Aumann Integral of Multifunctions Valued in Quasy-Banach Spaces and Some of its Properties

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Abstract
We will study the integral of Aumann when the values of multifunctions are subsets of a quasy-Banach separable space. Initially we will see the extension of the Aumann integral in the case of multifunctions with values in quasy-normed spaces and in some instances of its existence illustrating with examples.
We will see the following relation between measurable and integrable of multifunctions according to Aumann and some various operations which do not bring us out of class by Aumann integrable multifunctions. Here we can mention that, if the union action of integrable multifunctions by Aumann is again integrable, for cutting can't say the same thing. We see also that is true the property of linearity of the Aumann's integral.
Finally we are shifting focus to the limit and note that there is a rather interesting statement that in the case when the space X which defines sequence of multifunctions F_n is finite measure allows us to say that: Not only the limit of a sequence of integrable multifunctions by Aumann is the integrable multifunction but it's true the equality $\int F d\mu = \lim_{n\to+\infty} \int F_n d\mu$. This assertion is also based on a similar theorem known Fatou theorem, which noted that, can be extended to our case.

Keywords: Aumann integral, quasy-Banach space, convergence theorem of multifunctions

Introduction

A relation F: X→Y is a subset of X×Y. Otherwise, F can be considered as a function of X in all subsets of the set of Y. Domain of F is the set Dom (F) = $\{x \in X: F(x) \neq \emptyset\}$, the image of F is the union of the images F(x) when x ranges over X, Rang (F) = $\bigcup_{x\in Dom(F)} F(x)$ while graph of F is the set Graph (F) = $\{(x,t): t \in F(x), x \in Dom(F)\}$.
If we want to emphasize the properties of F as a subset of X×Y, referring to it's graph [1].
A multifunction is a relation with domain X. We denote it F:X ⇉ Y or $F:X\to 2^Y$.

Let X and Y be topological spaces.
(i) A multifunction $F:X\to 2^Y$ is called with closed, open or compact valued if, for every x∈X, F (x) is respectively, a closed, open or compact set in Y. Moreover, if Y is a topological vector space and for every x∈X, F (x) is a convex set on Y, then F is called with convex values.
(ii) A multifunction $F:X\to 2^Y$ is called closed, open or compact if the Graph(F) is respectively, a closed, open or compact set according to the product topology of X×Y. Moreover, if X and Y are topological vector spaces, then F is called convex if its graph is a convex set by product topology X×Y.

In multifunctios (relations) collection determined various mathematical operations.
Let's say we have the operation * that is defined as follows: $F_1 * F_2$: x→ $F_1(x) * F_2(x)$. In this way we can determine $F_1 \cap F_2$, $F_1 \cup F_2$, $F_1 + F_2$ (in vector space), $F_1 \times F_2$ etc. Similarly, if the function α: $2^Y \to 2^Y$ determine α(F):x→α(F(x)). For example, will use the notation $\bar{F}$, for the multifunction x→$\overline{F(x)}$ where, $\overline{F(x)}$ is denoted the closure of set F(x).

Also for multifunctions as for functions, we study continuity, their measurement and integration.

There are two ways to define the inverse image of a subset M by a multifunction F.

(i) $F^-(M) = \{x \in X: F(x) \cap M \neq \emptyset\}$
(ii) $F^+(M) = \{x \in X: F(x) \subset M\}$

The subset $F^-(M)$ is called the invers image of M by F and $F^+(M)$ is called the core of M by F.

A relation (multifunction) is called measurable (weakly measurable, B-measurable, K-measurable) only when the inverse image $F^-(B)$ is measurable for every closed set (respectively open set, Borel's set and compact set) B of Y.

Let $F_n: X \to 2^Y$ be a sequence of multifunctions. Easily shown that, for every subset A⊂Y is true the equality $(\bigcup_n F_n)^-(A) = \bigcup_n F_n^-(A)$.
So, the following proposition is true:

Proposition 1
If $F_n: X \to 2^Y$ are measurable (weakly measurable, B-measurable, K-measurable) multifunctions then $\bigcup_n F_n: X \to 2^Y$ is also a measurable (weakly measurable,B-measurable,K-measurable) multifunction.

We note also that a similar statement is true for the cutting of multifunctions.

Proposition 2
If $F_n: X \to 2^Y$ are measurable (weakly measurable, B-measurable, K-measurable) multifunctions then the multifunction $\bigcap_n F_n: X \to 2^Y$ is also a measurable (weakly measurable, B-measurable, K-measurable) multifunction.

