Solvability of Systems of Integral Equations by using the Mixed Monotone Property in Partially Ordered Metric Spaces

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Abstract- The purpose of this work is to present some existence results of solutions for a class of systems of mixed monotone mappings in partially ordered metric spaces. The results of this work are extensions and generalizations of known coupled and tripled fixed point results. The methods of proofs used in this work, show that most of new coupled and tripled fixed point results are merely reformulation of some fixed point results in the literature. As an interesting application of our results, we discuss the existence and uniqueness of solutions for a class of systems of nonlinear integral equations.

$$x_i(t) = f_i\left(t, x_1(t), \dots, x_n(t), \int_a^{\beta_i(t)} g_i\big(t, s, x_1(s), \dots, x_n(s)\big)ds\right).$$

Finally, an example is presented to show the efficiency of our results.

Keywords- System of Integral Equations, Mixed Monotone Property, Partially Ordered Set, Fixed Point.

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1. Introduction

In 2004, Ran and Reurings published a fixed point theorem for contractive type mapping in partially ordered metric spaces [20] and Bhaskar and Lakshmikantham [9] introduced the concepts of coupled fixed point and mixed monotone property for contractive operators and established some interesting coupled fixed point theorems under a weak contractivity.

Definition 1.1. An element $(x, y) \in X \times X$ is called a coupled fixed point of the mapping $f : X \times X \to X$ if $f(x, y) = x$ and $f(y, x) = y$.

Definition 1.2. Let $(X, \preccurlyeq)$ be a partially ordered set and $f: X \times X \to X$. The mapping f is said to has the mixed monotone property if f is monotone nondecreasing in its first argument and is monotone nonincreasing in its second argument, that is, for any $x, y \in X$,

$$x_1, x_2 \in X, x_1 \preccurlyeq x_2 \Rightarrow f(x_1, y) \preccurlyeq f(x_2, y)$$
$$y_1, y_2 \in X, y_1 \preccurlyeq y_2 \Rightarrow f(x, y_1) \succcurlyeq f(x, y_2)$$

Recently, Harjani, López and Sadarangani [12], Amini-Harandi and Emami [5], Harjani and Sadarangani [13], Nieto and Rodŕiguez - Lóopez [17] proved some fixed point theorems for nondecreasing mappings in partially ordered metric spaces.

Lakshmikantham and Ćirić [15] extended the results of Bhaskar and Lakshmikantham for a mixed monotone linear contractive mapping and to generalize the notion of a mixed monotone mapping.

Berinde [7] obtained the coupled fixed point theorems for mixed monotone operators which is essentially different and more natural. Luong and Thuan [16] presented some coupled fixed point theorems for a mixed monotone operator in a complete metric space endowed with a partial order by using altering distance functions.

Also Borcut and Berinde [10], Amini-Harandi [4] introduced the concept of tripled fixed point for nonlinear mappings in partially ordered complete metric spaces and obtained some existence and uniqueness theorems for contractive type mappings. Karapinar and Berinde [14] obtained existence and uniqueness results for quadruple fixed points of operators $F : X^4 \to X$. Very recently, Berzig and Samet [8] introduced the concept of fixed point of n-order for mappings $F : X^n \to X$, where $n \geq 2$ and X is an ordered set endowed with a metric d and proved n-order fixed point theorems. For some other papers devoted to the same or related topics, see [3, 11, 18, 21]. On the other hand, authors illustrated these important results by proving the existence and uniqueness of the solution for a periodic boundary value problem of the form.

$$x'(t) = f\big(t, x(t)\big), \tag{1.1}$$

and the general form of integral equations.

$$x(t) = \int_0^T G(t,s)\,[f(s, u(s)) + \lambda u(s)]ds, \tag{1.2}$$

$$x(t) = h(t) + \int_0^T [K_1(t,s) + K_2(t,s)]\big[f\big(s, x(s)\big) + g\big(s, x(s)\big)\big]ds. \tag{1.3}$$

In this work, we present some existence theorems for solution of a system of mixed monotone mappings in partially ordered metric spaces and the obtained results are extensions and generalizations of known coupled and tripled fixed point results in [4, 10, 15, 20]. Finally, as an application, we investigate the problem of existence of solutions for the system of integral equations.

$$x_i(t) = f_i\left(t, x_1(t), \dots, x_n(t), \int_a^{\beta_i(t)} g_i\big(t, s, x_1(s), \dots, x_n(s)\big)\, ds\right), \tag{1.4}$$

where $\in \mathbb{R}$, f_i , g_i and $\beta_i$satisfy in special conditions.

2. Main Results

In this section we introduce the concept of (A, B) - mixed monotone property for mappings of the form $F : X^n \to X$ and present some existence theorems for solution of a system of mixed monotone mappings in partially ordered metric spaces. We start by introducing some definitions. Here after by the set Ω_n we mean

$\Omega_n = \{(A, B) | A \cup B = \{1, 2, \dots, n\}, A \cap B = \emptyset\}$, for all $n \in \mathbb{N}$.

