Some Results on Semi-Compactness

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***Abstract* — We give necessary and sufficient conditions for a semi-regular space to be semi-compact and for a map to be semi-compact preserving (semi-compact) when domain (co-domain) of map is SCS.**

***Keywords* — semi-compact, semi-cluster, semi-regular, semi-convergence.**

I. Introduction

Since the introduction of semi-open sets by Levine [4], various authors have investigated the corresponding concepts of semi-compactness, semi-regularity of SCS spaces (see [2], [3], [4] etc). It is known that a regular space is compact if and only if there exist a dense set D in X such that every net in D has a cluster point in X [7]. In this paper, we give the corresponding result for semi-regular spaces (Theorem 2.1 below). Further we give necessary and sufficient condition for a map $f : X \rightarrow Y$ to be semi-compact preserving (semi-compact) by using the concept of semi-closure when domain (range) of map is SCS (Theorem 2.3 below). A subset A in a topological space X is said to be *semi-open* [4] if and only if $A \subset cl(Int(A))$, or equivalently, if there exists an open subset U of X such that $U \subset A \subset cl(U)$. A is called semi-closed if X-A is semi-open. The semi-closure scl(A) of a subset A of a space X is the intersection of all semi-closed subsets of X that contain A, or equivalently, the smallest semi-closed subset of X that contains A. A space X is called *semi-compact* [2] if any semi-open cover of X has a finite subcover. A space X is said to be *semi-regular* [3] if for each x in X and every semi-open set U containing x there exist semi-open set V containing x such that $V \subset scl(V) \subset U$. A net $\{x_\alpha\}$ is *semi-converges* [2] (*semi-clusters* [1]) at x if and only if $\{x_\alpha\}$is eventually (frequently) in every semi-open set containing x. A space is said to be *SCS* [8] if any subset of X which is semi-compact is semi-closed.

Notation: Throughout this paper, X and Y will denote arbitrary topological spaces. For a subset A of a space X, scl(A) will denote the semi-closure of A.

We will also make use of following results:

Theorem 1.1: **([1])** A space X is semi-compact if and only if every net in X has a semi-cluster point in X.

Theorem 1.2: **(**Theorem 2.4; **[5])** Let X be a topological space. Then D is dense in X if and only if scl(D) = X.

II. Results

We begin with the following definitions.

Definition 2.1. A subset A of X will be called *relatively semi-compact* if scl(A) is semi-compact..

REMARK 2.1. Since semi-closed subsets of semi-compact spaces are semi-compact [6], therefore every subset of semi-compact space is relatively semi-compact

DEFINITION 2.2. A map f : X → Y is called *semi-compact preserving (semi-compact)* if the image (inverse image) of a semi-compact subset of X (Y) is semi-compact in Y (X).

The following characterization of compactness for regular spaces is known.

THEOREM (Ex 201 and 202, Sec 7.2 of [7]). A regular space X is compact if and only if there exists a dense subset D of X such that every net in D has a cluster point in X.
Our first result gives a similar characterization of semi-compactness for semi-regular spaces.

THEOREM 2.1. A semi-regular space X is semi-compact if and only if there exists a dense subset D of X such that every net in D has a semi-cluster point in X.

PROOF: Equivalently, we prove if there exist a dense subset D of X such that every filterbase in D has a semi-cluster point in X then X is semi-compact. Let D be such a dense set. Assume X is not semi-compact, then there exist a cover $\{U_\alpha\}$ of semi-open set in X with no finite subcover. Since X is semi-regular, there exists semi-open cover $\{V_\beta\}$ of X such that for each β there exist α such that $scl(V_\beta) \subset U_\alpha$. By Theorem 1.2 above, since X = scl(D), $\{V_\beta\}$ is a semi-open cover of scl(D) with no finite subcover. Therefore, the collection **B** = $\{D - \cup V_{\beta i}$ i= 1 to n | for any positive integer n$\}$ is a filterbase in D. By assumption, **B** has a semi-cluster point x. Then $x \in scl(D)$ implies $x \in V_\beta$ for some β and so V_β is a semi-open set containing x. Then **(D -** V_β**)** $\cap V_\beta = \varnothing$ contradicts the fact that x is semi-cluster point of **B.** Hence scl(D) = X is semi-compact. The converse follows immediately from Theorem 1.1 above.
Above Theorem motivates the following Definition:

DEFINITION 2.1. A set A in a space X will be called semi-clustering if every net in A has a semi-cluster point in X.

COROLLARY 2.1. In a semi-regular space which is not semi-compact, every dense subset is non semi-clustering.

The following result characterizes relatively semi-compactness for semi-regular spaces which is corollary to the above Theorem 2.1.

THEOREM 2.2. In a semi-regular space X, a set A is relatively semi-compact if and only if A is semi-clustering in X.
For our next result, we make the following Definition.

DEFINITION. A net $\{x_\alpha\}$ in a space X will be called **relatively semi-compact** in X if its range is relatively semi-compact in X.

The following Theorem characterizes semi-compact preserving (semi-compact) maps $f : X \to Y$ in terms of semi-cluster point of relatively semi-compact nets, where X (Y) is assumed to be SCS.

THEOREM 2.3. Let $f : X \to Y$ be a map, where X (Y) is SCS. Then f is semi-compact preserving (semi-compact) if and only if for every relatively semi-compact net $\{x\alpha\}$ in X ($\{f(x\alpha)\}$ in Y) with range $S = \cup\alpha\{x\alpha\}$, ($S = \cup\alpha\{f(x\alpha)\}$), the net $\{f(x\alpha)\}$ has a semi-cluster point in f(scl(S)) (the net $\{x\alpha\}$ has a semi-cluster point in f–1(scl(S)).

PROOF: For arbitrary spaces X and Y, if f is semi-compact preserving (semi-compact) and $\{x_\alpha\}$ ($\{f(x_\alpha)\}$) is a relatively semi-compact net, then $\{f(x_\alpha)\}$ is a net in the semi-compact set f(scl(S)) ($\{x_\alpha\}$ is a net in the semi-compact set f^{-1}(scl(S)) and so has a semi-cluster point in f(scl(S)) (f^{-1}(scl(S)) by Theorem 1.1 above, proving the necessity of the condition. Conversely, let X (Y) be SCS and assume that for every net $\{x_\alpha\}$ in X ($\{f(x_\alpha)\}$ in Y) with relatively semi-compact range S, the net $\{f(x_\alpha)\}$ has a semi-cluster point in f(scl(S)) (the net $\{x_\alpha\}$ has a semi-cluster point in f^{-1}(scl(S)). Let K be a semi-compact subset of X (Y) and $\{f(x_{\alpha}$-)}be any net in f(K) ($\{x_\alpha\}$ be any net in f^{-1}(K)). Then $\{x_\alpha\}$ may be taken to be a net in K ($\{f(x_\alpha)\}$ is a net in K) and since X (Y) is SCS, K is semi-closed and so $\{x_\alpha\}$ ($\{f(x_\alpha)\}$) is a relatively semi-compact net. Therefore, by assumption, the net $\{f(x_\alpha)\}$ has a semi-cluster point in f(K) (the net $\{x_\alpha\}$ has a semi-cluster point in f^{-1}(K)) and so f(K) (f^{-1}(K)) is semi-compact. Hence f is semi-compact preserving (semi-compact).

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