Proof
Easily shown that, for every A⊂Y we can write $(\bigcap_n F_n)^+(A) = \bigcap_n F_n^+(A)$ (1).

On the other hand, for every n∈N is true the equality $F_n^-(A) = X\backslash F_n^+(Y\backslash A)$ (2).

Since for every n∈N the multifunction F_n is measurable (weakly measurable, B-measurable, K-measurable) then $F_n^-(M)$ is measurable for every closed (open, Borel's set) subset M⊂Y.

From equality (2) it follows that for every n∈N the set $F_n^+(Y\backslash M)$ is measurable. The set $Y\backslash M$ is open (closed, Borel's set) and the equality (1) allows us to say that the set $(\bigcap_n F_n)^+(A)$ is measurable for every open (closed, Borel's set) subset A⊂Y.

Applying equality (2) for the multifunction $\bigcap_n F_n$ conclude that, the set $(\bigcap_n F_n)^-(A)$ is measurable for every closed (open,Borel'set) subset A⊂Y.

Measurable of multifunctions is closely associated with the concept of measurable of his selections.

For a given multifunction F:X→2^Y,called selection a function f:X→Y such that f(x)∈F(x) almost everywhere in X.

Let (X,Σ,μ) be a complete σ-finite measure space and Y a separable quasy-Banach space.We denote by $\mathcal{F}$ the set of all integrable selections of F, so

$$\mathcal{F} = \{f \in L^1(X,Y,\mu): f(x) \in F(x) \text{ a.e in } X\}$$

(a.e is used for cutting of expression almost everywhere)

A multifunction F:X→2^Y is called integrably bounded if there exists a nonnegative function $f(x) \in L^1(X,R,\mu)$ such that $F(x) \subset f(x)B$ almost everywhere in X, where is denoted B the unit ball on Y [2].

Aumann did suggest definition of the integral of a multifunction in the following way:

The integral of F on X is the set of integrals of integrable selections of F:

$$\int_X F d\mu = \left\{\int_X f d\mu : f \in \mathcal{F}\right\}$$

(Integrals in this set are Bochner integrals of the function f. Remember that: Every measurable function f:X→Y is called Bochner integrable if there is a sequence of simple functions $f_n: X \to Y$ such that $\lim_{n\to+\infty} \int_X \|f_n(x) - f(x)\| d\mu(x) = 0$. In this case the Bochner integral of f on E is defined by the equation $\int_E f d\mu = \lim_{n\to+\infty} \int_E f_n d\mu$. Bochner integral extends to functions valued in the quasy-normed spaces. This is accomplished by following a path similar to the construction of this integral in the case of functions valued in normed spaces. Easily shown that the Bochner integral for functions valued in quasy normed spaces is linear.)

In the following we will say that F is integrable multifunction by Aumann if the set $\left\{\int_X f d\mu : f \in \mathcal{F}\right\}$ is not empty.

Referring to Aumann [7], the proposition is true

Proposition 3

If $F: T \to 2^{E^n}$ where, T = [0,1] and E^n is an n-dimensional Euclidean space, is Borel measurable and integrably bounded multifunction then $F: T \to 2^{E^n}$ is integrable by Aumann.

Terminology and preliminaries

The following (X,Σ,μ) will be a complete finite measurable space and (Y, ||.||) will be a quasy-Banach space. Let D be a subset of X, define diam (D) = $sup_{x,y\in D}\|x-y\|^p$ as the diameter set D.

Remember that:for every quasi-norm in Y there is a equivalent p-norm (Aoki-Rolewics theorem [4]). Therefore, the above constant p corresponds to the equivalent p-norm.

Let's be Σ^+ family of set A∈Σ with a positive measure and Σ_A^+ the collection of subsets of A that are part of Σ^+.

Definition 4

The multifunction F:X→2^Y satisfies the property (P) if for every ε> 0 and for every A∈Σ^+ there are B∈Σ_A^+ and D⊂X with diameter diam (D) ≤ ε such that F (x) ∩ D ≠ ∅ for every x∈B (ie B⊂F^-(D)).

As in the case of normed space [1] note that the following proposition is true:

Proposition 5

Let F:X→2^Y be a multifunction. The following propositions are true:

(i) If it is a multifunction G:X→2^Y that satisfies the property (P) and G(x) ⊂ F(x) almost everywhere according μ on X, then F satisfies the property (P).

