L - Magic labeling

V.L.Stella Arputha Mary [#1] S.Navaneethakrishnan [*2] A.Nagarajan [#3]

[#1] *Department of Mathematics, St.Mary's College, Tuticorin - 628 001.*

[*2,#3] *Department of Mathematics, V.O.C College, Tuticorin - 628 001.*

Tamil Nadu, India.

***Abstract* — Let $(L,\wedge,\vee)$ be a lattice. A graph $G(V,E)$ is said to be L-magic if there exists a labeling f of the edges of G with the elements of L induces the vertex labeling f^+ defined as $f^+(v) = \bigvee_{u\in V} f(uv)$ taken over all edges uv incident at v is a constant and the constant is nothing but the least upper bound of L and also induces another vertex labeling f^- defined as $f^-(v) = \bigwedge_{u\in V} f(uv)$ is also a constant and the constant is the greatest lower bound of L. A graph is said to be L-magic if it admits L - magic labeling.**

Keywords — L - magic labeling, L - magic graph, least upper bound, greatest lower bound.

I. INTRODUCTION

By a graph $G(V,E)$ we mean G is a finite, simple, and undirected graph. Magic labelings were introduced by Sedlacek in 1963. Kong, Lee and Sun [3] used the term magic labeling for the labeling of edges with non negative integers such that for each vertex v the sum of the labels of all edges incident at v is same for all v.

For any non trivial abelian group A, under addition, a graph G is said to be A magic if there exists a labeling f of the edges of G with non zero elements of A such that, the vertex labeling f^+ defined as $f^+(v) = \sum f(uv)$ taken over all edges uv incident at v is a constant.

This idea motivate us to define L - magic labeling. Let $(L,\wedge,\vee)$ be a Lattice, A graph $G(V,E)$ is said to be L - magic if there exists a labeling f of the edges of G with the elements of L such that the vertex labeling f^+ defined as $f^+(v) = \bigvee_{u\in V} f(uv)$ considered overall edges uv incident at v is a constant and the constant is the least upper bound of the set L and another vertex labeling f^- defined as $f^-(v) = \bigwedge_{u\in V} f(uv)$ taken over all edges incident at v is a constant and the constant is the greatest lower bound of L.

A graph is said to be L-magic if it admits L-magic labeling.

In this paper, we consider a lattice with $L = \{1,2,3\}$ and $\leq$ is the "less than or equal to" relationship among numbers. By lub and glb we mean the least upper bound and greatest lower bound.

II. BASIC DEFINITIONS

Definition 2.1

A non empty set A on which a partial ordering relationship,(generally denoted by $\leq$) is defined is called a partially ordered set or poset and it is written as $(A,\leq)$.

Definition 2.2 [5]

A Lattice is a poset (partially ordered set) $(L,\leq)$ in which every 2 - element subset $\{a,b\}$ has a lub and glb. That is, poset $(L,\leq)$ is a lattice if for every $a,b\in L$, $\text{lub}(a,b)$ and $\text{glb}(a,b)$ exist in L.

Definition 2.3

A nonempty set L closed under two binary operations $\wedge$ and $\vee$ is called a lattice $(L,\wedge,\vee)$ provided the following axioms hold.

1. *(i)* $a\wedge a=a$ *(ii)* $a\vee a=a$ *for all* $a\in L$
2. *(i)* $a\wedge b=b\wedge a$ *(ii)* $a\vee b=b\vee a$ *for all* $a,b\in L$
3. *(i)* $(a\wedge b)\wedge c=a\wedge(b\wedge c)$ *(ii)* $(a\vee b)\vee c=a\vee(b\vee c)$ *for all* $a,b,c\in L$
4. *(i)* $a\wedge(a\vee b)=a$ *(ii)* $a\vee(a\wedge b)=a$ *for all* $a,b,\in L$.

Definition 2.4

If $a\leq a\vee b$ and $b\leq a\vee b$ then $a\vee b$ is the upper bound of elements $a,b\in L$ Also if $a\leq c$ and $b\leq c$ then $a\vee b\leq c$ then $a\vee b=\text{lub}\{a,b\}$ for all $a,b\in L$. It is also denoted as $a\vee b=\sup(a\ b)$.

