GRACEFULNESS OF CYCLE OF CYCLES AND CYCLE OF COMPLETE BIPARTITE GRAPHS

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Abstract

In this paper we prove that cycle of cycles $C(C_{n_1}, C_{n_2}, \ldots, C_{n_t})$ is graceful, when $t \equiv 0 \pmod 2$, $n_i \equiv 0 \pmod 4$ $(1 \le i \le t)$ and $\sum_{i=1}^{\frac{t}{2}} n_i = \sum_{i=\frac{t}{2}}^{t} n_i$. We also prove that cycle of the complete bipartite graph $C(tK_{m,n})$ (t is an even integer) is graceful and $C(tC_n)$ is cordial, $\forall\ t, n \in N - \{1, 2\}$.

Key words : Graceful labeling, Cordial labeling, Cycle of cycles and cycle of complete bipartite graphs.

AMS **subject classification number :** 05C78.

1 Introduction :

Let $G = (V, E)$ be a simple, undirected graph of size (p, q). i.e. $|V| = p$ and $|E| = q$. For all terminology and notations we follows Harary (Harary 1972). We will give brief summary of definitions which are useful for this paper.

Definition−1.1: A function f is called *graceful labeling* of a graph $G = (V, E)$ if $f : V \longrightarrow \{0, 1, \ldots, q\}$ is injective and the induce function $f^\star : E \longrightarrow \{1, 2, \ldots, q\}$ defined as $f^\star(e) = |f(u) - f(v)|$ is bijective for every edge $e = (u, v) \in E$. A graph G is called *graceful graph* if it admits a graceful labeling.

Definition−1.2: A function $f : V \rightarrow \{0, 1\}$ is called *binary vertex labeling* of a graph G and $f(v)$ is called *label of the vertex* v of G under f.

For an edge $e = (u, v)$, the induced function $f^* : E \rightarrow \{0, 1\}$ defined as $f^*(e) = |f(u) - f(v)|$. Let $v_f(0)$, $v_f(1)$ be number of vertices of G having labels 0 and 1 respectively under f and let $e_f(0)$, $e_f(1)$ be number of edges of G having labels 0 and 1 respectively under f^*.

A binary vertex labeling f of a graph G is called *cordial labeling* if $|v_f(0) - v_f(1)| \leq 1$ and $|e_f(0) - e_f(1)| \leq 1$. A graph which admits cordial labeling is called *cordial graph.*

For detail survey of graph labeling refer Gallian (Gallian 2013) in which vast amount of literature is available on different type of labeling. The graceful labeling was introduced by Rosa (Rosa 1967, p. 349 − 355).

Kaneria, Makadia and Jariya (Kaneria et al. 2014) introduced cycle of graphs $C(G_1, G_2, \ldots, G_n)$ replacing each vertices of C_n by connected graphs $G_1, G_2, \ldots, G_n$. They have proved that $C(tC_n)$, the cycle graph with t copies of C_n and $C(tK_{n,n})$, the cycle graph with t copies of symmetric complete bipartite graph $K_{n,n}$ are graceful.

Definition−1.3: For a cycle C_n, each vertices of C_n is replace by connected graphs $G_1, G_2, \ldots, G_n$ is known as *cycle of graphs* and we shall denote it by $C(G_1, G_2, \ldots, G_n)$. If we replace each vertices by a graph G i.e. $G_1 = G$, $G_2 = G$, $\ldots$ $G_n = G$, such cycle of a graph G, we shall denote it by $C(nG)$.

In present paper we have proved that cycle of cycles $C(C_{n_1}, C_{n_2}, \ldots, C_{n_t})$ and cycle of t copies of the complete bipartite graphs $C(tK_{m,n})$ are graceful graphs. We also proved that cycle of cycles $C(tC_n)$ is cordial.