(ii) If F has a measurable selection then F satisfies the property (P).

Basic theorems on the existence of measurable selectors are Kuratowski-Ryll Nardzewski theorem and the Aumann theorem.

Theorem 6 (Kuratowski-Ryll Nardzewski)[3]

If (X,Σ) is a measurable space, Y is a metric space separable, F:X→2^Y is a weakly measurable multifunction, with closed values in Y then the multifunction F has a measurable selection f.

Theorem 7 (Aumann) [6]

If (X,Σ) is a σ finite measurable space , Y a Borel subset of a Polish space and F:X→2^Y is a multifunction with separable graph then there is a measurable function f:X →Y such that f(x)∈F(x) for x ∈ X almost everywhere according μ on X.

Given the fact that there are quasy-normed separable spaces, moreover any quasi-Banach separable space (X,q) with constant of quasy-norm (modul of concavity) C is linearly homeomorfe with quotient space ℓ p [0,1] where C =$2^{\frac{1}{p-1}}$ [5], we conclude that the theorem of Kuratowski-Ryll Nardzewski remains true even when the space Y is quasy-Banach separable space.

A Polish space is a separable completely metrizable topological space. On the other hand, every quasy- Banach separable space is a separable completely metrizable space. So, Aumann theorem remains true even when the space Y is quasy-Banach separable space.

The Results

Immediately from the definition of the Aumann's integrable multifunction note that:

- In the case of the function f:X→Y, so when values of multifunction are set with an element, note that the Aumann integral of f coincides with its Bochner integral. So, every function f:X→Y Aumann integrable is measurable.
- Each of integrable multifunction according Aumann has Bochner integrable selections, then there is measurable selections of it.
- Let there be (X,Σ,μ) a complete finite measurable space and F:X→2^Y a Aumann integrable multifunction. From proposition 5 (ii) we conclude that the multifunction F satisfies the property (P) of the definition 4.

Let us give an example of an integrable multifunction according to Aumann.

Example 8

Every constant multifunction F:X→2^Y is integrable according to Aumann.

Proof

Since the multifunction F is constant, then for every x ∈ X we have F (x) = B, where B ⊂ Y is a fixed subset. So, enough to take a constant selection f: X → Y such that for every x ∈ X we have f(x) = y_0 ∈ B. These selections can be seen as simple functions of the form f(x) = $y_0\chi_A$ where A is a measurable subset of X that contains x-in along with a neighbourhood. This ends the proof.

Based on the properties of Bochner integral of a function according to a measure μ easily shown that the propositions are true:

Proposition 9 [2]

If the multifunction F:X→2^Y is with convex valued then also the integral $\int_X Fd\mu$ a is convex set.

Proposition 10 [2]

If the multifunction F:X→2^Y is integrable according to Aumann and λ∈R then the multifunction λF:X→2^Y is also integrable according to Aumann and $\int_X \lambda F d\mu = \lambda \int_X F d\mu$.

The quasy-normed space is also a vectorial topological space . So, we can affirm that:

Propostion 11

If the multifunctions $F_1, F_2: X \to 2^Y$ are integrable according to Aumman then the multifunction $F_1 + F_2$ is integrable according to Aumman and $\int_X (F_1 + F_2)(x)d\mu = \int_X F_1(x)\,d\mu + \int_X F_2(x)\,d\mu$.

Proof

Since the multifunctions F_1 and F_2 are integrable according to Aumman then there are the functions $f_1, f_2: X \to Y$ that are respectively the selections of F_1 and F_2.

Denote $A = \{x \in X: f_1(x) \in F_1(x)\}$ and $B = \{x \in X: f_2(x) \in F_2(x)\}$.

By way of selecting functions f_1 and f_2 note that $\mu(X \setminus f_1^{-1}(A)) = 0$ and $\mu(X \setminus f_2^{-1}(B)) = 0$.

Thus, the equalities

$$\begin{aligned}(X \setminus f_1^{-1}(A)) \cup (X \setminus f_2^{-1}(B)) &= (X \cap (f_1^{-1}(A))^C) \cup (X \cap (f_2^{-1}(B))^C) = \\ &= X \cap ((f_1^{-1}(A))^C \cup (f_2^{-1}(B))^C) \\ &= X \cap (f_1^{-1}(A) \cap f_2^{-1}(B))^c \\ &= X \setminus (f_1^{-1}(A) \cap f_2^{-1}(B))\end{aligned}$$

lead us to the conclusion $\mu(X \setminus (f_1^{-1}(A) \cap f_2^{-1}(B))) = 0$.