Definition 2.1. Let $(X, \preccurlyeq)$ be a partially ordered set and $(A, B) \in \Omega_n$. We say that $F : X^n \to X$ has the (A, B) -mixed monotone property if F is nondecreasing in the A argument and nonincreasing in the B argument, i, e.,

If $(x_1, x_2, \dots, x_n)$, $(y_1, y_2, \dots, y_n) \in X^n, x_i \preccurlyeq y_i$ for $i \in A$ and $x_i \succcurlyeq y_i$ for $i \in B$ then we have

$F(x_1, x_2, \dots, x_n) \leq F(y_1, y_2, \dots, y_n)$.

Note that if $A = \{1\}, B = \{2\}, n = 2$, then Definition 2.1 reduces to Definition 1.2.

Definition 2.2. Let $(X, \preccurlyeq)$ be an ordered set and $F : X^n \to X$ be a given mapping having the (A, B) mixed monotone property, an element $(x_1, x_2, \dots, x_n) \in X^n$ is an n-order fixed point of $F : X^n \to X$ if there exist n maps $\emptyset_i : \{1 \dots, n\} \to \{1 \dots, n\}$ such that.

$$\begin{cases} i \in A \begin{cases} \emptyset_i(A) = A \\ \emptyset_i(B) = B \end{cases} \\ i \in B \begin{cases} \emptyset_i(A) \subseteq B \\ \emptyset_i(B) \subseteq A \end{cases} \end{cases} \qquad (2.1)$$

and

$F\left(x_{\emptyset_i(1)}, x_{\emptyset_i(2)}, \dots, x_{\emptyset_i(n)}\right) = x_i$, for all $1 \le i \le n$. Also an element $x \in X$ is called a fixed point of F if $F(x, x, \dots, x) = x$.

Definition 2.3. Let Φ denote all functions $\emptyset : [0, \infty)^n \to [0, \infty)$ which satisfy:

(i) nondecreasing in each argument.

(ii) $\emptyset(x_1 + y_1, \dots, x_n + y_n) \le \emptyset(x_1, \dots, x_n) + \emptyset(y_1, \dots, y_n)$,

(iii) $\lim_{m\to\infty} \emptyset(x_1, m, \dots, x_n, m) = 0 \Leftrightarrow x_{i,m} \to 0$ as $m \to \infty$ for all $1 \le i \le n$.

For example, functions $\emptyset_1(t_1, \dots, t_n) = \sum_{i=1}^{n} t_i$ and $\emptyset_2(t_1, \dots, t_n) = \max_i(t_i)$ satisfy in Definition 2.3.

Definition 2.4. Let(X, d) be a metric space and $\emptyset \in \boldsymbol{\Phi}$. We define a function $D_\emptyset : X^n \times X^n \to [0, \infty)$ by

$$D_\emptyset(X, Y) = D_\emptyset\left((x_1, \dots, x_n), (y_1, \dots, y_n)\right) = \emptyset\left(d(x_1, y_1), \dots, d(x_n, y_n)\right)$$

such that $X = (x_1, \dots, x_n), Y = (y_1, \dots, y_n) \in X^n$.

Proposition 2.1. If (X, d) is a complete metric space and $\emptyset \in \Phi$ then $(X^n, D_\emptyset)$ is a complete metric space.

Proof. The proof is obvious, we omit it. ∎

Definition 2.5. Let $(X, \preccurlyeq)$be a partially ordered set. We say that X has a condition H if X has the following properties:

(i) If (x_n) is a nondecreasing sequence that is convergent to x then $x_n \preccurlyeq x$ for all n,

(ii) If (y_n) is a nonincreasing sequence that is convergent to y then $y_n \succcurlyeq y$ for all n.

Now, we are ready to give our first main result. The following fixed point theorem will help us to do it. Incidentally, our method of proof shows that how we can establish an n-order fixed point result from a fixed point result.

Theorem 2.2. [20] Let $(X, \preccurlyeq)$be a partially ordered set and suppose there is a metric d on X such that (X, d) is a complete metric space. Suppose f is a nondecreasing mapping with

$d(f x, f y) \le kd(x, y)$, for all $x, y \in X$, $x \preccurlyeq y$, where $0 < k < 1$. Also suppose either:

(a) f is continuous or,

(b) If (x_n) is a nondecreasing sequence that is convergent to x then $x_n \preccurlyeq x$ for all n.

If there exists $x_0 \in X$ with $x_0 \preccurlyeq f(x_0)$ then f has a fixed point.