2 Main Results :

Theorem−2.1 : Cycle of cycles $C(C_{n_1}, C_{n_2}, \ldots, C_{n_t})$ is graceful, when $t \equiv 0 \pmod 2$, $n_i \equiv 0 \pmod 4$ $(1 \leq i \leq t)$ and $\sum_{i=1}^{\frac{t}{2}} n_i = \sum_{i=\frac{t}{2}}^{t} n_i$.

Proof : Let G be the graph $C(C_{n_1}, C_{n_2}, \ldots, C_{n_t})$ with $t \equiv 0 \pmod 2$, $n_i \equiv 0 \pmod 4$ $(1 \leq i \leq t)$ and $\sum_{i=1}^{\frac{t}{2}} n_i = \sum_{i=\frac{t}{2}}^{t} n_i$. Let $u_{i,j}$ $(1 \leq j \leq n_i)$ be the vertices of the cycle C_{n_i}, $\forall\ i = 1, 2, \ldots, t$. We join u_{i,n_i} with $u_{i+1,1}$, $\forall\ i = 1, 2, \ldots, t-1$ and u_{t,n_t} with $u_{1,1}$ to form the cycle graph $C(C_{n_1}, C_{n_2}, \ldots, C_{n_t})$.

Since $n_i \equiv 0 \pmod 4$, $\forall\ i = 1, 2, \ldots, t$, we know that each C_{n_i} 's are graceful graphs with graceful labelings $f_i : V(C_{n_i}) \longrightarrow \{0, 1, \ldots, n_i\}$ $(1 \leq i \leq t)$ defined as follows:

$$\begin{aligned}
f_i(u_{i,j}) &= \tfrac{j-1}{2}, && \forall\ j = 1, 3, \ldots, n_i - 1, \text{ when } i \leq \tfrac{t}{2} \text{ or} \\
& && \forall\ j = 1, 3, \ldots, \tfrac{n_i}{2} - 1, \text{when } i > \tfrac{t}{2} \\
&= \tfrac{j+1}{2}, && \forall\ j = \tfrac{n_i}{2} + 1, \tfrac{n_i}{2} + 3, \ldots, n_i - 1, \text{ when } i > \tfrac{t}{2} \\
&= n_i - \tfrac{j-2}{2}, && \forall\ j = 2, 4, \ldots, \tfrac{n_i}{2}, \text{ when } i \leq \tfrac{t}{2} \text{ or} \\
& && \forall\ j = 2, 4, \ldots, n_i, \text{when } i > \tfrac{t}{2} \\
&= n_i - \tfrac{j}{2}, && \forall\ j = \tfrac{n_i}{2} + 2, \tfrac{n_i}{2} + 4, \ldots, n_i, \text{ when } i \leq \tfrac{t}{2}
\end{aligned}$$

$$\begin{aligned}
\text{i.e. } f_i(u_{i,j}) &= \tfrac{j-1}{2}, && \text{when } j = 1, 3, \ldots, n_i - 1 \\
&= n_i - \tfrac{j-2}{2}, && \text{when } j = 2, 4, \ldots, \tfrac{n_i}{2} \\
&= n_i - \tfrac{j}{2}, && \text{when } j = \tfrac{n_i}{2} + 2, \tfrac{n_i}{2} + 4, \ldots, n_i; \\
& \forall\ i = 1, 2, \ldots, \tfrac{t}{2} && \text{and}
\end{aligned}$$

$$\begin{aligned}
f_i(u_{i,j}) &= \tfrac{j-1}{2}, && \text{when } j = 1, 3, \ldots, \tfrac{n_i}{2} - 1 \\
&= \tfrac{j+1}{2}, && \text{when } j = \tfrac{n_i}{2} + 1, \tfrac{n_i}{2} + 3, \ldots, n_i - 1 \\
&= n_i - \tfrac{j-2}{2}, && \text{when } j = 2, 4, \ldots, n_i; \\
& \forall\ i = \tfrac{t}{2} + 1, \tfrac{t}{2} + 2, \ldots, t.
\end{aligned}$$

