Rational Diophantine Quadruples

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ABSTRACT

This paper concerns with the study of constructing strong rational diophantine quadruples with suitable property.

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I.INTRODUCTION

Let q be a non-zero rational number. A set $\{a_1, a_2, ...a_m\}$ of non-zero rational is called a rational $D(q)$-mtuple, if $a_i a_j + q$ is a square of a rational number for all $1 \le i \le j \le m$. The mathematician Diophantus of Alexandria considered a variety of problems on indeterminant equations with rational or integers solutions. In particular, one of the problems was to find the sets of distinct positive rational numbers such that the product of any two numbers is one less than a rational square [14] and Diophantus found four positive rationals $\frac{1}{16}, \frac{33}{16}, \frac{17}{4}, \frac{105}{16}$ [4,5]. The first set of four positive integers with the same property, the set $\{1,3,8,120\}$ was found by Fermat. It was proved in 1969 by Baker and Davenport [3] that a fifth positive integer cannot be added to this set and one may refer [6, 7,11] for generalization. However, Euler discovered that a fifth rational number can be added to give the following rational Diophantine quintuple $\left\{1,3,8,120,\frac{777480}{8288641}\right\}$. Rational sextuples with two equal elements have been given in [2]. In this 1999, Gibs [13] found several examples of rational Diophantine sextuples, eg., $\left\{\frac{11}{192}, \frac{35}{192}, \frac{155}{27}, \frac{512}{27}, \frac{1235}{48}, \frac{180873}{16}\right\}$, $\left\{\frac{17}{448}, \frac{265}{448}, \frac{2145}{448}, 252, \frac{23460}{7}, \frac{2352}{7921}\right\}$.

All known Diophantine quadruples are regular and it has been conjectured that there are no irregular Diophantine quadruples [1,13] (this is known to be true for polynomials with integer co-efficients [8]). If so then there are no Diophantine quintuples. However there are infinitely many irregular rational Diophantine quadruples. The smallest is $\frac{1}{4}, 5, \frac{33}{4}, \frac{105}{4}$. Many of these irregular quadruples are examples of another common type for which two of the subtriples are regular i.e., $\{a,b,c,d\}$ is an irregular rational Diophantine quadruple, while $\{a,b,c\}$ and $\{a,b,\mathrm{d}\}$ are regular Diophantine triples. These are known as semi – regular rational Diophantine quadruples. These are only finitely many of these for any given common denominator l and they can readily found.

Moreover in [12], it has been proved that $D(\mp k^2)$ - triple $\{k^2, k^2 \pm 1, 4k^2 \pm 1\}$ cannot be extended to a $D(\mp k^2)$ - quintuple. In [10], it has been proved that $D(-k^2)$ - triple $\{1, k^2+1, k^2+4\}$ cannot be extended to a

$D(-k^2)$ - quadruple if $k \geq 5$.Also one may refer [1-20] for diophantine quadruples.These result motivated us to search for strong and almost strong rational Diophantine quadruples.

II.METHOD OF ANALYSIS

Section A:

In this section we search for distinct rational quadruple P,Q,R,S such that the product of any two of them added with $p^2 \mathrm{n}^2 + q^2$ is a perfect square

Consider $P = \frac{a}{b}$ and $Q = \frac{b}{a}(2npq)$

Note that $PQ + p^2n^2 + q^2$ is a perfect square

Let R be any non-zero rational integer such that

$$PR + p^2n^2 + q^2 = \alpha^2 \tag{1}$$

$$QR + p^2n^2 + q^2 = \beta^2 \tag{2}$$

From (1), we have

$$R = \frac{\alpha^2 - p^2n^2 - q^2}{P} \tag{3}$$

Assume $\alpha = X + \frac{a}{b}T$ (4)

$$\beta = X + \frac{b}{a}(2npq)T \tag{5}$$

On substituting (3) in (2) and using (4) and (5), we get

$$X^2 = \frac{a}{b}\left(\frac{b}{a}2npq\right)T^2 + p^2n^2 + q^2$$

Whose initial solution is $T_0 = 1,\ \mathrm{X}_0 = p\mathrm{n} + q$

Thus $\alpha = pn + q + \frac{a}{b}$

$$\beta = pn + q + \frac{b}{a}2npq$$

Therefore from (3)

$$R = \frac{a^2 + 2npqb^2 + 2ab\left(pn + q\right)}{ab}$$

Let S be any non – zero rational integer such that

$$PS + p^2n^2 + q^2 = \alpha^2 \tag{6}$$

$$QS + p^2n^2 + q^2 = \beta^2 \tag{7}$$

$$RS + p^2n^2 + q^2 = \gamma^2 \tag{8}$$

From (7)

$$S = \frac{\beta^2 - p^2n^2 - q^2}{Q} \tag{9}$$

Let $$\beta = X + \frac{b}{a}(2npq)T \qquad (10)$$

And $$\gamma = X + \frac{a^2 + 2npqb^2 + 2ab\left(pn+q\right)}{ab}T \qquad (11)$$

On substituting (9) in (8) and employing (10) and (11) it leads to

$$X^2 = \left(\frac{a^2 + 2npqb^2 + 2ab\left(pn+q\right)}{ab}\right)\left(\frac{b}{a}2npq\right)T^2 + p^2n^2 + q^2$$

Whose initial solution is $T_0 = 1$ and $X_0 = \dfrac{a(pn+q)+2npqb}{a}$

Thus $\beta = \dfrac{a(pn+q)+4npqb}{a}$

Therefore from (9)

$$S = \frac{\left(a(pn+q)+4npqb\right)^2 - (p^2n^2+q^2)a^2}{2npqab}$$

Hence (P,Q,R,S) is a strong rational diophantine quadruple .