Definition 2.5

If $a\wedge b\leq a$ and $a\wedge b\leq b$ then $a\wedge b$ is the lower bound of elements $a,b\in L$ Also if $c\leq a$ and $c\leq b$ then $c\leq a\wedge b$ then $a\wedge b=\text{glb}\{a,b\}$ for all $a,b\in L$. It is also denoted as $a\wedge b=\inf(a\ b)$.

Definition 2.6

Let $G_1(V_1,E_1)$ and $G_2(V_2,E_2)$ be two graphs. Then their union $G=G_1G\cup G_2$ is a graph with the vertex set $V=V_1\cup V_2$ and edge set $E=E_1\cup E_2$.

Definition 2.7

The join G_1+G_2 of $G_1$and G_2 consists of $G_1\cup G_2$ and all lines joining V_1 with V_2. The graph P_n+K_1 is called a fan P_n+2K_1 is called the double fan. It is denoted as DF_n. The graph C_n+K_1 is called a cone or a wheel W_n with n spokes and the grah C_n+2K_1 is called the double cone. It is denoted as DC_n.

Definition 2.8

The helm H_n is the graph obtained from a W_n by attaching a pendant edge at each vertex of the *n-cycle* of the wheel.

III MAIN RESULTS

Let us learn through the following theorem about *L*-magic labeling

Theorem 3.1.

C_n is *L*-magic for $n\equiv 0 \pmod 2$.

Proof. Let $f: E(C_n)\to L$

$f(u_{2i-1}u_{2i})=1,\quad 1\leq i\leq n/2$

$f(u_{2i}u_{2i+1})=3,\quad 1\leq i\leq n/2\quad (u_{n+1}\equiv u_1)$

Let $f^+: V(c_n)\to L$

By Definition

$$f^+(u_i) = \bigvee_{u \in V} f(uu_i) = f(u_{i-1}u_i) \vee f(u_i u_{i+1})$$
$$= 1 \vee 3 = lub\{1,3\}$$
$$= 3,\ 1 \le i \le n\ \ (u_0 = u_n)$$

Let $f^- : V(c_n) \to L$

$$f^-(u_i) = \bigwedge_{u \in V} f(uu_i) = f(u_i u_{i+1}) \wedge f(u_{i-1}u_i)$$
$$= 1 \wedge 3$$
$$= glb\{1,3\} = 1,\ 1 \le i \le n\ \ (u_0 = u_n)$$

$f^+(v)$ is a constant and $f^-(v)$ is also a constant for all $v \in C_n$. Therefore C_n *is* L - magic for $n \equiv 0\,(mod\ 2)$.

Example 3.2. L - magic labeling is given for C_6.

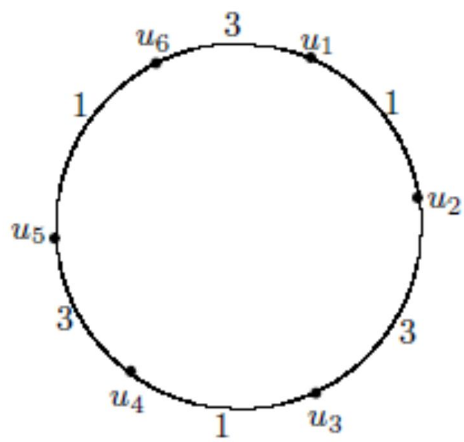

Fig. 1 L - magic labeling of C_6

Theorem 3.3.

W_n *is* L -magic for $n \ge 3$.

Proof. Let $V(W_n) = \{u\} \cup \{u_i / 1 \le i \le n\}$.

$E(W_n) = \{uu_i / 1 \le i \le n\} \cup \{u_i u_{i+1} / 1 \le i \le n\}[u_{n+1} \equiv u_1]$

case 1 $n \equiv 1\,(mod\ 2)$

Let $f : E(W_n) \to L$ be defined as

$$f(uu_1) = 1 \ \text{ and } \ f(uu_n) = 3$$
$$f(uu_i) = 2,\ 2 \le i \le n-1$$
$$f(u_{2i-i}u_{2i}) = 3,\ 1 \le i \le \frac{n-1}{2}$$

$$f(u_{2i}u_{2i+1}) = 1,\ 1 \le i \le \frac{n-1}{2}$$

$$f(u_n u_1) = 2.$$

Now, $f^+ : V(W_n) \to L$

$$f^+(u_i) = \bigvee_{u \in V} f(uu_i)$$
$$= f(u_{i-1}u_i) \vee f(u_i u_{i+1}) \vee f(uu_i)$$
$$= 2 \vee 3 \vee 1$$
$$= lub\{1,2,3\} = 3,\ 1 \le i \le n.$$

$$f^+(u) = \bigvee_{u\,i \in V} f(uu_i)$$
$$= f(uu_1) \vee f(uu_2) \vee \ldots \vee f(uu_n)$$
$$= 1 \vee 2 \vee \ldots \vee 3$$
$$= lub\{1,2,3\} = 3.$$

