Numerical Evaluation of Integrals of Analytic Functions

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Abstract: **Some quadrature rules involving values of the integrand and its derivatives at a certain set of nodes in the domain of analyticity of an analytic function have been constructed for the numerical evaluation of the contour integral of the function along a directed line segment. The degree of precision of the quadrature rules has been enhanced by examining the truncation error associated with the quadrature rules.**

Key words: **Analytic functions, quadrature rules, degree of precision.**

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I. INTRODUCTION

The integral for which numerical evaluation has been discussed in this paper is prescribed as

$$I(f) = \int_L f(z)dz \qquad (1)$$

where $f(z)$ is an analytic function in the disk $\Omega = \{z: |z - z_0| \le \rho, \rho > |h|\}$ and L is a directed line segment from the point $z_0 - h$ to the point $z_0 + h$. Several authors viz. Birkhoff and Young [1], Tosic [2], Lether [3], Acharya, Acharya and Nayak [4], Milovanovic [5] etc have considered the problem of numerical evaluation of the integral $I(f)$. The generalized seven point rule formulated by Milovanovic [5] has degree of precision at least seven. Examining the error terms in the generalized rule the degree of precision has been enhanced to eleven. Similarly the rule due to Acharya, Acharya and Nayak [4] requiring nine function values and the values of the derivatives has degree of precision nine.

Our aim in the present paper is to construct an interpolatory quadrature rule meant for the integral $I(f)$ which requires at most nine evaluations of the function and its derivatives having degree of precision at least nine. The particular cases of the rule shall be discussed such that the number of function evaluations is reduced and the degree of precision is enhanced.

II. CONSTRUCTION OF THE GENERAL RULE

Let t, r be two positive real parameters in the interval $(0,1]$. Let S be the set of following points:

$$S = \{z_0, z_0 \pm th, z_0 \pm ith, z_0 \pm rh, z_0 \pm irh\} \qquad (2)$$

where $i = \sqrt{-1}$. Considering S as the set of nodes let the proposed interpolatory quadrature rule meant for the integral $I(f)$ be given by

$$\begin{aligned}\tilde{R}(f;t,r) = {} & C_0 f(z_0) + C_1\{f(z_0 + th) + f(z_0 - th)\} \\ & + C_2\{f(z_0 + ith) + f(z_0 - ith)\} \\ & + C_3 rh\{f'(z_0 + rh) - f'(z_0 - rh)\} \\ & + C_4 irh\{f'(z_0 + irh) - f'(z_0 - irh)\} \qquad (3)\end{aligned}$$

where the coefficient $C_j, j = 0(1)4$ are to be determined. It is noteworthy that the rule $\tilde{R}(f;t,r)$ integrates all odd degree monomials $(z - z_0)^{2n-1}, n \in \mathbb{N}$ exactly. By making the rule $\tilde{R}(f;t,r)$ exact i.e.

$$I(f) = \tilde{R}(f;t,r) \qquad (4)$$

for the even degree monomials $f(z) = (z - z_0)^{2n}, n = 0,1,2,3,4$ we arrive at the following system of equations:

$$\left.\begin{aligned} &C_0 + 2C_1 + 2C_2 = 2h, \\ &(C_1 - C_2)t^2 + 2(C_3 - C_4)r^2 = h/3, \\ &(C_1 + C_2)t^4 + 4(C_3 + C_4)r^4 = h/5, \\ &(C_1 - C_2)t^6 + 6(C_3 - C_4)r^6 = h/7, \\ &(C_1 + C_2)t^8 + 8(C_3 + C_4)r^8 = h/9. \end{aligned}\right\} \quad (5)$$

for which the solutions as follows.

$$\left.\begin{aligned} C_0 &= 2h\left\{1 - \frac{18r^4 - 5}{45t^4(2r^4 - t^4)}\right\}, \\ C_1 &= \frac{h}{2t^2}\left\{\frac{18r^4 - 5}{45t^2(2r^4 - t^4)} + \frac{7r^4 - 1}{7(3r^4 - t^4)}\right\}, \\ C_2 &= \frac{h}{2t^2}\left\{\frac{18r^4 - 5}{45t^2(2r^4 - t^4)} - \frac{7r^4 - 1}{7(3r^4 - t^4)}\right\}, \\ C_3 &= \frac{h}{12r^2}\left\{\frac{5 - 9t^4}{30r^2(2r^4 - t^4)} + \frac{3 - 7t^4}{7(3r^4 - t^4)}\right\}, \\ C_4 &= \frac{h}{12r^2}\left\{\frac{5 - 9t^4}{30r^2(2r^4 - t^4)} - \frac{3 - 7t^4}{7(3r^4 - t^4)}\right\}. \end{aligned}\right\} \quad (6)$$

The following theorem is now evident from equations (1), (3) – (6):

Theorem 1: The rule $\tilde{R}(f;t,r)$ has degree of precision at least nine.

The truncation error associated with the rule $\tilde{R}(f;t,r)$ is given by

$$\tilde{E}(f;t,r) = I(f) - \tilde{R}(f;t,r). \quad (7)$$

Using the Taylor series expansion in ascending powers of $(z-z_0)^n$ for the function $f(z)$ in the disk Ω in equation (7) we have after simplification the following:

$$\tilde{E}(f;t,r) = \gamma a_{10}h^{11} + \delta b_{12}h^{13} + \mathrm{O}(h^{15}) \quad (8)$$

where

$$\left.\begin{aligned} \gamma &= 2\left\{\frac{1}{11} - \frac{3t^8(7r^4 - 1) + 5r^8(3 - 7t^4)}{21(3r^4 - t^4)}\right\}, \\ \delta &= 2\left\{\frac{1}{13} - \frac{t^8(18r^4 - 5) - 3r^8(5 - 9t^4)}{45(2r^4 - t^4)}\right\} \end{aligned}\right\} \quad (9)$$

and a_n's are the Taylor coefficients given by $a_n = f^{(n)}(z_0)/n!$.