Theorem 2.3. Let $(X, d, \preccurlyeq)$ be a complete partially ordered metric space, $(A, B) \in \Omega_n$ and $F_i : X^n \to X$ $(i = 1, \dots n)$be an operator such that F_i is a (A, B) -mixed monotone property for $i \in A$, F_j is a (B, A)- mixed monotone property for $j \in B$ and

$$d\left(F_i(x_1, \dots, x_n), F_i(y_1, \dots, y_n)\right) \le k\max_i\{d(x_i, y_i)\}, \qquad (2.2)$$

for all $(x_1, \dots, x_n), (y_1, \dots, y_n) \in X^n$ with $x_i \preccurlyeq y_i$ for $i \in A$ and $x_i \succcurlyeq y_i$ for $i \in B$, where $0 < k < 1$. Also suppose either:

(a) F_i is continuous, or

(b) X has condition on H.

If there exists $x_0 = (x_1^0, x_2^0, \dots, x_n^0) \in X^n$ such that

$$x_i^0 \preccurlyeq F_i(x_1^0, \dots, x_n^0) \text{ and } x_j^0 \succcurlyeq \in F_j(x_1^0, \dots, x_n^0), \qquad (2.3)$$

for all $i \in A$ and $j \in B$ then there exist $(x_1^*, x_2^*, \dots, x_n^*) \in X^n$ such that for all $1 \le i \le n$

$F_i(x_1^*, x_2^*, \dots, x_n^*) = x_i^*$.

Proof. We define $G: X^n \to X^n$ by

$$G(x_1, \dots, x_n) = \big(F_1(x_1, \dots, x_n), \dots, F_n(x_1, \dots, x_n)\big) \qquad (2.4)$$

It is straightforward to show that G has a fixed point in X^n if and only if there exist $(x_1^*, x_2^*, \dots, x_n^*) \in X^n$ such that for all $1 \le i \le n$

$F_i(x_1^*, x_2^*, \dots, x_n^*) = x_i^*$.

Consider $D_\emptyset: X^n \times X^n \to [0, \infty)$ which is defined by Definition 2.4, $\emptyset(t_1 \dots, t_n) = \max_i(t_i)$. From Proposition 2.1, $(X^n, D_\emptyset)$ is a complete metric space. Now, we establish a new order " $\preccurlyeq_{(A,B)}$ " on X^n by

$$(x_1 \dots, x_n) \preccurlyeq_{(A,B)} (y_1 \dots, y_n) \Leftrightarrow \begin{cases} x_i \preccurlyeq y_i & i \in A \\ x_i \succcurlyeq y_i & i \in B \end{cases}$$

It is easy to see that $\big(X^n, D_\emptyset, \preccurlyeq_{(A,B)}\big)$ is a complete partially ordered metric space. We show that G is a nondecreasing operator. To do this fix arbitrary $(x_1 \dots, x_n), (y_1 \dots, y_n) \in X^n$ such that $(x_1 \dots, x_n) \preccurlyeq_{(A,B)} (y_1 \dots, y_n)$ then we have

$$\begin{aligned} (x_1 \dots x_n) \preccurlyeq_{(A,B)} (y_1 \dots y_n) &\Rightarrow \begin{cases} x_i \preccurlyeq y_i & i \in A \\ x_i \succcurlyeq y_i & i \in B \end{cases} \\ &\Rightarrow \begin{cases} F_i(x_1 \dots x_n) \preccurlyeq F_i(y_1 \dots y_n) & i \in A \\ F_i(x_1 \dots x_n) \succcurlyeq F_i(y_1 \dots y_n) & i \in B \end{cases} \\ &\Rightarrow G(x_1 \dots x_n) \preccurlyeq G(y_1 \dots y_n) \end{aligned} \qquad (2.6)$$

It follows that G is a nondecreasing operator. Moreover, by (2.2) we have

$$\max_i\{d\big(F_i(x_1 \dots, x_n), F_i(y_1 \dots, y_n)\big)\} \le k \max_i\{d(x_i, y_i)\}. \qquad (2.7)$$

And $D_\emptyset\big(G\,X, G\,Y\big) \le kD_\emptyset\big(X, Y\big)$, for all $X, Y \in X^n, X \preccurlyeq_{(A,B)} Y$. Now, if there exists $X_0 = (x_1^0, x_2^0, \dots, x_n^0) \in X^n$ satisfying (2.3) then from (2.5) and definition of G we get $X_0 \preccurlyeq_{(A,B)} G(X_0)$.

Also, if F_i is continuous, or X has condition H, then we have either (a) or (b) in Theorem 2.2. Since operator G satisfies all the conditions appearing in Theorem 2.2 so, G has a fixed point and the proof is complete. ■

In [17] the authors also considered some additional conditions to ensure the uniqueness of the fixed point.

Theorem 2.4. [17] In addition to hypotheses of Theorem 2.2, suppose that for each $x, y \in X$ there exists $z \in X$ which is comparable to x and y, then f has a unique fixed point.

Similarly, we can prove the following uniqueness theorem for n-order fixed points.