Let q is the number of edges for the cycle of cycles $C(C_{n_1}, C_{n_2}, \ldots, C_{n_t})$. Note that $q = \sum_{i=1}^{t} (n_i + 1)$. Take $l_1 = q - n_1$, $l_1' = 0$,

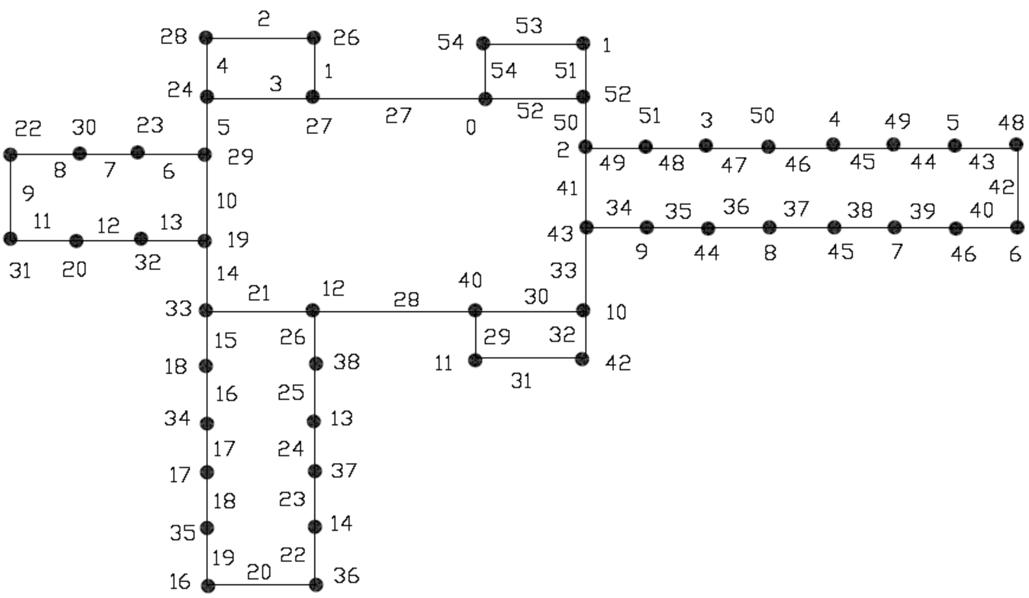
$$\begin{aligned}
l_i &= q - \tfrac{1}{2}\textstyle\sum_{s=1}^{i-1} n_s - n_i - (i-1), && \forall\ i = 2, 3, \ldots, \tfrac{t}{2} \\
l_i' &= \textstyle\sum_{s=1}^{i-1} \tfrac{n_s}{2}, && \forall\ i = 2, 3, \ldots, \tfrac{t}{2} \\
l_j &= q - \tfrac{1}{2}\textstyle\sum_{s=1}^{j-1} n_s - n_j - (\tfrac{t}{2} + 1), && \forall\ i = \tfrac{t}{2} + 1, \tfrac{t}{2} + 2, \ldots, t \\
l_j' &= \textstyle\sum_{s=1}^{j-1} \tfrac{n_s}{2} + (j - \tfrac{t}{2} - 1), && \forall\ i = \tfrac{t}{2} + 1, \tfrac{t}{2} + 2, \ldots, t.
\end{aligned}$$

We define the labeling function $f : V(G) \longrightarrow \{0, 1, \ldots, q\}$, where $q = \sum_{i=1}^{t} (n_i + 1)$ for the graph $G = C(C_{n_1}, C_{n_2}, \ldots, C_{n_t})$ as follows:

$$f(u_{i,j}) = f_i(u_{i,j}) + l_i', \qquad \text{when } j \equiv 1 \pmod 2$$
$$= f_i(u_{i,j}) + l_i, \qquad \text{when } j \equiv 0 \pmod 2, \qquad \forall\, i = 1, 2, \ldots, t.$$

Above labeling pattern give rise graceful labeling to the graph G.