On the other hand, for every x∈$f_1^{-1}(A) \cap f_2^{-1}(B)$ is true the relation $(f_1(x) + f_2(x)) \in (F_1 + F_2)(x)$.

Therefore the function g:X→Y such that g(x) = $f_1(x) + f_2(x)$ is a selection of multifunction $F_1 + F_2$ and more than the function g is integrable according to Bochner. So we have shown that the multifunction $F_1 + F_2$ is integrable according to Aumann.

Denote respectively $\mathcal{F}_1, \mathcal{F}_2, \mathcal{F}_{12}$ sets of integrable according to Bochner selection for multifunctions $F_1, F_2, F_1 + F_2$. From the above reasoning it is clear that $\mathcal{F}_1 + \mathcal{F}_2 = \mathcal{F}_{12}$.

In these conditions by definition of Aumann integral and linearity of the Bochner integral conclude that $\int_X (F_1 + F_2)(x)d\mu = \int_X F_1(x)\,d\mu + \int_X F_2(x)\,d\mu$.

Now see how we can formulate a similar proposition to 3 in our case.

Initially, we note that it is true that lemma:

Lemma 12

Let (X,Σ,μ) be a measurable space with finite measure. If F:X→2^Y is an integrably bounded multifunction that has measurable selections then F:X→2^Y is integrable according to Aumann.

Proof

Since F is an integrably bounded multifunction then there is a nonnegative function $f(x) \in L^1(X, R, \mu)$ such that $F(x) \subset f(x)B$ almost everywhere according μ on X (B is denoted unit ball in Y). On the other hand under the assumption of Lemma, there is a measurable selection g of F. So, g(x)∈f(x)B almost everywhere according μ on X. This fact tells us that ||g(x)|| ≤ f(x) almost everywhere according μ on X and therefore the real value function||g(x)|| is integrable according to Lebesgue.

Since g is a measurable function then there is a sequence of simple functions $(g_n)_{n\in N}$ such that $\lim_{n\to+\infty} g_n = g$ almost everywhere according μ on X. Therefore, for every ε > 0 there is a $n_0 \in N$ such that for every $n \geq n_0$ is true the inequality $\|g_n(x) - g(x)\| < \varepsilon$.

From the third property of quasy norm we have: $\|g_n(x)\| \leq K\|g_n(x) - g(x)\| + K\|g(x)\| < K\varepsilon + Kf(x)$ almost everywhere according μ on X. The function K(ε + f(x)) is also non-negative integrable and since $(\|g_n(x)\|)_{n\in N}$ is a sequence of real valued functions then from the Lebesgue integration theory we conclude that the sequence $(\|g_n(x)\|)_{n\in N}$ is integrable according to Lebesgue.

On the other hand inequality $\|g_n(x) - g(x)\| \leq K\|g_n(x)\| + K\|g(x)\|$ and integrable according to Lebesgue of functions $\|g_n(x)\|$ and $\|g(x)\|$ guarantee integrable according to Lebesgue of functions $\|g_n(x) - g(x)\|$ for any n∈N.

So is true the inequality $\int_X \|g_n(x) - g(x)\|d\mu < \varepsilon\mu(X) = \varepsilon'$ where ε' can be a positive number however small while the measure of X is finite. Therefore we conclude that the function g is Bochner-integrable, which is the same as the fact that F is integrable by Aumann.

By Lema above and Kuratowski-Ryll Nardzewski theorem it follows that the proposition is true:

Proposition 13
Let (X,Σ,μ) be a measurable space with finite measure. If F:X→2^Y is a weakly measurable multifunction with closed valued and integrably bounded then it is integrable according to Aumann.

Also Lemma 12 and Aumann theorem on the existence of a measurable selection of multifunction allow us to affirm that:

Proposition 14
Let (X,Σ,μ) be a measurable space with finite measure and Y a Borel subset of a Polish space. If F:X→2^Y is a multifunction with separable graph and integrably bounded then it is integrable according to Aumann.