Theorem 2.5. In addition to hypotheses of Theorem 2.3, suppose that for every $X, Y \in \big(X^n, \preccurlyeq_{(A,B)}\big)$ (" $\preccurlyeq_{(A,B)}$ ") is given by (2.5)), there exists $Z \in X^n$ that is comparable to X, Y then F_i has a unique n-order fixed point.

Corollary 2.6. (Theorem 2.1, 2.2 of [9]) Let $(X, \preccurlyeq)$ be a partially ordered set and suppose there exists a metric d on X such that (X, d) is a complete metric space. Let $f: X \times X \to X$ be a mixed monotone property on X such that there exists a $k \in [0, 1)$ with

$$d(f(x, y), f(u, v)) \le \frac{k}{2}[d(x, u) + d(y, v)]$$

For all $x \succcurlyeq u$ and $y \preccurlyeq v$. Also suppose either:

(a) f is continuous or,

(b) X has the following properties:

(i) If (x_n) is a nondecreasing sequence that is convergent to x then $x_n \preccurlyeq x$ for all n,

(ii) If (y_n) is a nonincreasing sequence that is convergent to y then $y_n \succcurlyeq y$ for all n.

If there exist two elements $x_0, y_0 \in X$ with $x_0 \preccurlyeq f(x_0, y_0)$ and $y_0 \succcurlyeq f(y_0, x_0)$ then there exist $x, y \in X$ such that $x = f(x, y)$ and $y = f(y, x)$.

Proof. In the special case $n = 2, F_1(x, y) = f(x, y), F_2(x, y) = f(y, x), A = \{1\}$ and $B = \{2\}$. Using assumptions of corollary there exists a $k \in [0,1)$ with

$$d(f(x,y), f(u,v)) \leq \frac{k}{2}[d(x,u) + d(y,v)]$$

for all $x \succcurlyeq u$ and $y \preccurlyeq v$, so

$$d(F_1(x,y), F_1(u,v)) \leq \frac{k}{2}[d(x,u) + d(y,v)] \leq k \max\{d(x,u), d(y,v)\}$$

for all $x \succcurlyeq u$ and $y \preccurlyeq v$, Also, since

$$d(f(y,x), f(v,u)) \leq \frac{k}{2}[d(u,x) + d(v,y)]$$

for all $x \succcurlyeq u$ and $y \preccurlyeq v$, so

$$d(F_2(x,y), F_2(u,v)) \leq \frac{k}{2}[d(u,x) + d(v,y)] \leq k \max\{d(x,u), d(y,v)\}$$

for all $x \succcurlyeq u$ and $y \preccurlyeq v$ which proves that f verifies the contraction condition (2.2) in Theorem 2.3. Now, the proof follows from Theorem 2.3. ■

The following corollary generalized some main results appearing in [4, 20, 8].

Corollary 2.7. Let $(X, d, \preccurlyeq)$ be a complete partially ordered matric space and $(A, B) \in \Omega_n$. Let $F : X^n \to X$ be a mapping having the (A, B)-mixed monotone property such that

$$d(F(x_1, \dots, x_n), F(y_1, \dots, y_n)) \leq k \max_i \{d(x_i, y_i)\}, \qquad (2.8)$$

for all $(x_1, \dots, x_n), (y_1, \dots, y_n) \in X^n$ with $x_i \preccurlyeq y_i$ for $i \in A$ and $x_i \succcurlyeq y_i$ for $i \in B$, where $0 < k < 1$. Also suppose either:

(a) F is continuous, or

(b) X has condition H.

If there exists $X_0 = (x_1^0, x_2^0, \dots, x_n^0) \in X^n$ such that

$$x_i^0 \preccurlyeq F(x^0_{\emptyset_i(1)}, \dots, x^0_{\emptyset_i(n)}) \text{ and } x_j^0 \succcurlyeq F(x^0_{\emptyset_j(1)}, \dots, x^0_{\emptyset_j(n)}), \qquad (2.9)$$

for all $i \in A$ and $j \in B$ where $\emptyset_i: \{1 \cdots, n\} \to \{1 \cdots, n\}$ satisfy condition (2.1) for all $1 \leq i \leq n$ then F has an n-order fixed point.

Proof. We define $F_i: X^n \to X$ by $F_i(x_1, \dots, x_n) = F(x_{\emptyset_i(1)}, \dots, x_{\emptyset_i(n)})$.