Illustration-2.2: $C(C_4, C_{16}, C_4, C_{12}, C_8, C_4)$ and its graceful labeling shown in following figure.



Theorem-2.3 Cycle of complete bipartite graphs $C(tK_{m,n})$, $t \equiv 0 \pmod 2$, $m, n \in N$ is graceful.

Proof : Let $G = C(tK_{m,n})$ be a graph which contains t copies of the complete bipartite graph $K_{m,n}$, where $t \equiv 0 \pmod 2$ and $m, n \in N$. Let $u_{i,j}$ $(1 \leq j \leq m)$ and $w_{i,j}$ $(1 \leq j \leq n)$ be vertices of i^{th} copy of $K_{m,n}$, $\forall\, i = 1, 2, \ldots, t$. Now join $w_{i,n}$ with $u_{i+1,1}$ $(1 \leq i \leq \frac{t}{2} - 1)$, join $w_{\frac{t}{2},n}$ with $w_{\frac{t}{2}+1,1}$, join $u_{i,m}$ with $w_{i+1,1}$ $(\frac{t}{2} + 1 \leq i \leq t - 1)$ and join $u_{t,m}$ with $u_{1,1}$ by an edge to form cycle of graphs $G = C_t(K_{m,n})$.

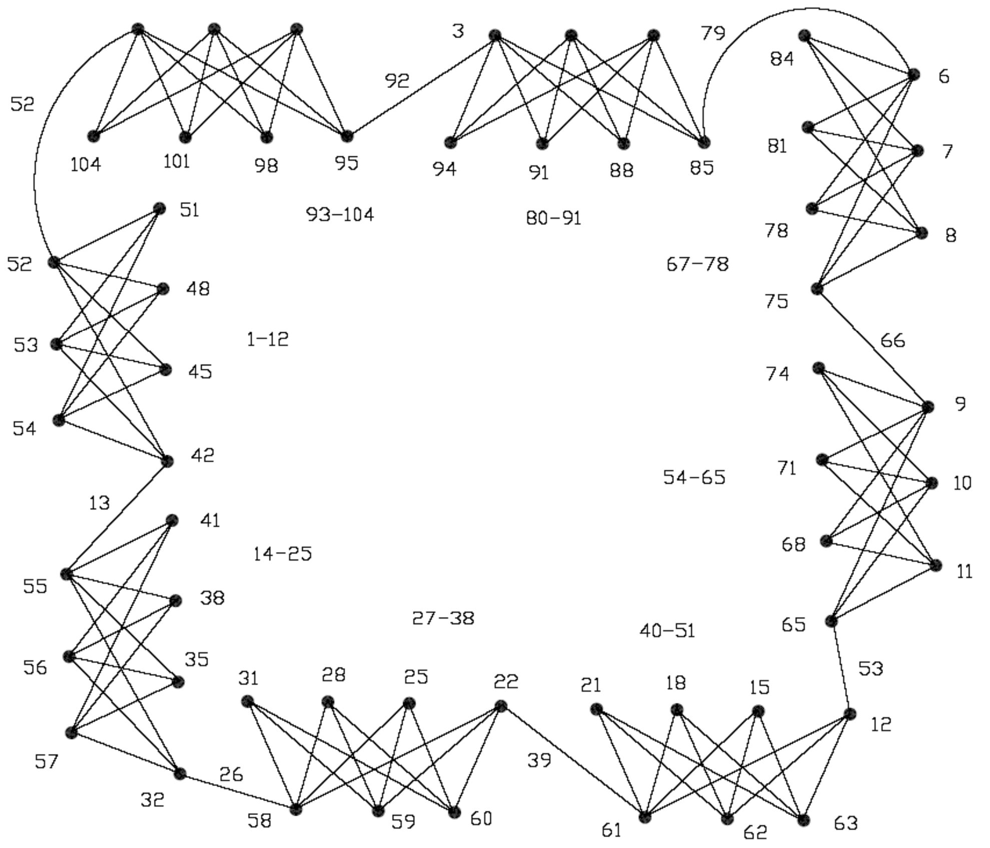
We define the labeling function $f : V(G) \longrightarrow \{0, 1, \ldots, q\}$, where $q = (mn + 1)t$ as follows:

