } x,y \in X \qquad \text{---------(2.7)}$$

where $\emptyset \in \Phi(P,C)$ and $a_1,a_2,a_3,a_4 \in [0,1)$ satisfying $a_1 + a_2 + a_3 + 2a_4 < 1$. If $f(X) \cup g(X) \subset h(X)$ and $h(X)$ is complete subspace of X, then the maps f, g *and* h have a unique point of coincidence in X. Moreover, if (f,h) and (g,h) are weakly compatible pairs then f, g and h have a unique common fixed point.

Proof. Suppose x_0 be any arbitrary point of X. Since $f(X) \cup g(X) \subset h(X)$, starting with x_0 we define a sequence $\{y_n\}$ such that

$y_{2n} = fx_{2n} = hx_{2n+1}$ and $y_{2n+1} = gx_{2n+1} = hx_{2n+2}$, for all $n \geq 0$. We shall prove that $\{y_n\}$ is a Cauchy sequence in X.

If $y_n = y_{n+1}$ for some n, e.g. if $y_{2n} = y_{2n+1}$, then from (2.7) we obtain

$\emptyset[d(y_{2n+2}, y_{2n+1})] = \emptyset[d(fx_{2n+2}, gx_{2n+1})]$

$\leq a_1\emptyset[d(hx_{2n+2}, hx_{2n+1})] + a_2\emptyset[d(fx_{2n+2}, hx_{2n+2})]$
$+a_3\emptyset[d(gx_{2n+1}, hx_{2n+1})] + a_4\emptyset[d(fx_{2n+2}, hx_{2n+1}) + d(gx_{2n+1}, hx_{2n+2})]$

$= a_1\emptyset[d(y_{2n+1}, y_{2n})] + a_2\emptyset[d(y_{2n+2}, y_{2n+1})]$

$+a_3\emptyset[d(y_{2n+1}, y_{2n})] + a_4\emptyset[d(y_{2n+2}, y_{2n}) + d(y_{2n+1}, y_{2n+1})]$

Since $y_{2n} = y_{2n+1}$ it follows from above inequality that

$\emptyset[d(y_{2n+2}, y_{2n+1})] \leq (a_2 + a_4)\emptyset[d(y_{2n+2}, y_{2n+1})]$.

As $a_2 + a_4 < 1$ and from (vi) of remark 1.1 we obtain $\emptyset[d(y_{2n+2}, y_{2n+1})] = 0_B$ also $\emptyset \in \Phi(P, C)$ therefore we have $d(y_{2n+2}, y_{2n+1}) = 0_E$ ie. $y_{2n+2} = y_{2n+1}$.

Similarly we obtain that

$y_{2n} = y_{2n+1} = y_{2n+2} = - - - -= \vartheta$ (say)

Therefore $\{y_n\}$ is a Cauchy sequence.

Suppose $y_n \neq y_{n+1}$ for all n. Then from (2.7) it follows that

$\emptyset[d(y_{2n}, y_{2n+1})] = \emptyset[d(fx_{2n}, gx_{2n+1})]$

$\leq a_1\emptyset[d(hx_{2n}, hx_{2n+1})] + a_2\emptyset[d(fx_{2n}, hx_{2n})]$

$+a_3\emptyset[d(gx_{2n+1}, hx_{2n+1})] + a_4\emptyset[d(fx_{2n}, hx_{2n+1}) + d(gx_{2n+1}, hx_{2n})]$

$= a_1\emptyset[d(y_{2n-1}, y_{2n})] + a_2\emptyset[d(y_{2n}, y_{2n-1})]$

$+a_3\emptyset[d(y_{2n+1}, y_{2n})] + a_4\emptyset[d(y_{2n}, y_{2n}) + d(y_{2n+1}, y_{2n-1})]$

$\leq (a_1 + a_2 + a_4)\emptyset[d(y_{2n-1}, y_{2n})] + (a_3 + a_4)\emptyset[d(y_{2n}, y_{2n+1})]$

i.e. $\emptyset[d(y_{2n}, y_{2n+1})] \leq \frac{a_1+a_2+a_4}{1-a_3-a_4}\emptyset[d(y_{2n-1}, y_{2n})]$

$= \lambda\emptyset[d(y_{2n-1}, y_{2n})]$

where $\lambda = \frac{a_1+a_2+a_4}{1-a_3-a_4} < 1(since\ a_1 + a_2 + a_3 + 2a_4 < 1)$.