Now, we show that F_i is a (A, B)-mixed monotone mapping for all $i \in A$ and F_i is a (B, A)-mixed monotone mapping for all $i \in B$ To do this fix arbitrary $i_0 \in A$ and $(x_1, \dots, x_n), (y_1, \dots, y_n) \in X^n$ such that $(x_1, \dots, x_n) \preccurlyeq_{(A,B)} (y_1, \dots, y_n)$. Since $i_0 \in A$ so by using (2.1) we have

$$(x_1,\dots,x_n) \preccurlyeq_{(A,B)} (y_1,\dots,y_n) \Rightarrow \begin{cases} x_i \preccurlyeq y_i & i \in A \\ x_i \preccurlyeq y_i & i \in B \end{cases}$$
$$\Rightarrow \begin{cases} \emptyset_{i_0}(x_i) \preccurlyeq \emptyset_{i_0}(y_i) & i \in A \\ \emptyset_{i_0}(x_i) \succcurlyeq \emptyset_{i_0}(y_i) & i \in B \end{cases}$$
$$\Rightarrow F(\emptyset_{i_0}(x_1),\dots,\emptyset_{i_0}(x_n)) \preccurlyeq F(\emptyset_{i_0}(y_1),\dots,\emptyset_{i_0}(y_n))$$
$$\Rightarrow F_{i_0}(x_1,\dots,x_n) \preccurlyeq F_{i_0}(y_1,\dots,y_n)$$

Therefore F_i is a (A,B)-mixed monotone mapping for all $i \in A$. By similar reasoning one can show that F_i is a (B,A) -mixed monotone mapping for all $i \in B$. It is easy to verify that (2.8) and (2.9) imply (2.2) and (2.3), respectively. Then the proof is completed. ■

Agarwal, El-Gebeily and O'Regan in [1] presented the following result for mappings satisfying a φ - contraction in the setting of complete partially ordered metric spaces.

Theorem 2.8. Let $(X,\preccurlyeq)$ be a partially ordered set and suppose there is a metric d on X such that (X,d) is a complete metric space. Suppose f is a nondecreasing mapping with

$d(f\,x,\ f\,y) \leq \varphi(d(x,y))$, for all $x,y \in X$, $x \preccurlyeq y$, where $\varphi : [0,\infty) \to [0,\infty)$ is a nondecreasing and continious function such that $\varphi(t) < t$ for all $t > 0$ and $\varphi(0) = 0$. Also, suppose either:

(a) f is continuous or,

(b) If (x_n) is a nondecreasing sequence that is convergent to x then $x_n \preccurlyeq x$ for all n.

If there exists $x_0 \in X$ with $x_0 \preccurlyeq f(x_0)$ then f has a fixed point.

Besides, if for each $x,y \in X$ there exists $z \in X$ which is comparable to x and y, then f has a unique fixed point.

Now, we present an n-order fixed point result for mappings satisfying a φ-contraction in the setting of complete partially ordered metric spaces.

Theorem 2.9. Let $(X,d,\preccurlyeq)$ be a complete partially ordered metric space, $(A,B) \in \Omega_n$ and $F_i : X^n \to X$ $(i = 1,\cdots,n)$ be an operator such that $F_i$is a (A,B)-mixed monotone property for $i \in A$, F_j is a (B,A)- mixed monotone property for $j \in B$ and

$$d\big(F_i(x_1,\dots,x_n),F_i(y_1,\dots,y_n)\big) \leq \varphi\left(\max_i\{d(x_i,y_i)\}\right), \tag{2.10}$$

for all $(x_1,\dots,x_n),(y_1,\dots,y_n) \in X^n$ with $x_i \preccurlyeq y_i$ for $j \in A$ and $x_i \succcurlyeq y_i$ for $i \in B$, where $\varphi : [0,\infty) \to [0,\infty)$ is a nondecreasing and continious function such that $\varphi(t) < t$ for all $t > 0$ *and* $\varphi(0) = 0$. Also, suppose either:

(a) $F_i$is continuous, or

(b) X has condition H.

If there exists $X_0 = (x_1^0,x_2^0,\cdots,x_n^0) \in X^n$ such that

$$x_i^0 \preccurlyeq F_i(x_1^0,\cdots,x_n^0) \text{ and } x_j^0 \succcurlyeq F_j(x_1^0,\cdots,x_n^0), \tag{2.11}$$

for all $i \in A$ and $j \in B$ then there exist $(x_1^*,x_2^*,\cdots,x_n^*) \in X^n$ such that for all $1 \leq i \leq n$

$$F_i(x_1^*,x_2^*,\cdots,x_n^*) = x_i^*.$$

Besides, if for each $x,y \in X$ there exists $z_1,z_2 \in X$ such that $z_1 \preccurlyeq x \preccurlyeq z_2$ and $z_1 \preccurlyeq y \preccurlyeq z_2$, then $(x_1^*,x_2^*,\cdots,x_n^*)$ is uniqueness.