$$f(u_{1,j}) = j - 1, \qquad \forall\, j = 1, 2, \ldots, m$$
$$f(w_{1,j}) = q - (j - 1)m, \qquad \forall\, j = 1, 2, \ldots, n$$

$$f(u_{l,j}) = f(u_{l-1,m}) + j, \qquad \forall\ j = 1, 2, \ldots, m,\ \forall\ l = 2, 3, \ldots, \tfrac{t}{2}$$
$$f(w_{l,j}) = f(w_{l-1,n}) - 1 - (j-1)m, \qquad \forall\ j = 1, 2, \ldots, n,\ \forall\ l = 2, 3, \ldots, \tfrac{t}{2}$$
$$f(u_{\frac{t}{2}+1,j}) = f(w_{\frac{t}{2},n}) - (1 + j), \qquad \forall\ j = 1, 2, \ldots, m$$
$$f(w_{\frac{t}{2}+1,j}) = f(u_{\frac{t}{2},m}) + 1 + (j-1)m, \qquad \forall\ j = 1, 2, \ldots, n$$
$$f(u_{l,j}) = f(u_{l-1,m}) - j, \qquad \forall\ j = 1, 2, \ldots, m,\ l = \tfrac{t}{2} + 2, \tfrac{t}{2} + 3, \ldots, t$$
$$f(w_{l,j}) = f(w_{l-1,n}) + 1 + (j-1)m), \qquad \forall\ j = 1, 2, \ldots, n,\ l = \tfrac{t}{2} + 2, \tfrac{t}{2} + 3, \ldots, t.$$

Above labeling pattern give rise graceful labeling to the graph $G = C(tK_{m,n})$.

Illustration−2.4: $C(8K_{3,4})$ and its graceful labeling shown in following figure.



Theorem−2.5 $C(tC_n)$ is cordial, $\forall\ t, n \in N - \{1, 2\}$.

Proof : Let G be the cycle of cycle $C(tC_n)$. Let $u_{i,j}$ $(1 \leq j \leq n)$ be vertices of i^{th} copy of the cycle C_n, $\forall\ i = 1, 2, \ldots, t$. Now join $u_{i,n}$ with $u_{i+1,1}$ $(1 \leq i \leq t-1)$ and join $u_{t,n}$ with $u_{1,1}$ by an edge to form the cycle of cycle $C(tC_n)$.

To define the labeling function $f : V(G) \longrightarrow \{0, 1\}$, we consider following four cases.

Case–I : $n \equiv 3 \pmod 4$.

$$
\begin{aligned}
f(u_{i,j}) &= 0, && \text{when } j \equiv 1,2 \pmod 4 \text{ and } i \equiv 1 \pmod 2 \text{ or} \\
& && \quad j \equiv 0,3 \pmod 4 \text{ and } i \equiv 0 \pmod 2 \\
&= 1, && \text{when } j \equiv 0,3 \pmod 4 \text{ and } i \equiv 1 \pmod 2 \text{ or} \\
& && \quad j \equiv 1,2 \pmod 4 \text{ and } i \equiv 0 \pmod 2; \\
& \forall\, j = 1, 2, \ldots, n,\ \forall\, i = 1, 2, \ldots, 2 \cdot \lfloor \tfrac{t}{2} \rfloor, \\
f(u_{t,n}) &= 0, && \text{when } t \text{ is odd} \\
f(u_{t,j}) &= 0, && \text{when } j \equiv 0,1 \pmod 4 \text{ and } t \text{ is odd} \\
&= 1, && \text{when } j \equiv 2,3 \pmod 4 \text{ and } t \text{ is odd} \\
& \forall\, j = 1, 2, \ldots, n-1.
\end{aligned}
$$