$f^- : V(W_n) \to L$

$$f^-(u_i) = \bigwedge_{u \in V} f(uu_i)$$
$$= f(u_{i-1}u_i) \wedge f(u_i u_{i+1}) \wedge f(uu_i)$$
$$= 2 \wedge 3 \wedge 1$$
$$= glb\{1,2,3\} = 1,\ 1 \le i \le n.$$

$$f^-(u) = \bigwedge_{u\,i \in V} f(uu_i)$$
$$= f(uu_1) \wedge f(uu_2) \wedge \ldots \wedge f(uu_n)$$
$$= 1 \wedge 2 \wedge \ldots \wedge 3$$
$$= glb\{1,2,3\} = 1.$$

Hence $f^+(v)$ and $f^-(v)$ are constant for all $v \in V$.

case 2 $n \equiv 0\ (mod\ 2)$

Let $f : E(W_n) \to L$ be defined as

$$f(uu_1) = 1 \text{ and } f(uu_n) = 3$$

$$f(uu_i) = 2,\ 2 \le i \le n-1$$

$$f(u_{2i-1}u_{2i}) = 3,\ 1 \le i \le n/2$$

$$f(u_{2i}u_{2i+1}) = 1,\ 1 \le i \le n/2$$

Now, $f^+ : V(W_n) \to L$

$$f^+(u_i) = \bigvee_{u \in V} f(uu_i)$$
$$= f(u_{i-1}u_i) \vee f(u_i u_{i+1}) \vee f(uu_i)$$
$$= 3 \vee 1 \vee 2$$
$$= lub\{1,2,3\}$$

$$= 3, \; 2 \le i \le n-1$$

$$\begin{aligned} f^{+}(u_1) &= f(u_n u_1) \vee f(u_1 u_2) \vee f(uu_1) \\ &= 1 \vee 3 \vee 1 \\ &= lub\{1,3\} \\ &= 3 \end{aligned}$$

$$\begin{aligned} f^{+}(u_n) &= f(u_{n-1} u_n) \vee f(u_n u_1) \vee f(uu_n) \\ &= 3 \vee 1 \vee 3 \\ &= lub\{1,3\} \\ &= 3. \end{aligned}$$

$$\begin{aligned} f^{+}(u) &= \bigvee_{u i \in V} f(uu_i) \\ &= f(uu_1) \vee f(uu_2) \vee \ldots \vee f(uu_n) \\ &= 1 \vee 2 \vee \ldots \vee 3 \\ &= lub\{1,2,3\} \\ &= 3. \end{aligned}$$

Now, $f^{-} : V(W_n) \to L$

$$\begin{aligned} f^{-}(u_i) &= \bigwedge_{u \in V} f(uu_i) \\ &= f(u_{i-1} u_i) \wedge f(u_i u_{i+1}) \wedge f(uu_i) \\ &= 3 \wedge 1 \wedge 2 \\ &= glb\{1,2,3\} \\ &= 1, \; 2 \le i \le n-1. \end{aligned}$$

$$\begin{aligned} f^{-}(u_1) &= f(u_n u_1) \wedge f(u_1 u_2) \wedge f(uu_1) \\ &= 1 \wedge 3 \wedge 1 \\ &= glb\{1,3\} = 1 \end{aligned}$$

$$\begin{aligned} f^{-}(u_n) &= f(u_{n-1} u_n) \wedge f(u_n u_1) \wedge f(uu_n) \\ &= 3 \wedge 1 \wedge 3 \\ &= glb\{1,3\} = 1 \end{aligned}$$

$$\begin{aligned} f^{-}(u) &= \bigwedge f(uu_i) \\ &= f(uu_1) \wedge f(uu_2) \wedge \ldots \wedge f(uu_n) \\ &= 1 \wedge 2 \wedge \ldots \wedge 3 \\ &= glb\{1,2,3\} = 1. \end{aligned}$$

Hence $f^{+}(v)$ and $f^{-}(v)$ are constant for all $v \in V(W_n)$

$f^{+}(v) = 3$ which is the least upper bound of L and $f^{-}(v) = 1$ which is the greatest lower bound of L for all $v \in V(W_n)$.