Proof. Define $G : X^n \to X^n$ as in the proof Theorem 2.3 by which is given by (2.4). We know that G has a fixed point in X^n if and only if there exist $(x_1^*,x_2^*,\cdots,x_n^*) \in X^n$ such that for all $1 \leq i \leq n$

$$F_i(x_1^*,x_2^*,\cdots,x_n^*) = x_i^*.$$

Consider $D_{\emptyset}: X^n \times X^n \to [0,\infty)$ which is defined by Definition 2.4, $\emptyset(t_1, \dots, t_n) = \max_i(t_i)$ and $\preccurlyeq_{(A,B)}$ by (2.5). From the assumptions we get

$$D_{\emptyset}(G\,X, G\,Y) \le \varphi\big(D_{\emptyset}(X,Y)\big).$$

Using Theorem 2.8 and in a manner similar to the proof of Theorem 2.3, we can prove that G has a fixed point. Also for the uniqueness of n-order fixed point the the proof is similar to the proof of Theorem 2.5. ■

In [2] Aghajani et al. proved that for every nondecreasing and upper semicontinuous function $\varphi: \mathbb{R}_+ \to \mathbb{R}_+$ the following two conditions are equivalent:

(I) $\varphi(t) < t$, for any $t > 0$.

(II) $\lim_{n\to\infty} \varphi^n(t) = 0$, for any $t > 0$.

Thus Theorem 2.9 is true if (I) is replaced by (II). Furthermore, notice that if "$\frac{\sum_{i=1}^n d(x_i,y_i)}{n}$" be in place of "$\max_i\{d(x_i,y_i)\}$". Then Theorem 2.9 will still be true.

Corollary 2.2 of [15] is a special case of the following corollary.

Corollary 2.10. Let $(X, d, \preccurlyeq)$ be a complete partially ordered metric space and $(A,B) \in \Omega_n$. Let $F: X^n \to X^n$ be a mapping having the (A,B)-mixed monotone property such that

$$d\big(F(x_1, \dots, x_n), F(y_1, \dots, y_n)\big) \le \varphi(\max_i\{d(x_i,y_i)\}), \tag{2.12}$$

for all $(x_1, \dots, x_n), (y_1, \dots, y_n) \in X^n$ with $x_i \preccurlyeq y_i$ for $i \in A$ and $x_i \succcurlyeq y_i$ for $i \in B$, where $\varphi: [0,\infty) \to [0,\infty)$ is a nondecreasing and continuous function such that $\varphi(t) < t$ for all $t > 0$ and $\varphi(0) = 0$. Also, suppose either:

a) F is continuous, or

(b) X has condition H.

If there exists $X_0 = (x_1^0, x_2^0, \cdots, x_n^0) \in X^n$ such that

$$x_i^0 \preccurlyeq F\big(x^0_{\emptyset_i(1)}, \cdots, x^0_{\emptyset_i(n)}\big) \text{ and } x_j^0 \succcurlyeq F\left(x^0_{\emptyset_j(1)}, \cdots, x^0_{\emptyset_j(n)}\right), \tag{2.13}$$

for all $i \in A$ and $j \in B$ where $\emptyset_i: \{1 \cdots, n\} \to \{1 \cdots, n\}$ satisfy condition (2.1) for all $1 \le i \le n$ then F has an n-order fixed point.

Proof. The logic of the proof is similar to the proof of corollary 2.7. ■

3. Application

In the following section, we considered the space $X = C[a,b]$ of continuous function defined on $[a,b]$ with the standard metric given by $d(x,y) = \sup_{t\in[a,b]} |x(t), y(t)|$, for $x, y \in C[a,b]$.

This space can also be equipped with a partial order given by

$x, y \in C[a,b]$, $x \preccurlyeq y \Leftrightarrow x(t) \le y(t)$, for any $t \in [a,b]$.

Moreover, in [17] it is proved that $(C[a,b], \preccurlyeq)$ with the above mentioned metric satisfies condition H.

Now, we formulate our result.

Theorem 3.1. Assume that the following conditions are satisfied:

(i) $\beta_i: [a,b] \to [a,b]$ $(i = 1,2,\cdots,n)$ are continuous functions.

(ii) $f_i: [a,b] \times \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}$ $(i = 1,2,\cdots,n)$ are continuous and nondecreasing on $\mathbb{R}^n$ and $\mathbb{R}$ such that

$|f_i(t,x_1,\cdots,x_n,u)-f_i(t,y_1,\cdots,y_n,v)|\le \lambda_i\varphi_i(\max_i|x_i-y_i|)+|u-v|,$ (3.1)

for all $(x_1,\cdots,x_n),(y_1,\cdots,y_n)\in\mathbb{R}^n$ with $x_i\preccurlyeq y_i$, where nondecreasing functions $\varphi:\mathbb{R}_+\to\mathbb{R}_+$ satisfy the hypotheses of Theorem 2.9 and $0<\lambda_1,\cdots,\lambda_n<1$.

(iii) $g_i:[a,b]\times[a,b]\times\mathbb{R}^n\to\mathbb{R}$ $(i=1,2,\cdots,n)$ are nondecreasing on $\mathbb{R}^n$ such that

$|g_i(t,s,x_1,\cdots,x_n)-g_i(t,s,y_1,\cdots,y_n)|\le \xi_i\vartheta_i(\max_i|x_i-y_i|),$ (3.2)

for all $(x_1,\cdots,x_n),(y_1,\cdots,y_n)\in\mathbb{R}^n$ with $x_i\preccurlyeq y_i$, where nondecreasing functions $\vartheta_i:\mathbb{R}_+\to\mathbb{R}_+$ satisfy the hypotheses of Theorem 2.9 and $0<\xi_1,\cdots,\xi_n\le 1$.