Case–II : $n \equiv 0 \pmod 4$.

$$
\begin{aligned}
f(u_{i,j}) &= 0, && \text{when } j \equiv 1,2 \pmod 4 \text{ and } i \equiv 0,1 \pmod 4 \text{ or} \\
& && \quad j \equiv 0,3 \pmod 4 \text{ and } i \equiv 2,3 \pmod 4 \\
&= 1, && \text{when } j \equiv 0,3 \pmod 4 \text{ and } i \equiv 0,1 \pmod 4 \text{ or} \\
& && \quad j \equiv 1,2 \pmod 4 \text{ and } i \equiv 2,3 \pmod 4; \\
& \forall\, j = 1, 2, \ldots, n,\ \forall\, i = 1, 2, \ldots, t.
\end{aligned}
$$

Case–III : $n \equiv 1 \pmod 4$.

$$
\begin{aligned}
f(u_{i,j}) &= 0, && \text{when } j \equiv 0,1 \pmod 4 \text{ and } i \equiv 1 \pmod 2 \text{ or} \\
& && \quad j \equiv 2,3 \pmod 4 \text{ and } i \equiv 0 \pmod 2 \\
&= 1, && \text{when } j \equiv 2,3 \pmod 4 \text{ and } i \equiv 1 \pmod 2 \text{ or} \\
& && \quad j \equiv 0,1 \pmod 4 \text{ and } i \equiv 0 \pmod 2; \\
& \forall\, j = 1, 2, \ldots, n,\ \forall\, i = 1, 2, \ldots, t-1, \\
f(u_{t,1}) &= 0, \\
f(u_{t,j}) &= 0, && \text{when } j \equiv 2,3 \pmod 4 \\
&= 1, && \text{when } j \equiv 0,1 \pmod 4; \\
& \forall\, j = 2, 3, \ldots, n.
\end{aligned}
$$

Case−IV : $n \equiv 2 \pmod 4$.

$$f(u_{i,1}) = 0,\ f(u_{i,2}) = 1,\ \forall\ i = 1, 2, \ldots, t,$$

$$\begin{aligned} f(u_{i,j}) &= 0, && \text{when } j \equiv 1, 2 \pmod 4 \text{ and } i \equiv 1 \pmod 2 \text{ or} \\ &&& j \equiv 0, 1 \pmod 4 \text{ and } i \equiv 0 \pmod 2 \\ &= 1, && \text{when } j \equiv 0, 3 \pmod 4 \text{ and } i \equiv 1 \pmod 2 \text{ or} \\ &&& j \equiv 2, 3 \pmod 4 \text{ and } i \equiv 0 \pmod 2; \end{aligned}$$