It is true the following proposition:
Proposition 15
If G:X→2^Y is integrable according to Aumann and G(x) ⊂ F(x) almost everywhere according μ on X then the multifunction F is also integrable according to Aumann.
The proof is immediate from the fact that the integrable according to Aumann multifunctions are multifunctions that have integrable according to Bochner selections and construction of F colection.

Corollary 16
a) If F:X→2^Y is integrable according to Aumann then $\bar{F}$:X→2^Y is also integrable according to Aumann.
b) If the sequence F_n:X→2^Y consisting of integrable according to Aumann multifunctions then $\bigcup_{n\in N} F_n$ is also integrable according to Aumann.

Note that the cutting operation of two integrable according to Aumann multifunctions it can draw us out of this class multifunctions.
Example 17
Let $F_1, F_2: [0,1] \to 2^R$ be two multifunctions such that:

$F_1(x) = \begin{cases} \{3\} & x \in A \cap C \\ [3,5[& x \in [0,1] \setminus A \cap C \end{cases}$ and $F_2(x) = \begin{cases} \{3\} & x \in [0,1] \setminus A \\ ]2,3] & x \in A \end{cases}$ where A is a immeasurable set according to Lebesgue measure and C is Cantor set.
Note that $(F_1 \cap F_2)(x) = \{3\}$ for every x∈[0,1] and the function f(x) = 3 for every x∈[0,1] serves as a integrable selection of each of them, therefore these multifunctions are integrable according to Aumann in [0,1]. So we built an example when cutting two of integrable multifunctions is also integrable.

Take $F_1(x) = \begin{cases} \{3\} & x \in A \cap C \\ (0,2) & x \in [0,1] \setminus (A \cap C) \end{cases}$ and $F_2(x) = \begin{cases} \{2\} & x \in [0,1] \setminus A \\ ]1,4] & x \in A \end{cases}$ where A is a immeasurable set according to Lebesgue measure and C is Cantor set. Note that the functions f(x) = 3 and g(x) = 2 for every x∈[0,1] serve respectively as Bochner integrable selections, and hence are integrable according to Aumann.
Note that $(F_1 \cap F_2)(x) = \begin{cases} \{3\} & x \in A \cap C \\ (1,2) & x \in A \setminus A \cap C \\ \emptyset & x \in [0,1] \setminus A \end{cases}$ and however constructed a function f :[0,1]→R such that f(x)∈ $(F_1 \cap F_2)(x)$ almost everywhere according to Lebesgue measure in R, it is true that $f^{-1}(-\infty, 2) = A \setminus (A \cap C)$. The set $A \setminus (A \cap C)$ is immeasurable according to Lebesgue measure and so the function f is not Bochner integrable. Therefore $F_1 \cap F_2$ is not integrable according to Aumann.

Aumann [7] has shown that, if the multifunctions $F_n: T \to 2^{E^n}$ where, T = [0,1] and E^n is an n-dimensional Euclidean space, are bounded from the same integrable function then $\int limsup F_n d\mu \supset limsup \int F_n d\mu$.
In our case, we notice that this proposition is true:
Proposition 18
If $F_n: X \to 2^Y$ is a sequence of integrable according to Aumann multifunctions and Aumann integral of $limsup F_n$ exists then we have $\int limsup F_n d\mu \supseteq limsup \int F_n d\mu$.
Proof
Starting from the equation $limsup F_n(x) = \bigcap_{n\in N}(\bigcup_{k\geq n} F_k(x))$ and the fact that not always the cutting of two integrable multifunctions is again integrable is clear that, there is not always Aumann's integral of the $limsup F_n$.
So, firstly we show that if $\int limsup F_n d\mu = \emptyset$ then $limsup \int F_n d\mu = \emptyset$.
Suppose that $\int limsup F_n d\mu = \emptyset$, so there is no function f:X→Y that is integrable according to Bochner such that f(x)∈$limsup F_n(x)$ almost everywhere according to μ on X.
Therefore, for any Bochner integrable function f:X→Y there is a subset A⊂X with positive measure whose points satisfy the condition $f(x) \notin limsup F_n(x)$. So, for every x∈A have that: found $n_0 \in N$ such that for every k ≥ n_0 it is true the relation $f(x) \notin F_k(x)$.