(iv) There exists $x_i^0\in C[a,b]$ $(i=1,\cdots,n)$ such that

$x_i^0\preccurlyeq f_i\left(t,x_1^0(t),\cdots,x_n^0(t),\int_a^{\beta_i(t)}g_i\,(t,s,x_1^0(s),\cdots,x_n^0(s))ds\right).$ (3.3)

(v) For all $i=1,\cdots,n$

$\lambda_i+(b-a)\xi_i\le 1.$ (3.4)

Then the system of integral equations (1.4) has a unique solution in the space $C[a,b]^n$.

Proof. We define the operators $F_i:C[a,b]^n\to C[a,b]$ by

$F_i(x_1,\cdots,x_n)(t)=f_i\left(t,x_1(t),\cdots,x_n(t),\int_a^{\beta_i(t)}g_i\,(t,s,x_1(s),\cdots,x_n(s))ds\right),$

for all $t\in[a,b]$ and $1\le i\le n$. Let us fix arbitrarily$1\le i\le n$. Since f_i,g_i,x_i and β_i are continuous, so F_i is continuous operator from $C[a,b]^n$ to $C[a,b]$. Moreover, F_i has the (A,B)-mixed monotone property such thst $A=\{1,\cdots,n\}$ an $B=\emptyset$. Now, we show that F_i satisfies the hypothesis (2.10) of Theorem 2.9. To do this, take $x_j,y_j\in C[a,b]$ such that $x_j\preccurlyeq y_j$ for all $j=1,\cdots,n$. Then we have

$$\begin{aligned}
d&=d\big(F_i(x_1,\cdots,x_n),F_i(y_1,\cdots,y_n)\big)\\
&=\sup_{t\in[a,b]}|F_i(x_1,\cdots,x_n)(t)-F_i(y_1,\cdots,y_n)(t)|\\
&=\sup_{t\in[a,b]}\Bigg|f_i\left(t,x_1(t),\cdots,x_n(t),\int_a^{\beta_i(t)}g_i\,(t,s,x_1(s),\cdots,x_n(s))ds\right)\\
&\qquad -f_i\left(t,y_1(t),\cdots,y_n(t),\int_a^{\beta_i(t)}g_i\,(t,s,y_1(s),\cdots,y_n(s))ds\right)\Bigg|
\end{aligned}$$

Due to (3.1), (3.2) and since $\varphi_i$and ϑ_i are nondecreasing, we have

$$\begin{aligned}
d&\le \sup_{t\in[a,b]}\lambda_i\varphi_i\left(\max_i|x_i(t)-y_i(t)|\right)\\
&\quad+\sup_{t\in[a,b]}\left|\int_a^{\beta_i(t)}g_i\,(t,s,x_1(s),\cdots,x_n(s))ds-\int_a^{\beta_i(t)}g_i\,(t,s,y_1(s),\cdots,y_n(s))ds\right|\\
&\le \lambda_i\varphi_i\left(\max_i\|x_i-y_i\|+\sup_{t\in[a,b]}\left|\int_a^{\beta_i(t)}g_i\,(t,s,x_1(s),\cdots,x_n(s))-g_i(t,s,y_1(s),\cdots,y_n(s))ds\right|\right)\\
&\le \lambda_i\varphi_i\left(\max_i\|x_i-y_i\|\right)+\int_a^b\xi_i\vartheta_i\left(\max_i\|x_i-y_i\|\right)ds
\end{aligned}$$

$\le \lambda_i\varphi_i\left(\max_i\|x_i-y_i\|\right)+(b-a)\xi_i\vartheta_i\left(\max_i\|x_i-y_i\|\right).$

If we define $\varphi=\max\{\varphi_1,\cdots,\varphi_n,\vartheta_1,\cdots,\vartheta_n\}$ then the last inequality and assumption (v) give us

$$d\big(F_i(x_1,\cdots,x_n),F_i(y_1,\cdots,y_n)\big) \le \varphi\left(\max_i \|x_i - y_i\|\right).$$