Writing $d_n = \emptyset[d(y_n, y_{n+1})]$, we obtain

$d_{2n} \leq \boldsymbol{\lambda} d_{2n-1}$ ----------(2.8)

Again

$\emptyset[d(y_{2n+2}, y_{2n+1})] = \emptyset[d(fx_{2n+2}, gx_{2n+1})]$

$\leq a_1\emptyset[d(hx_{2n+2}, hx_{2n+1})] + a_2\emptyset[d(fx_{2n+2}, hx_{2n+2})]$

$+a_3\emptyset[d(gx_{2n+1}, hx_{2n+1})] + a_4\emptyset[d(fx_{2n+2}, hx_{2n+1}) + d(gx_{2n+1}, hx_{2n+2})]$

$= a_1\emptyset[d(y_{2n+1}, y_{2n})] + a_2\emptyset[d(y_{2n+2}, y_{2n+1})]$

$+a_3\emptyset[d(y_{2n+1}, y_{2n})] + a_4\emptyset[d(y_{2n+2}, y_{2n}) + d(y_{2n+1}, y_{2n+1})]$

$\leq (a_1 + a_3 + a_4)\emptyset[d(y_{2n+1}, y_{2n})] + (a_2 + a_4)\emptyset[d(y_{2n+1}, y_{2n+2})]$

i.e. $\emptyset[d(y_{2n+2}, y_{2n+1})] \leq \frac{a_1+a_3+a_4}{1-a_2-a_4}\emptyset[d(y_{2n+1}, y_{2n})]$

$$= \mu\emptyset[d(y_{2n+1}, y_{2n})]$$

where $\mu = \frac{a_1+a_3+a_4}{1-a_2-a_4} < 1(since\ a_1 + a_2 + a_3 + 2a_4 < 1)$.

Therefore $d_{2n+1} \leq \mu d_{2n}$ ---------(2.9)

From (2.8) and (2.9) we get

$$d_{2n} \leq \lambda d_{2n-1} \leq \lambda\mu d_{2n-2} \leq ----\leq \lambda^n\mu^n d_0$$

and

$$d_{2n+1} \leq \mu d_{2n} \leq \lambda\mu d_{2n-1} \leq ----\leq \lambda^n\mu^{n+1} d_0.$$

Thus

$$d_{2n} + d_{2n+1} \leq \lambda^n\mu^n(1+\mu)d_0 \quad \text{--------(2.10)}$$

and

$$d_{2n+1} + d_{2n+2} \leq \lambda^n\mu^{n+1}(1+\lambda)d_0 \quad \text{-------(2.11)}$$

Let $n, m \in N$, then for the sequence $\{y_n\}$ we consider $\emptyset[d(y_n, y_m)]$ in two cases.

If n is even and $m > n$, then using (1.1) and (2.10) we obtain

$$\emptyset[d(y_n, y_m)] \leq k\emptyset[d(y_n, y_{n+1})] + k\emptyset[d(y_{n+1}, y_{n+2})] +$$

$$-------+ k\emptyset[d(y_{m-1}, y_m)]$$

$$\leq k[d_n + d_{n+1} + d_{n+2} + d_{n+3} + ----]$$

$$\leq k[\lambda^{\frac{n}{2}}\mu^{\frac{n}{2}}(1+\mu)d_0 + \lambda^{\frac{n+2}{2}}\mu^{\frac{n+2}{2}}(1+\mu)d_0 + ---]$$

$$\emptyset[d(y_n, y_m)] \leq \frac{k(\lambda\mu)^{\frac{n}{2}}(1+\mu)}{1-\lambda\mu}d_0.$$