Assuming that $limsup \int F_n d\mu \neq \emptyset$ we have found a function f_0:X→Y that is integrable according to Bochner such that: for every n∈N there is $k_0 \in N$ such that $k_0 \geq n$ and $\int f_0 d\mu \in \int F_{k_0} d\mu$. So, integrable function f_0 satisfies the relation $f_0(x) \in F_{k_0}(x)$ for $k_0 \geq n$ almost everywhere according to μ on X. Therefore for $n_0 \in N$ we find $k_0 \geq n_0$ such that $f_0(x) \in F_{k_0}(x)$ almost everywhere according to μ on X or in other words, we find a point in the set A for which $f_0(x) \in F_{k_0}(x)$.
The latter, by the above, contradicts the fact that $\int limsup F_n d\mu = \emptyset$. Thus we have shown that, if $\int limsup F_n d\mu = \emptyset$ then $limsup \int F_n d\mu = \emptyset$.
Let us turn now to the case when $\int limsup F_n d\mu \neq \emptyset$.
Take the function f:X→Y such that $\int f\, d\mu \in limsup \int F_n d\mu$ and we can show that $\int f\, d\mu \in \int limsup F_n d\mu$.
By reasoning in paragraph above we conclude that under Bochner integrable function f:X→Y we have that for every n∈N there is $k_0 \in N$ such that $k_0 \geq n$ and $f(x) \in F_{k_0}(x)$ almost everywhere according to μ on X. In these conditions $f(x) \in limsup F_n(x) = \cap_{n\in N}(\cup_{k\geq n} F_k(x))$ almost everywhere according to μ on X and the function f is integrable according to Bochner.
So we have shown that $\int f\, d\mu \in \int limsup F_n d\mu$.

Similarly to the proof of the claim above show a similar proposition with Fatou theorem.

Proposition 19

If $F_n: X \to 2^Y$ is a sequence of integrable according to Aumann multifunctions and Aumann integral of $liminf F_n$ exists then we have $\int liminf F_n d\mu \subseteq liminf \int F_n d\mu$.

Remark 20

We can find sequences of integrable according to Aumann multifunctions for which there is Aumann integral of $liminf F_n$ and $limsup F_n$ but are not true the relations $\int limsup F_n d\mu \subseteq limsup \int F_n d\mu$ and $\int liminf F_n d\mu \subseteq liminf \int F_n d\mu$.
Take the sequence of multifunctions $F_n: [0,1] \to R$ such that:

$$F_n(x) = \begin{cases} \{1\} & x \in \left[0, \frac{1}{n}\right] \\ \left[0, \frac{1}{n}\right[\cup \{1\} & x \in \left]\frac{1}{n}, 1\right] \end{cases}$$

where R is equipped with Lebesgue measure.
The function f(x) = 1 for every x∈[0,1] serves as the Bochner integrable selection for any multifunction F_n. So every multifunction F_n is integrable according to Aumann.
Find now $limsup F_n(x) = \cap_{n\in N}(\cup_{k\geq n} F_k(x))$ and $liminf F_n(x) = \cup_{n\in N}(\cap_{k\geq n} F_k(x))$.
If x = 0 then for every n∈N the set $F_k(x) = \{1\}$. So that $limsup F_n(x) = \{1\}$ and $liminf F_n(x) = \{1\}$.
If x = 1 then for every k∈N the set $F_k(x) = \left[0, \frac{1}{k}\right[\cup \{1\}$. So that are true the equalities $\cup_{k\geq n} F_k(x) = \left[0, \frac{1}{n}\right[\cup \{1\}$ and $\cap_{k\geq n} F_k(x) = \{0,1\}$. Therefore we have $limsup F_n(x) = liminf F_n(x) = \{0,1\}$.
Let us take x∈(0,1) and to find $limsup F_n(x)$ and $liminf F_n(x)$. For these x there is $n_0(x) \in N$ such that for every $n \geq n_0(x)$ is real inequality $\frac{1}{n} < x$. So are true the equalities $\cup_{k\geq n} F_k(x) = \left[0, \frac{1}{n_0(x)}\right[\cup \{1\}$ and $\cap_{k\geq n} F_k(x) = \{0,1\}$. Therefore we have $limsup F_n(x) = \left[0, \frac{1}{n_0(x)}\right[\cup \{1\}$ because of $n_0(x)$ is a constant and $liminf F_n(x) = \{0,1\}$. So finally have:

$$limsup F_n(x) = \begin{cases} \{1\} & x = 0 \\ \left[0, \frac{1}{n_0(x)}\right[\cup \{1\} & x \in]0,1[\\ \{0,1\} & x = 1 \end{cases} \quad \text{and}$$

$$liminf F_n(x) = \begin{cases} \{1\} & x = 0 \\ \{0,1\} & x \in]0,1] \end{cases}$$

Thus the function f (x) = 0 for all x∈[0,1] is an Bochner integrable selection of $limsup F_n(x)$ and $liminf F_n(x)$. On the other hand $\int f d\mu \notin limsup \int F_n d\mu$ because this function is not a selection for none of multifunctions F_n (Because the points of the segment $\left[0, \frac{1}{n}\right]$ where f (x) does not included in the set $F_n(x)$ have positive measure).

