# EXPLICIT EXPRESSION FOR FIRST INTEGRAL OF A RATIONAL TYPE OF KOLMOGOROV SYSTEMS

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ABSTRACT. In this paper we cherecterize the integrability and introduce an explicit expression of first integral then consequently the non-existence of periodic orbits for rational type of the planar Kolmogorove systems of the form

$$\begin{aligned} \dot{x} &= x\left(\frac{P_{n_1}(x,y)}{P_{n_2}(x,y)} + \frac{R_{k_1}(x,y)}{R_{k_2}(x,y)}\right) \\ \dot{y} &= y\left(\frac{Q_{m_1}(x,y)}{Q_{m_2}(x,y)} + \frac{R_{k_1}(x,y)}{R_{k_2}(x,y)}\right) \end{aligned}$$

where $n_1$, $n_2$, $m_1$, $m_2$, $k_1$, and $k_2$ are positive integers and $P_i$, $Q_j$, and $R_k$ are homogeneous polynomials of degree $i$, $j$ and $k$ respectively such that $n_1 - n_2 = m_1 - m_2$. We also present an example in order to illustrate the applicability of the result.

## 1. Introduction

A two dimensional rational vector field defined on the real plane is a vector field of the form

$$X(x,y) = P(x,y)\frac{\partial}{\partial x} + Q(x,y)\frac{\partial}{\partial y} \tag{1.1}$$

where $P$, $Q$ are rational functions means division of two homogeneous polynomials

$$P(x,y) = \frac{P_{n_1}(x,y)}{P_{n_2}(x,y)}, \qquad Q(x,y) = \frac{Q_{m_1}(x,y)}{Q_{m_2}(x,y)}$$

and $P_i(x,y)$, $Q_j(x,y)$ are homogeneous polynomials of degree $i$, $j$ respectively, and $P(x,y)$, $Q(x,y)$ are coprime in the ring $R[x,y]$.

Let $U$ be an open subset of $R^2$. If there exists a non-constant $C^1$ function $H : U \to R$, which is constant on all the solutions of $X$ contained in $U$, then we say that $H$ is a first integral of $X$ on $U$, and that $X$ is integrable on $U$. Means we have $\frac{\partial H}{\partial x}P + \frac{\partial H}{\partial y}Q = 0$ on $U$. For more details about first integral see for instance [1,2,4,5,6,7,11], see also the references quoted in those articles.

If $f \in R[x,y]$, then $f(x,y) = 0$ is an algebraic curve. We say that $f = 0$ is invariant if $Xf = Kf$, $K \in R[x,y]$. In this case $K$ is called the cofactor of $f$. Its degree is lower than $n_1 - n_2$.The expression which defines $K$ is written as

$$\frac{\partial f}{\partial x}P + \frac{\partial f}{\partial y}Q = Kf \tag{1.2}$$

Recall that a limit cycle of system 1.1 is an isolated periodic solution in the set of all periodic solutions of the system. On the points of the algebraic curve

*Date*: August 10, 2014.
1991 *Mathematics Subject Classification.* 34C05, 34C07, 34C25.
*Key words and phrases.* Intergability, Kolmogorov system, periodic orbit, rational function.

$f(x,y)=0$, one can see from 1.2 that the gradient of $f$ is orthogonal to the vector field $(P,Q)$ . Hence at every point of $f=0$ the vector field $(P,Q)$ is tangent to the curve $f=0$, so the curve $f=0$ is formed by trajectories of the vector field $(P,Q)$.

In [10] the authors characterize the integrability and an explicit expression of first integral and the non-existence of periodic orbits for the 2-dimensional Kolmogorov systems of polynomial form

$$\begin{aligned} \dot{x} &= x(P_n(x,y)+R_m(x,y)) \\ \dot{x} &= y(Q_n(x,y)+R_m(x,y)) \end{aligned} \tag{1.3}$$

where $n$ and $m$ are positive integers and $P_n$, $Q_n$ and $R_m$ are homogeneous polynomials of degree $n$, $n$, and $m$ respectively.

In this paper we extend these results to characterize integrability and presentan an explicit expression of first integral and then the non-existence of periodic orbits to systems with $P_n$, $Q_n$ and $R_m$ are not necessarily polynomials, but rational functions of the form

$$\begin{aligned} \dot{x} &= x(\frac{P_{n_1}(x,y)}{P_{n_2}(x,y)}+\frac{R_{k_1}(x,y)}{R_{k_2}(x,y)}) \\ \dot{y} &= y(\frac{Q_{m_1}(x,y)}{Q_{m_2}(x,y)}+\frac{R_{k_1}(x,y)}{R_{k_2}(x,y)}) \end{aligned} \tag{1.4}$$

with $n_1-n_2=m_1-m_2=n$ and $k_1-k_2=k$.