Hence, W_n is L-magic for $n \ge 3$.

Example 3.4 The L - magic labeling of W_5 and W_8 are shown below.

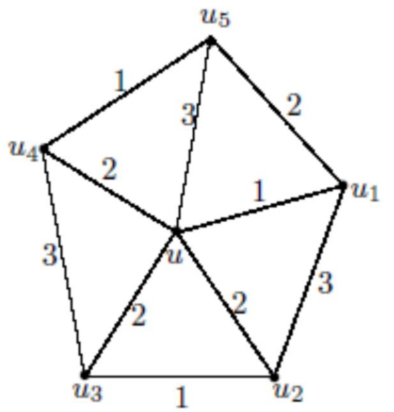

Fig. 2 L -magic labeling of W_5

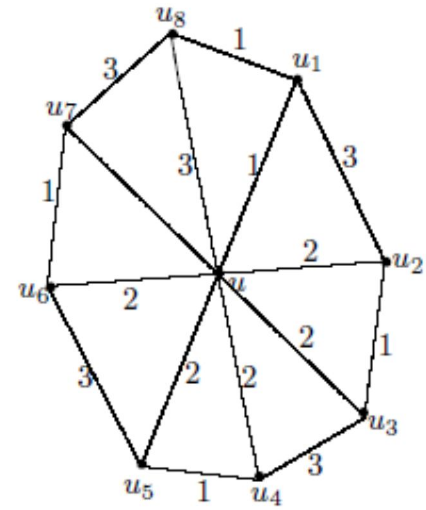

Fig. 3 L -magic labeling of W_8

Observation 3.5 In a similar way of labeling $f(vu_i)$ as that of $f(uu_i)$ in both the cases $1 \le i \le n$ we can prove the graph double cone is also L - magic.

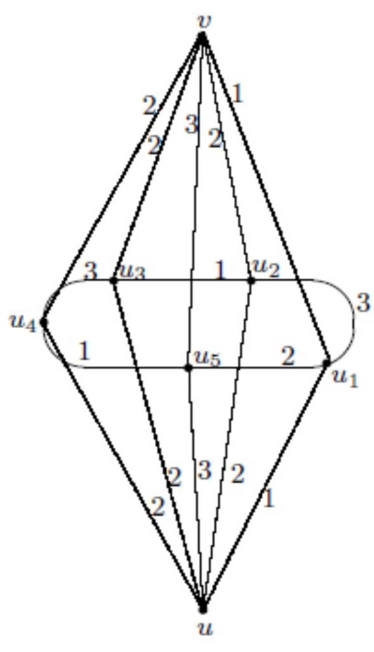

Fig. 4 L -magic labeling of DC_5

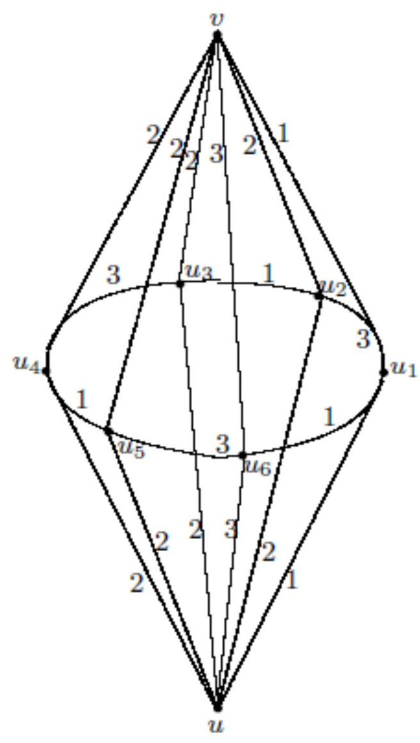

Fig. 5 L -magic labeling of DC_6

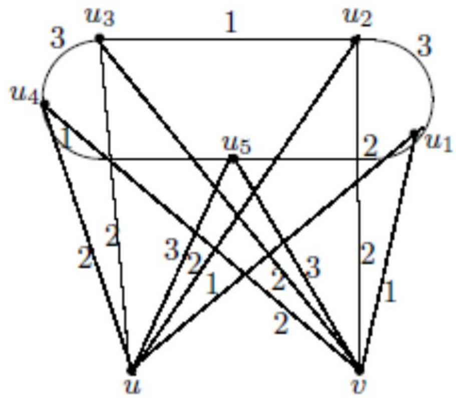

Fig. 6 L -magic labeling of DC_5

Theorem 3.6

F_n is L -magic for $n \geq 3$.