Similarly if n is odd and $m > n$, then again using (1.1) and (2.11) we obtain

$$\emptyset[d(y_n, y_m)] \leq \frac{k(\lambda\mu)^{\frac{n-1}{2}}(1+\lambda)}{1-\lambda\mu}d_0.$$

Since $\lambda < 1, \mu < 1$ therefore $\lambda\mu < 1$, so in both the cases $\emptyset[d(y_n, y_m)] \to 0_B$ as $n \to \infty$, and since $\emptyset \in \Phi(P, C)$ we have $d(y_n, y_m) \to 0_E$ as $n \to \infty$. So by lemma 1.1, $\{y_n\} = \{hx_{n-1}\}$ is a Cauchy sequence.

Since $h(X)$ is complete, there exists $\vartheta \in h(X)$ and $u \in X$ such that $lim_{n\to\infty} y_n = \vartheta$ and $\vartheta = hu$.We shall show that u is a coincidence point of pairs (f, h) and (g, h) i.e. $fu = gu = hu$.

If $fu \neq hu$ then $0_E \prec d(fu, hu)$. Using (2.7) we obtain

$$\emptyset[d(fu, y_{2n+1})] = \emptyset[d(fu, gx_{2n+1})]$$

$$\leq a_1\emptyset[d(hu, hx_{2n+1})] + a_2\emptyset[d(fu, hu)]$$

$$+a_3\emptyset[d(gx_{2n+1}, hx_{2n+1})] + a_4\emptyset[d(fu, hx_{2n+1}) + d(gx_{2n+1}, hu)]$$

$$= a_1\emptyset[d(hu, y_{2n})] + a_2\emptyset[d(fu, hu)] + a_3\emptyset[d(y_{2n+1}, y_{2n})]$$

$$+a_4\emptyset[d(fu, y_{2n}) + d(y_{2n+1}, hu)]$$

Since $y_{2n} \to hu, d_{2n} \to 0_B, d(fu, y_{2n+1}) \to d(fu, hu)$as $n \to \infty$ and $\emptyset \in \Phi(P, C)$, therefore letting $n \to \infty$ in above inequality and using remark 1.1 we get

$$\emptyset[d(fu, hu)] \leq (a_2 + a_4)\emptyset[d(fu, hu)]$$

$$< \emptyset[d(fu, hu)] \text{ (since } a_2 + a_4 < 1),$$

a contradiction. Therefore $fu = hu$. Similarly it can be shown that $gu = hu$.

Therefore,

$$fu = gu = hu = \vartheta \qquad \text{-------------(2.12)}$$

Thus ϑ is point of coincidence of pairs (f, h) and (g, h). We shall show that it is unique.

Suppose w is another point of coincidence of these pairs i.e. $fz = gz = hz = w$ for some $z \in X$.

Then from (2.7) it follows that

$$\emptyset[d(w, \vartheta)] = \emptyset[d(fz, gu)]$$

$$\leq a_1\emptyset[d(hz, hu)] + a_2\emptyset[d(fz, hz)] + a_3\emptyset[d(gu, hu)]$$

$$+a_4\emptyset[d(fz, hu) + d(hz, gu)]$$

$$= a_1\emptyset[d(w, v)] + a_2\emptyset[d(w, w)] + a_3\emptyset[d(\vartheta, \vartheta)]$$

$$+a_4\emptyset[d(w, \vartheta) + d(\vartheta, w)]$$

$$= (a_1 + 2a_4)\emptyset[d(w, \vartheta)]$$

Since $a_1 + 2a_4 < 1$, by remark 1.1 we obtain

$\emptyset[d(w, \vartheta)] = 0_B$ i.e. $w = \vartheta$. Thus point of coincidence is unique.

If pairs (f, h) and (g, h) are weakly compatible, from (2.12) we have $f\vartheta = fhu = hfu = h\vartheta$ and $g\vartheta = ghu = hgu = h\vartheta$, therefore $f\vartheta = g\vartheta = h\vartheta = p$ (say). This shows that p is another point of coincidence, therefore by uniqueness, we must have $p = \vartheta$ i.e.

$$f\vartheta = g\vartheta = h\vartheta = \vartheta.$$

Thus ϑ is unique common fixed point of self maps f, g and h.