These systems, as mentioned in [10] , are called Kolmogorov systems which appear in applications that the per unit of change $\frac{dx_i}{dt}/x_i$ of the dependent variables $x_i(t)$ are given functions $f_i(x_i,...,x_n)$ of these variables at any time These systems are also called Lotka-Volterra systems because were started to be studied by them in [12] and in [14], respectively. Later on Kolmogorov came, giving some generalization in [8] and then some authors denote them by Kolmogorov systems. There are many natural phenomena which can be modeled by the Kolmogorov systems such as mathematical ecology and population dynamics [12] ,chemical reactions, plasma physics [9], hydrodynamics [3] economics, etc.

## 2. Mean Result

Before we present the Theorem, define the trigonometric functions

$$\begin{aligned} F(\theta) &= \cos^2 x\frac{P_{n_1}(\cos\theta,\sin\theta)}{P_{n_2}(\cos\theta,\sin\theta)}+\sin^2 x\frac{Q_{m_1}(\cos\theta,\sin\theta)}{Q_{m_2}(\cos\theta,\sin\theta)} \\ G(\theta) &= \cos\theta\sin\theta\left[\frac{Q_{m_1}(\cos\theta,\sin\theta)}{Q_{m_2}(\cos\theta,\sin\theta)}-\frac{P_{n_1}(\cos\theta,\sin\theta)}{P_{n_2}(\cos\theta,\sin\theta)}\right] \\ R(\theta) &= \frac{R_{k_1}(\cos\theta,\sin\theta)}{R_{k_2}(\cos\theta,\sin\theta)} \end{aligned} \tag{2.1}$$

**Theorem 1.** *Consider system 1.4. Then the following statements hold.*

*(a) If $G(\theta) \neq 0$ and $n \neq k$, then system 1.4 has the first integral*

$$H(x,y) = \left(x^2+y^2\right)^{\frac{k-n}{2}} \exp\left((k-n)\int^{\arctan\frac{y}{x}} J(s)ds\right) + (k-n)\int^{\arctan\frac{y}{x}} K(u)\exp\left((k-n)\int^{u} J(s)ds\right)du \tag{2.2}$$

*where*

$$J(\theta) = \frac{F(\theta)}{G(\theta)} \qquad \text{and} \qquad K(\theta) = \frac{R(\theta)}{G(\theta)} \tag{2.3}$$

*(b) If $G(\theta) \neq 0$ and $n = k$, then system 1.4 has the first integral*

$$H(x,y) = \left(x^2+y^2\right)^{\frac{1}{2}} \exp\left(-\int^{\arctan\frac{y}{x}} (J(s)+K(s))ds\right) \tag{2.4}$$

*(c) If $G(\theta) \equiv 0$, then system 1.4 has the first integral $H(x,y) = \frac{y}{x}$.*
*(d) System 1.4 has no periodic orbits.*

*Proof.* (a)
System 1.4 in the polar coordinates $x = r\cos\theta,\ \ y = r\sin\theta$ is in the form

$$\begin{aligned} \dot{r} &= r^{n+1}F(\theta) + r^{k+1}R(\theta) \\ \dot{\theta} &= r^{n}G(\theta) \end{aligned} \tag{2.5}$$

where $F$, $G$ and $R$ are given in 2.1, $n = n_1 - n_2 = m_1 - m_2$ and $k = k_1 - k_2$.

Here we assume that $G(\theta) \neq 0$, then if we consider $r$ as dependent variable of the independent variable $\theta$, system 2.5 becomes the differential equation

$$\frac{dr}{d\theta} = rJ(\theta) + r^{k+1-n}K(\theta) \tag{2.6}$$

where $J(\theta)$ and $K(\theta)$ are given in 2.3.

Equation 2.6 is a Bernoulli differential equation. Make use of the transformation $\rho = r^{n-k}$ in order to transform equation 2.6 to the following linear differential equation

$$\frac{d\rho}{d\theta} - (n-k)J(\theta)\rho = (n-k)K(\theta)$$

This equation has the first integral 2.2 given in the statement of the Theorem. Hence statement (a) of the Theorem is proved. □

*Proof.* (b)
Suppose now that $G(\theta) \neq 0$ and $n = k$. Then the differential equation 2.6 becomes

$$\frac{dr}{d\theta} = r\left[J(\theta) + K(\theta)\right]$$

which has the first integral 2.4 given in the statement of the Theorem. Hence statement (b) of the Theorem is proved. □

*Proof.* (c)

Now assume that $G(\theta) \equiv 0$. Then from 2.5 we have $\dot{\theta} = 0$. So the straight lines through the origin of coordinates of the differential system 1.4 are invariant by the flow of this system. Hence $h = \frac{y}{x}$ is a first integral of the system. This completes the proof of statement (c). □

*Proof.* (d)

Assume that system 1.4 has an equilibrium point $a$ . Since the axes $x$ and $y$ are formed by trajectories of the system, surrounding the equilibrium located on the axes cannot be periodic orbit.