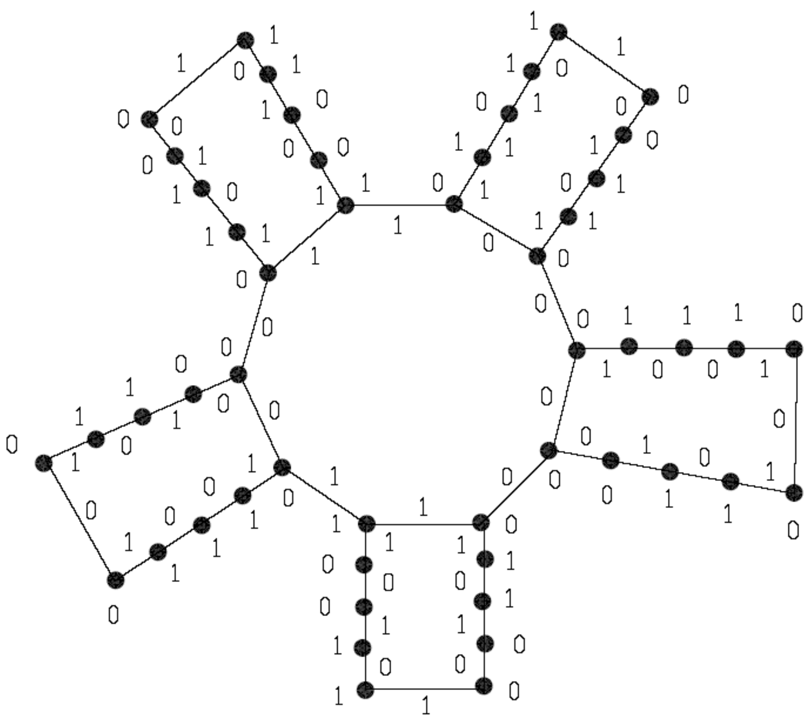
$$\forall\ j = 3, 4, \ldots, n,\ \forall\ i = 1, 2, \ldots, t-1,$$

$$\begin{aligned} f(u_{t,j}) &= 0, && \text{when } j \equiv 0, 1 \pmod 4 \text{ and } t \text{ is even or} \\ &&& j \equiv 2, 3 \pmod 4 \text{ and } t \text{ is odd} \\ &= 1, && \text{when } j \equiv 2, 3 \pmod 4 \text{ and } t \text{ is even or} \\ &&& j \equiv 0, 1 \pmod 4 \text{ and } t \text{ is odd;} \end{aligned}$$

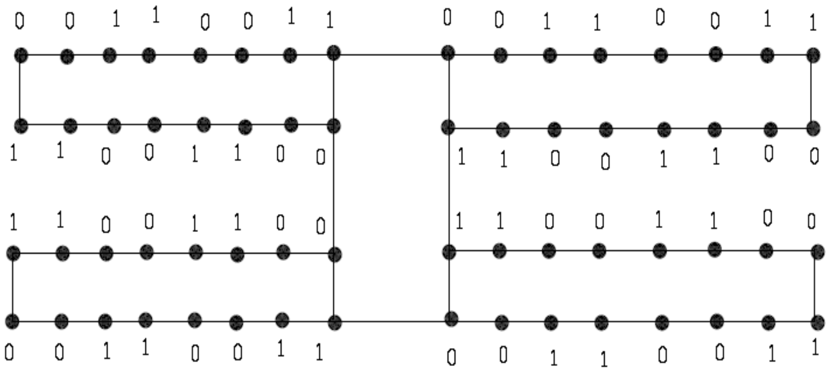
$$\forall\ j = 3, 4, \ldots, n.$$

Above labeling pattern give rise cordial labeling to the given graph G, as it satisfies $|v_f(0) - v_f(1)| \leq 1$ and $|e_f(0) - e_f(1)| \leq 1$ in above cases. Thus $G = C(tC_n)$ is a cordial graph, $\forall\ t, n \in N - \{1, 2\}$.

Illustration−2.6: $C(5C_{10})$ (it is related with case-IV) and its cordial labeling shown in following figure.



Illustration−2.7: $C(4C_{12})$ (it is related with case-II) and its cordial labeling shown in following figure.



3 Concluding Remarks :

We discussed here graceful labeling for cycle of cycles and cycle of complete bipartite graphs. We also discussed cordial labeling for cycle of cycles. The present work contribute three new results. The labeling pattern is illustrated for derived results. Kaneria, Makadia and Jariya (Kaneria et al. 2014) proved $C(tC_n)$ is graceful which is particular case of *Theorem*−2.1 by taking $n_i = n$, $\forall\ i = 1, 2, \dots, t$ and $C(tK_{n,n})$ is graceful which is particular case of *Theorem*−2.3 by taking $m = n$.

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