III. GENERATION OF PARTICULAR RULES FROM $\tilde{\boldsymbol{R}}(\boldsymbol{f};\boldsymbol{t},\boldsymbol{r})$

It is noteworthy that the rule $\tilde{R}(f;t,r)$ reduces to ninth degree rule due to Acharya, Acharya and Nayak [4] if $t = r$. Further by making $r \to \infty$, the $\tilde{R}(f;t,r)$ reduces to the rules due to Birkhoff and Young [1], Tosic [2] and Lether [3] according as the parameter $t = 1, (3/7)^{0.25}, \sqrt{3/5}$ respectively.

The number of function evaluations in $\tilde{R}(f;t,r)$ can be reduced if

$$C_0 = 0, \quad C_4 = 0 \quad (10)$$

where the coefficients C_0 and C_4 are given in equations (6). Solving the equations (10) the admissible solutions i.e. solutions which lie in the interval $(0,1]$ have been found out and presented in Table-1. The pair of solutions $t_j, r_j, j = 1,2$ yield the following rules.

$$\begin{aligned} Q_j = {} & C_1\{f(z_0 + t_j h) + f(z_0 - t_j h)\} \\ & + C_2\{f(z_0 + it_j h) + f(z_0 - it_j h)\} \\ & + C_3 r_j h\{f'(z_0 + r_j h) - f'(z_0 - r_j h)\} \end{aligned} \quad (11)$$

Table-1

t	r
0.7952800160735923434106 5882542246	0.5913036965139735635138 1477218414
0.4989541049847640133196 7744770451	0.7895420878596878252541 6782809340

It is pertinent to note that the degree of precision of the $\tilde{R}(f;t,r)$ rises from 9 to 11 and from 9 to 13 according as the error coefficient $\gamma = 0$ or $\gamma = \delta = 0$ respectively. The equations $\gamma = 0$ and $C_4 = 0$ are solved simultaneously and the following three pairs of admissible solutions $t_j, r_j, j = 3,4,5$ are obtained and presented in Table-2.

Table-2

t	r
0.904635786593110980242712253055 25	0.37116193561078920821498757469025
0.862190731946721777901385892908 12	0.72379949498675400120748118273143
0.648262853694974999101605268661 83	0.85011219519470214825422117682031

Hence each of the rules Q_j given by

$$Q_j = C_0 f(z_0) + C_1\{f(z_0 + t_j h) + f(z_0 - t_j h)\}$$
$$+C_2\{f(z_0 + it_j h) + f(z_0 - it_j h)\}$$
$$+C_3 r_j h\{f'(z_0 + r_j h) - f'(z_0 - r_j h)\} \qquad (12)$$

has degree of precision 11 and requires 7 function evaluations where the values of t_j and r_j have been prescribed in Table-2.

Finally by solving the equations $\gamma = 0$ and $\delta = 0$ simultaneously we get the admissible solutions t_j and r_j, $j = 6,7,8$ which have been presented in Table-3.

Table-3

t	r
0.920786752920730627040730847122 81	0.44005887469377999833466837554778
0.798321941611901252980031002375 39	0.60022786945797071585005618744525
0.863443973912205475944498343464 40	0.72669236847018413340824109257588

Therefore each of the three rules $Q_j = \tilde{R}(f; t_j, r_j)$, $j = 6,7,8$ is a thirteenth degree rule requiring nine function evaluations.

IV. NUMERICAL EXPERIMENTS AND CONCLUSION

For the purpose of numerical verification the two following integrals I_1 and I_2 are considered.

$$I_1 = \int_{L_1} e^z dz$$
$$= 0.152170648331146 + 1.664009370491679i,$$

$$I_2 = \int_{L_2} sinz\, dz$$
$$= -1.198992981888516 + 2.063000093388935i$$

where L_1 and L_2 are directed line segments from $0.5 - i/2$ to $0.6 + i/2$ and from $1 + i$ to $1 + 2i$ respectively. The integrals I_1 and I_2 have been computed by the rules Q_j, $j = 1(1)8$, Q_{AAN} for $t = r = (3/7)^{0.25}$ and the maximum accuracy derivative free formula Q_M due to Milovanovic [5]. It is noteworthy that the rule Q_{AAN} due to Acharya, Acharya and Nayak [4] has maximum accuracy for $t = r = (3/7)^{0.25}$. It is also noteworthy that the rule Q_{AAN} for $t = r = (3/7)^{0.25}$ can be regarded as a modification of the rule due to Tosic [2]. The computed values of the absolute errors associated with the evaluation of the integrals I_1 and I_2 by all these rules have been appended in Table-4.

Table-4

Rules	$\lvert Error \rvert$ in Computation of I_1	$\lvert Error \rvert$ in Computation of I_2
Q_1	6.471×10^{-12}	8.142×10^{-12}
Q_2	3.173×10^{-12}	4.000×10^{-12}
Q_3	2.950×10^{-15}	3.972×10^{-15}
Q_4	3.401×10^{-15}	2.483×10^{-15}
Q_5	1.724×10^{-15}	2.483×10^{-15}
Q_6	8.496×10^{-16}	9.930×10^{-16}
Q_7	1.923×10^{-14}	2.308×10^{-14}
Q_8	1.223×10^{-14}	1.542×10^{-14}
Q_{AAN}	1.460×10^{-11}	1.831×10^{-11}
Q_M	9.159×10^{-16}	1.897×10^{-15}

The computed values exhibit that the rule Q_6 is slightly more accurate than the formula Q_M due to Milovanovic [5].

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