Suppose the equilibrium point $a$ located off the axes $x$ , $y$, means in one of the open quadrants.

Let $\gamma$ be a periodic orbit surrounding the equilibrium point $a$ . Let $H(\gamma) = h_\gamma$.

Assume that $G(\theta) \neq 0$ and $n \neq k$, then the curve $H = h$ with $h \in R$, which are formed by trajectories of the differential system 2.5 can be written as

$$r(\theta) = \left[ h \exp\left( (n-k) \int^{\theta} J(s)ds \right) - (k-n) \exp\left( (n-k) \int^{\theta} J(s)ds \right) \right.$$

$$\left. \int^{\theta} K(u) \exp\left( (k-n) \int^{u} J(s)ds \right) du \right]^{\frac{1}{n-k}}$$

Hence the orbit $\gamma$ is contained in the curve

$$r(\theta) = \left[ h_\gamma \exp\left( (n-k) \int^{\theta} J(s)ds \right) - (k-n) \exp\left( (n-k) \int^{\theta} J(s)ds \right) \right.$$

$$\left. \int^{\theta} K(u) \exp\left( (k-n) \int^{u} J(s)ds \right) du \right]^{\frac{1}{n-k}}$$

But this curve cannot contain the periodic orbit $\gamma$ contained in one of the open quadrants because this curve has at most a unique point on every ray $\theta = \theta^*$ for all $\theta^* \in [0, 2\pi)$.

Now suppose that $G(\theta) \neq 0$ and $n = k$. From 2.4, the curves $H = h$ with $h \in R$ can be written as

$$r(\theta) = h \exp\left( \int^{\theta} (J(s) + K(s))ds \right)$$

So the period orbit $\gamma$ must be contained in the curve

$$r(\theta) = h_\gamma \exp\left( \int^{\theta} (J(s) + K(s))ds \right)$$

Again this curve cannot contain the periodic orbit $\gamma$ for same reason that in the previous case.

Finally assume that $G(\theta) \equiv 0$. Then since all the straight lines through the origin are formed by trajectories, clearly the system has no periodic orbits. This completes the proof of statement (d). □

## 3. Example

The example is presented to illustrate the applicability of the Theorem 1.

Consider the following differential system

$$\begin{aligned} \dot{x} &= \frac{-x^2+y^2}{y^2} \\ \dot{y} &= \frac{-2x}{y} \end{aligned} \tag{3.1}$$

In order to rewrite the system 3.1 in the canonical form 1.4, we may write

$$\begin{aligned} \dot{x} &= x\left(\frac{x}{y^2} - \frac{2x^2-y^2}{xy^2}\right) \\ \dot{y} &= y\left(\frac{-1}{x} - \frac{2x^2-y^2}{xy^2}\right) \end{aligned}$$

Therefore

$$\begin{aligned} F(\theta) &= \frac{\cos^2\theta - \sin^2\theta}{\sin^2\theta\cos\theta}, \qquad G(\theta) = \frac{-1}{\sin\theta}, \\ J(\theta) &= \frac{\sin^2\theta - \cos^2\theta}{\sin\theta\cos\theta}, \qquad K(\theta) = \frac{2\cos^2\theta - \sin^2\theta}{\sin\theta\cos\theta} \end{aligned}$$

Here $n_1 = 1$, $n_2 = 2$, $m_1 = 0$, $m_2 = 1$, $k_1 = 2$, and $k_2 = 3$. So $n_1 - n_2 = m_1 - m_2 = -1 = n$, and $k_1 - k_2 = -1 = k$, therefore $n = k$, so it is the case (b) of the Theorem.

Hence by Theorem(b) we conclude that

$$\begin{aligned} H(x,y) &= \left(x^2+y^2\right)^{\frac{1}{2}} \exp\left(-\int^{\arctan\frac{y}{x}} (J(s)+K(s))ds\right) \\ &= \frac{x^2+y^2}{y} \end{aligned}$$

It is clear, by direct calculation, that

$$\frac{dH}{dt}(=\frac{\partial H}{\partial x}\dot{x} + \frac{\partial H}{\partial y}\dot{y}) = 0$$

This justifies the applicability of the Theorem (b).

**Acknowledgement 1.** *The author thanks the University of Sharjah for its support.*

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