Note that S satisfies (6) provided $n = \dfrac{-aq}{p(a+3qb)}$.

Section B

In this section, we search for distinct rational quadruple (P,Q,R,S) such that product of any two minus unity is a perfect square.

Assume $P = \dfrac{a}{b}$

and $Q = \dfrac{b}{a}(\mathrm{n}^2 \pm 2\,\mathrm{n} + 2)$

where $a = 3r^2 - s^2 + 4rs(n \pm 1)$

and $\mathrm{b} = 2\,\mathrm{rs}$ (r, s ≠0)

Let R be any non – zero rational integer such that

$$PR - 1 = \alpha^2$$
$$QR - 1 = \beta^2$$

Repeating the same procedure as mentioned in section (A), we get

$$R = \frac{\left(3r^2 - s^2 + 4rs(n\pm1)\right)^2 + 4r^2s^2(n^2 \pm 2n + 2) + 4\left(3r^2 - s^2 + 4rs(n\pm1)\right)\left(rs\right)\left(n\pm1\right)}{\left(3r^2 - s^2 + 4rs(n\pm1)\right)\left(2rs\right)}$$

Let S be any non – zero rational integer such that

$$PS - 1 = \alpha^2$$
$$QS - 1 = \beta^2$$

$$RS-1=\gamma^2$$

Following the same procedure as mentioned in section (A) we get

$$S=\frac{\left(3r^2-s^2+4rs(n\pm1)\right)^2+16r^2s^2(n^2\pm2n+2)+8\left(3r^2-s^2+4rs(n\pm1)\right)(rs)(n\pm1)}{\left(3r^2-s^2+4rs(n\pm1)\right)(2rs)}$$

Hence (P, Q, R, S) is a strong rational diophantine quadruple in which the product of any two minus unity is a perfect square.

Remark:

If we take Q in different form we can generate different quadruple. Some of them are given below

1) If $Q=\frac{b}{a}(n^2+2n)$ with the property $\mathrm{D}(1)$ then quadruple is

$$\left(\frac{3r^2+s^2-4rs(n+1)}{2rs},\frac{2rs}{3r^2+s^2-4rs(n+1)}(n^2+2n),\right.$$
$$\frac{\left(3r^2+s^2-4rs(n+1)\right)^2+4r^2s^2n(n+2)+4rs\left(3r^2+s^2-4rs(n+1)\right)(n+1)}{\left(3r^2+s^2-4rs(n+1)\right)(2rs)},$$
$$\left.\frac{\left(3r^2+s^2-4rs(n+1)\right)^2+16r^2s^2n(n+2)+8rs\left(3r^2+s^2-4rs(n+1)\right)(n+1)}{\left(3r^2+s^2-4rs(n+1)\right)(2rs)}\right)$$

2) If $Q=\frac{b}{a}2n^2k^2$ with the property $\mathrm{D}\left[(g^2+2g-1)n^2k^2\right]$ then quadruple is

$$\left(\frac{a}{b},\frac{b}{a}2n^2k^2,\frac{a^2+2n^2k^2b^2+2nkab(g+1)}{ab},\frac{a^2+8n^2k^2b^2+4(g+1)abnk}{ab}\right)$$

where $a=3(g^2+2g-1)r^2+s^2-4rs(g+1)$ and $b=\frac{2rs}{nk}$

3) If $Q=\frac{b}{a}(-2n^2k^2)$ with the property $\mathrm{D}\left[(g^2+2g+3)n^2k^2\right]$ then quadruple is

$$\left(\frac{a}{b},\frac{b}{a}(-2n^2k^2),\frac{a^2-2n^2k^2b^2+2nkab(g+1)}{ab},\frac{a^2-8n^2k^2b^2+4(g+1)abnk}{ab}\right)$$

where $a=3(g^2+2g+3)r^2+s^2-4rs(g+1)$ and $b=\frac{2rs}{nk}$

4) If $Q=\frac{b}{a}(\pm2nk)$ with the property $\mathrm{D}(\mp2nk)$ then quadruple is

$$\left(\frac{6nkr^2\mp s^2}{2rs},\frac{4nkrs}{6nkr^2\mp s^2},\frac{(6nkr^2\mp s^2)+8nkr^2s^2}{(6nkr^2\mp s^2)2rs},\frac{(6nkr^2\mp s^2)+32nkr^2s^2}{(6nkr^2\mp s^2)2rs}\right)$$

5) If $Q = \frac{b}{a}\left(p^2n^2 + q^2\right)$ with the property $D(2npq)$ then quadruple is

$$\left(\frac{a}{b}, \frac{b}{a}\left(p^2n^2 + q^2\right), \frac{a^2 + (p^2n^2 + q^2)b^2 + 2(pn+q)ab}{ab}, \frac{a^2 + 4(p^2n^2 + q^2)b^2 + 4(pn+q)ab}{ab}\right)$$

where $a = 6npqr^2 + s^2 + 4rs(pn+q)$ and $b = 2rs$

6) If $Q = \frac{b}{a}q^2$ with the property $D(p^2n^2 + 2npq)$ then quadruple is

$$\left(\frac{a}{b}, \frac{b}{a}q^2, \frac{a^2 + q^2b^2 + 2(pn+q)ab}{ab}, \frac{a^2 + 4q^2b^2 + 4(pn+q)ab}{ab}\right)$$

where $a = 3(p^2n^2 + 2npq)r^2 + s^2 + 4rs(pn+q)$ and $b = 2rs$

III.CONCLUSION

To conclude, one may search for other families of strong t Diophantine quadruples .

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