Corollary 21

Let (X,Σ,μ) be a measurable space with finite measure. If F_n:X→2^Y is a sequence of weakly measurable mulifunctions with closed valued and integrably bounded from the same function f:X→R such that converges to multifunction F:X→2^Y almost everywhere according to μ on X then F is integrable according to Aumann and $\int F d\mu = \lim_{n\to+\infty} \int F_n d\mu$.

Proof

By proposition 13 note that the multifunctions F_n are integrable according to Aumann. Also from propositions 1 and 2 it follows that $liminf F_n$ and $limsup F_n$ are weakly measurable. On the other hand, since multifunctions F_n are integrably bounded from the same function f:X→R then the equalities $limsup F_n(x) = \cap_{n\in N}(\cup_{k\geq n} F_k(x))$ and $liminf F_n(x) = \cup_{n\in N}(\cap_{k\geq n} F_k(x))$ tell us that the multifunctions $liminf F_n$ and $limsup F_n$ are also integrably bounded form function f:X→R.
Since the sequence of multifunctions F_n:X→2^Y converges to multifunction F:X→2^Y almost everywhere according to μ on X then it true that $limsup F_n(x) = liminf F_n(x) = F(x)$ almost everywhere according to μ on X.
Also, for every n∈N the multifunction $\cap_{k\geq n} F_k(x)$ is weakly measurable and with closed valued. This multifunction is also integrably bounded from the function f:X→R. So from the proposition 13 we conclude that the multifunction $\cap_{k\geq n} F_k(x)$ is integrable according to Aumann. The Corollary 16 allows us to say that the multifunction $liminf F_n(x) = \cup_{n\in N}(\cap_{k\geq n} F_k(x))$ is integrable according to Aumann. So multifunction F(x) to which converges the sequence of multifunctions F_n is integrable according to

Aumann.With a similar reasoning as in Theorem 5 of [7] noted that in these conditions is true the equality $\int F d\mu = \lim_{n\to+\infty} \int F_n d\mu$.
More specifically, from propositions 18 and 19 we write:

$$limsup \int F_n(x)d\mu \subseteq \int limsup\, F_n(x)d\mu = \int F d\mu$$
$$= \int liminf\, F_n(x)d\mu \subseteq liminf \int F_n(x)d\mu$$

Therefore it is clear that we have shown the equality $liminf \int F_n(x)d\mu = limsup \int F_n(x)d\mu$ which guarantees us the existence of the limit of integrals sequence $\int F_n(x)d\mu$ and for more $\int F d\mu = \lim_{n\to+\infty} \int F_n d\mu$.

References

[1]B.Cascale, V.Kadets and J.Rodriguez, ''Measurability and selections of multi-functions in Banach spaces'', 2000 Mathematics subject classification; 1-10

[2]Jean-Pierre Aubin, Hélène Frankowska, ''Set-valued Analysis '', AMS Subject classification (1985), © Birkhäuser Boston 1990; 33-39

[3]K. Kuratowski and C. Ryll-Nardzewski, ''A general theorem on selectors'', Bull. Acad. Polon.
Sci. Sér. Sci. Math. Astronom. Phys. 13 (1965), 397-403

[4]Miroslav Pavlović ,''Introduction to function spaces on the disc'', Matematički institut SANU, Beograd 2004; 1-11

[5] Nuno C.Freire and M.F.Veiga, ''On a problem by N.Kalton and the space $l^p(I)$'', Acta mathematica Vietnamica, Vol.33, nr.1, 2008; 39-44

[6]R.J.Aumann, ''Measurable utility and the measurable choice theorem'' proc.int.colloq.,La decision, C.N.R.S., Aix-en-provence, 1967, 15-26.

[7]R.J.Aumann,''Integrals of set-valued functions'', Journal of Mathematics Analysis and applications 12, 1965, 1-12.