Therefore, F_i satisfies (2.10). Finally, let $x_1^0, x_2^0, \cdots, x_n^0$ be the functions appearing in assumption (iv), so by (iv), we get

$$x_i^0 \preccurlyeq F_i(x_1^0,\cdots,x_n^0)$$

The conclusion follows now from Theorem 2.9. ■

Example 3.1. Consider the following system of functional integral equations

$$\begin{cases} x_1(t) = \dfrac{t^2}{t^4+1}\ln\left(\dfrac{|x_1(t)+x_2(t)+x_3(t)|}{3}+1\right) + \dfrac{1}{12}\displaystyle\int_0^t se^{-t}\dfrac{|x_1(s)+x_2(s)+x_3(s)|}{1+|x_1(s)+x_2(s)+x_3(s)|}ds \\ x_2(t) = e^t + \dfrac{3|x_2(t)|}{4+4|x_2(t)|} + \displaystyle\int_0^{\sqrt{t}} \dfrac{s}{16s+1}\big(x_1(s)+x_2(s)+x_3(s)\big)ds \\ x_3(t) = \log(t) + \displaystyle\int_0^{t^2} s^2\ln(|x_1(s)+x_2(s)+x_3(s)|+1)ds, \end{cases}$$

for $t \in [0,1]$. Observe that Eq. (3.5) is a special case of Eq. (1.4) where

$$\beta_1(t) = t, \beta_2(t) = \sqrt{t}, \beta_3(t) = t^2,$$

$$f_1(t,x_1,x_2,x_3,y) = \frac{t^2}{t^4+1}\ln\left(\frac{1}{3}|x_1+x_2+x_3|+1\right)+y$$

$$f_2(t,x_1,x_2,x_3,y) = e^t + \frac{3|x_2|}{4|x_2|+4}+y,$$

$$f_3(t,x_1,x_2,x_3,y) = \log t + y,$$

$$g_1(t,s,x_1,x_2,x_3) = \frac{1}{12}se^{-t}\frac{|x_1+x_2+x_3|}{1+|x_1+x_2+x_3|}$$

$$g_2(t,s,x_1,x_2,x_3) = \frac{s(x_1+x_2+x_3)}{16s+1}$$

$$g_3(t,s,x_1,x_2,x_3) = s^2\ln(|x_1+x_2+x_3|+1)$$

From the definitions of β_i hypothesis (i) of Theorem 3.1 are obviously satisfied. Also, we have

$$|f_1(t,x_1,x_2,x_3,u) - f_1(t,y_1,y_2,y_3,v)| \le$$

$$\le \left|\frac{t^2}{t^4+1}\ln\left(\frac{1}{3}|x_1+x_2+x_3|+1\right)+u-\frac{t^2}{t^4+1}\ln\left(\frac{1}{3}|y_1+y_2+y_3|+1\right)-v\right|$$

$$\le \frac{t^2}{t^4+1}\left|\ln\left(\frac{1}{3}\frac{|x_1+x_2+x_3|+1}{|y_1+y_2+y_3|+1}\right)\right| + |u-v|$$

$$\le \frac{1}{2}\left|\ln\left(\frac{1}{3}\frac{|x_1+x_2+x_3|-|y_1+y_2+y_3|}{|y_1+y_2+y_3|+1}+1\right)\right| + |u-v|$$

$$\le \frac{1}{2}\left|\ln\left(\frac{|x_1-y_1|+|x_2-y_2|+|x_3-y_3|}{3}+1\right)\right| + |u-v|$$

$\le \frac{1}{2}\ln(\max_i|x_i-y_i|+1)|u-v|.$

Similarly,

$$f_2(t,x_1,x_2,x_3,u) - f_2(t,y_1,y_2,y_3,v) \le \frac{3\quad \max_i|x_i-y_i|}{4\max_i|x_i-y_i|+1} + |u-v|$$

$$|f_3(t,x_1,x_2,x_3,u) - f_3(t,y_1,y_2,y_3,v)| \le |u - v|$$

$$|g_1(t,s,x_1,x_2,x_3) - g_1(t,s,y_1,y_2,y_3)| \le \frac{1}{4}\frac{\max_i |x_i - y_i|}{\max_i |x_i - y_i| + 1}$$

$$|g_2(t,s,x_1,x_2,x_3) - g_2(t,s,y_1,y_2,y_3)| \le \frac{3}{16}\max_i |x_i - y_i|$$

$$|g_3(t,s,x_1,x_2,x_3) - g_3(t,s,y_1,y_2,y_3)| \le \ln\left(\max_i |x_i - y_i| + 1\right).$$

Thus, by taking

$\varphi_1(t) = \vartheta_3(t) = \ln(t+1)$, $\varphi_2(t) = \vartheta_1(t) = \frac{t}{t+1}$, $\varphi_3(t) = 0$, $\vartheta_2(t) = \frac{3}{4}t$, $\lambda_1 = \frac{1}{2}$, $\lambda_2 = \frac{3}{4}$, $\lambda_3 = 0$, $\xi_1 = \frac{1}{4}$, $\xi_2 = \frac{1}{4}$ and $\xi_3 = 1$, the functions f_i and g_i satisfy assumptions (ii) and (iii) of Theorem 3.1 and the hypothesis (3.4) holds. Finally, if we consider $x_1(t) = x_2(t) = x_3(t) = 0$ then the hypothesis (3.3) holds. Hence by using Theorem 3.1 the system of integral equations (3.5) has a unique solution in the space $C[0,1]^3$.

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