Proof. Let $V(F_n) = \{v_i / 1 \leq i \leq n\} \cup \{u\}$ and $E(F_n) = \{uv_i / 1 \leq i \leq n\} \cup \{v_i v_{i+1} / 1 \leq i \leq n-1\}$

case 1 : n be even

Let $f : E(F(n) \to L$ be defined as

$$f(v_{2i-1}v_{2i}) = 1,\ 1 \leq i \leq n/2$$

$$f(v_{2i}v_{2i+1}) = 3,\ 1 \leq i \leq n/2 - 1$$

$$f(uv_i) = 3,\ 2 \leq i \leq n-2 \text{ and } f(uv_{n-1}) = 1$$

$$f(uv_1) = f(uv_n) = 3$$

Let $f^+ : V(F_n) \to L$

$$\begin{aligned} f^+(u) &= \bigvee f(uv_i) \\ &= f(uv_1) \vee f(uv_2) \vee ... \vee f(uv_{n-1}) \vee f(uv_n) \\ &= 3 \vee 2 \vee ... \vee 1 \vee 3 \\ &= lub\{1,2,3\} = 3 \end{aligned}$$

$$\begin{aligned} f^+(v_i) &= \bigvee f(uv_i) \\ &= f(v_{i-1}v_i) \vee f(v_i v_{i+1}) \vee f(uv_i),\ 2 \leq i \leq n-2 \\ &= 1 \vee 3 \vee 2 \\ &= lub\{1,2,3\} \\ &= 3,\ 2 \leq i \leq n-2. \end{aligned}$$

$$\begin{aligned} f^+(v_{n-1}) &= f(v_{n-2}v_{n-1}) \vee f(v_{n-1}v_n) \vee f(uv_{n-1}) \\ &= 3 \vee 1 \vee 1 \\ &= lub\{1,3\} = 3 \end{aligned}$$

$$\begin{aligned} f^+(v_1) &= f(uv_1) \vee f(v_1 v_2) \\ &= 3 \vee 1 = lub\{1,3\} = 3 \end{aligned}$$

$$\begin{aligned} f^{+}(v_n) &= f(uv_n) \vee f(v_{n-1}v_n) \\ &= 3 \vee 1 \\ &= lub\{1,3\} = 3 \end{aligned}$$

Let $f^{-}: V(F_n) \to L$

$$\begin{aligned} f^{-}(u) &= \bigwedge_{vi \in V} f(uv_i) \\ &= f(uv_1) \wedge f(uv_2) \wedge f(uv_3) \ldots \wedge f(uv_{n-1}) \wedge f(uv_n) \\ &= 3 \wedge 2 \wedge \ldots \wedge 1 \wedge 3 \\ &= glb\{1,2,3\} = 1 \end{aligned}$$

$$\begin{aligned} f^{-}(v_i) &= f(v_{i-1}v_i) \wedge f(v_i v_{i+1}) \wedge f(uv_i) \quad 2 \le i \le n-2 \\ &= 1 \wedge 3 \wedge 2 \\ &= glb\{1,2,3\} = 1 \quad 2 \le i \le n-2 \end{aligned}$$

$$\begin{aligned} f^{-}(v_{n-1}) &= f(v_{n-2}v_{n-1}) \wedge f(v_{n-1}v_n) \wedge f(v_{n-1}v_n) \wedge f(uv_{n-1}) \\ &= 3 \wedge 1 \wedge 1 \\ &= glb\{1,3\} = 1 \end{aligned}$$

$$\begin{aligned} f^{-}(v_1) &= f(uv_1) \wedge f(v_1 v_2) \\ &= 3 \wedge 1 \\ &= glb\{1,3\} = 1 \end{aligned}$$

$$\begin{aligned} f^{-}(v_n) &= f(uv_n) \wedge f(v_{n-1}v_n) \\ &= 3 \wedge 1 \\ &= glb\{1,3\} = 1. \end{aligned}$$