Corollary 2.2. Let f *and* h be self maps on a cone metric space X with P be a normal cone and K is normal constant, satisfying $f(X) \subset h(X)$ and

$$\emptyset[d(fx, fy)] \leq a_1\emptyset[d(hx, hy)] + a_2\emptyset[d(fx, hx)]$$

$$+a_3\emptyset[d(fy, hy)] + a_4\emptyset[d(fx, hy) + d(fy, hx)]$$

for all $x, y \in X$,where $\emptyset \in \Phi(P, C)$ and $a_1, a_2, a_3, a_4 \in [0,1)$ satisfying $a_1 + a_2 + a_3 + 2a_4 < 1$. If one of $f(X)$ or $h(X)$ is complete subspace of X, then the maps f *and* h have a unique point of coincidence in X. Moreover, if f *and* h are weakly compatible, then f and h have a unique common fixed point.

Corollary 2.3. Let f be a self map on a cone metric space X with normal cone P and normal constant K satisfying

$$\emptyset[d(fx, fy)] \leq a_1\emptyset[d(x, y)] + a_2\emptyset[d(fx, x)]$$

$$+a_3\emptyset[d(fy, y)] + a_4\emptyset[d(fx, y) + d(fy, x)]$$

for all $x, y \in X$, where $\emptyset \in \Phi(P, C)$ and $a_1, a_2, a_3, a_4 \in [0,1)$ satisfying $a_1 + a_2 + a_3 + 2a_4 < 1$. If $f(X)$ is complete subspace of X, then f has a unique fixed point in X.

References

1. L.G.Huang, X.Zhang, Cone metric spaces and fixed point theorems of contractive mappings, J. Math. Anal. Appl. ,332(2)(2007), 1468-1476.
2. Sh.Rezapour, R.Hamlbarani, Some notes on the paper “Cone metric spaces and fixed point theorems of contractive mappings”, J. Math. Anal. Appl., 345(2) (2008), 719-724.
3. S.K. Malhotra, S. Shukla and R. Sen, Some coincidence and common fixed point theorems in cone metric spaces, Bulletin of Mathematical Analysis and Applications, Vol.4, Issue 2 (2012), 64-71.
4. M. Abbas and G. Jungck, Common fixed point results for noncommuting mappings without continuity in cone metric spaces, J. Math. Anal. Appl., 341(2008), 416-420.
5. M. Arshad, A. Azam and P. Vetro, Some common fixed point results in cone metric spaces, Fixed Point Theory and Applications, Volume 2009, Article ID 493965, 11 pages.
6. J. Olaleru, Common Fixed Points of Three Self-Mappings in Cone Metric Spaces, Applied Mathematics E-Notes, 11 (2011) 41-49.
7. M. Abbas and B.E. Rhoades, Fixed and periodic point results in cone metric spaces, Appl. Math. Lett. 21(2008), 511-515.
8. D. Ilic and V. Rakocevic, Common fixed points for maps on cone metric space, J. Math. Anal. Appl., 341(2008), 876-882.
9. P. Vetro, Common fixed points in cone metric spaces, Rend. Circ. Mat. Palermo 56(2007), 464-468.
10. Stojan Redenovic, Common fixed points under contractive conditions in cone metric spaces, Computers and Mathematics with Applications 58(2009),1273-1278.
11. M. Rangamma and K. Prudhvi, Common fixed points under contractive conditions for three maps in cone metric spaces, Bulletin of Mathematical analysis and applications ,Vol. 4 Issue 1 (2012), 174-180.
12. G. Jungck, S. Radenovic, S. Radojevic and V. Rakocevic, Common fixed point theorems for weakly compatible pairs on cone metric spaces, Fixed Point Theory and Applications, 57 (2009) article ID 643840, 13 pages.
13. A.K.Dubey, Rita Shukla, R.P.Dubey, Some common fixed point results of three self-mappings in cone metric spaces, Journal of Advances in Mathematics, Vol 7, No. 3(2014),pp 1277-1284.