case 2 : Let n be odd

Let $f : E(F_n) \to L$ be defined as

$$f(v_{2i-1}v_{2i}) = 1, \ 1 \le i \le \frac{n-1}{2}$$

$$f(v_{2i}v_{2i+1}) = 3, \ 1 \le i \le \frac{n-1}{2}$$

$$f(uv_i) = 2, \ 2 \le i \le n-1$$

$$f(uv_1) = 3$$

$$f(uv_n) = 1$$

Now $f^{+}: V(F_n) \to L$

$$\begin{aligned} f^{+}(u) &= \bigvee_{vi \in V} f(uv_i) \\ &= f(uv_1) \vee f(uv_2) \vee \ldots \vee f(uv_n) \\ &= 3 \vee 2 \vee \ldots \vee 1 \\ &= lub\{1,2,3\} = 3 \end{aligned}$$

$$\begin{aligned} f^{+}(v_i) &= f(v_{i-1}v_i) \vee f(v_i v_{i+1}) \vee f(uv_i) \quad 2 \le i \le n-1 \\ &= 1 \vee 3 \vee 2 \end{aligned}$$

$= lub\{1,2,3\} = 3$

$f^+(v_1) = f(uv_1) \vee f(v_1v_2)$

$= 3 \vee 1$

$= lub\{1,3\} = 3$

$f^+(v_n) = f(uv_n) \vee f(v_{n-1}v_n)$

$= 1 \vee 3$

$= lub\{1,3\} = 3.$

$f^- : V(F_n) \to L$

$f^-(u) = f(uv_1) \wedge f(uv_2) \wedge ... \wedge f(uv_n)$

$= 3 \wedge 2 \wedge ... \wedge 1$

$= glb\{1,2,3\} = 1$

$f^-(v_i) = f(v_{i-1}v_i) \wedge f(v_iv_{i+1}) \wedge f(uv_i)$ $\quad 2 \leq i \leq n-1$

$= 1 \wedge 3 \wedge 2$

$= glb\{1,2,3\} = 1$,

$f^-(v_1) = f(uv_1) \wedge f(v_1v_2)$

$= 3 \wedge 1$

$= glb\{1,3\} = 1$

$f^-(v_n) = f(uv_n) \wedge f(v_{n-1}v_n)$

$= 1 \wedge 3$

$= glb\{1,3\} = 1$

In both the cases $f^+(v) = 3$ and $f^-(v) = 1$ for all $v \in V(F_n)$. Hence, F_n is L -magic, for $n \geq 3$.

Example 3.7.

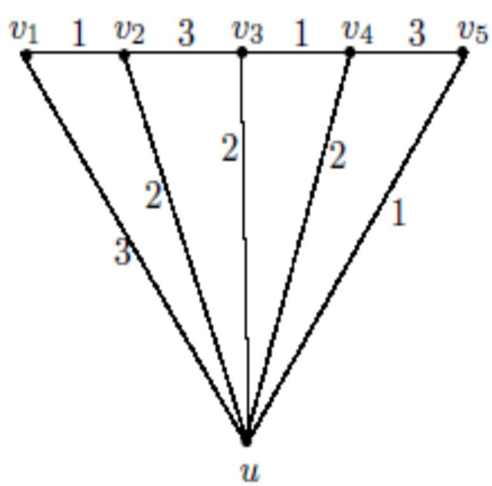

Fig.7 L - magic labeling of F_5

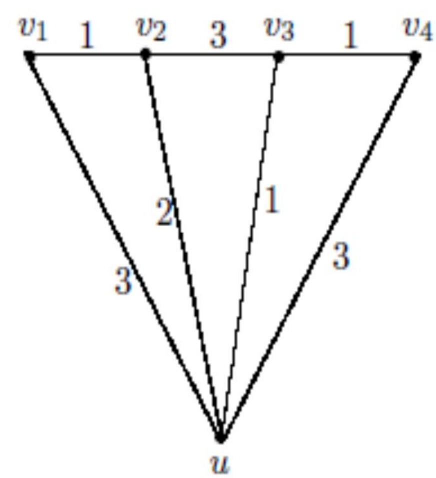

Fig. 8 L - magic labeling of F_4

Theorem 3.8

The graph *(double fan)* DF_n is L - magic $n \geq 3$

Proof. The graph has the following vertex set and Edge set.

$V(DFn) = \{u\} \cup \{v\} \cup \{v_i / 1 \leq i \leq n\}$ and

$E(DFn) = \{uv_i / 1 \leq i \leq m\} \cup \{v_i v_{i+1} / 1 \leq i \leq n-1\} \cup \{vv_i / 1 \leq i \leq n\}$.

The labeling of edges are same like the previous theorem (3.6).
In addition to the labeling of case 1 of theorem (3.6) we add the labeling.

$f(vv_i) = 2, \ 2 \leq i \leq n-2$

$f(vv_{n-1}) = 1,$

$f(vv_1) = f(vv_n) = 3.$

We get $f^+(v), f^-(v)$ are constant for all $v \in V(DFn)$ and in case 2, we have to add, additionally the following label as

$f(vv_i) = 2, \ 2 \leq i \leq n-1$

$f(vv_i) = 3 \ and \ f(vv_n) = 1.$

We can verify $f^+(v), f^-(v)$ are constant for all $v \in V(DFn)$.
Hence, DFn is L - magic.

Example 3.9.

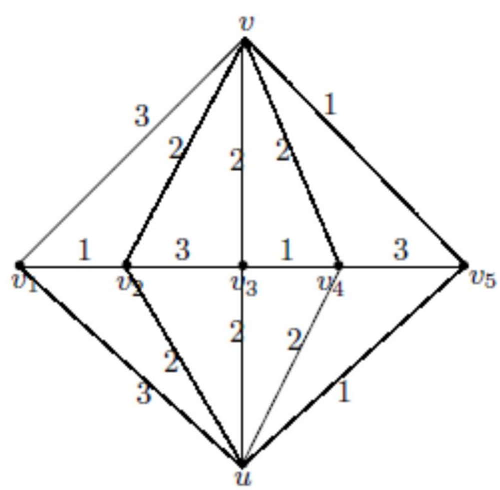

Fig.9 L - magic labeling of DF_5

Observation 3.10

1.The above labeling can be extended to any lattice L of integers , for the same relationship "less than or equal to" we can prove the above graphs are L -magic provided the labeling of the edges having "3 and 1" should be replaced by the greatest and lowest element of L , and the edges having the label "2" may be labeled by any number in between the glb and lub of L .
2.Similarly, a suitable edge labeling is possible for any lattice, according to the relationship defined in the poset, the glb and the lub are found.
For example, if $L = (1,2,3,4,6,12)$ and the relation is "divisibility" then $a \vee b$ and $a \wedge b$ are defined as $a \vee b$ = least

common multiplie of {a,b} and $a \wedge b$ = greatest common divisor of $\{a,b\}$, then it can be easily checked that the above graphs are L -magic.

Theorem 3.11

The graph having pendant edge(s) can not be L -magic.

Proof. While we are labeling the edges of the graph, the pendant vertex (vertices) will get only one labeling as one edge incident to it (them) So for any pendant vertex v, $f^+(v)$ and $f^-(v)$ are the same constant. It can be either lub of the lattice or glb of the lattice but for other vertices of the graph the vertex labelings f^+, f^- give two different constants namely lub of the lattice and glb of the lattice.
Hence, the graph having pendant edge(s) can not be L -magic.

Example 3.12 We show P_4 and H_5 are not L -magic.

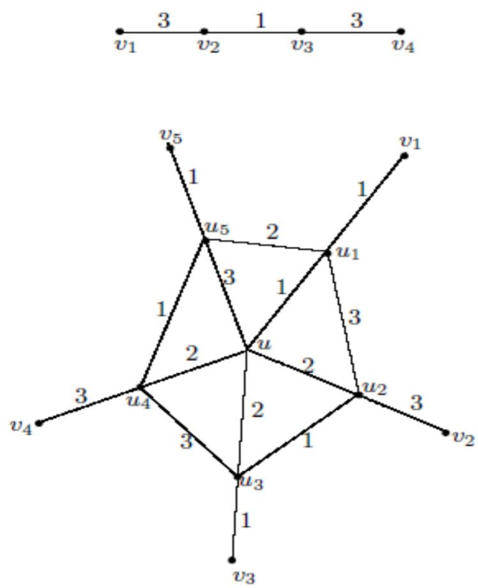

Fig. 10

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