Quasi Conformal Curvature Tensor on a P-Sasakian Einstein Manifold

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***Abstract*— In this paper, we have studied p-Sasakian Einstein manifold which satisfy the condition r –n(n - 1), a + 2(n - 1)b ≠ 0 i. e. the constant scalar curvature r. also the p-Sasakian Einstein manifold satisfying div $\tilde{C}$ = 0 have studied. where $\tilde{C}$ is quasi-conformal curvature tensor and r is the scalar curvature.**

***Keywords*—P-Sasakian manifold, Quasi-conformal curvature tensor, Einstein manifold.**

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1. PRELIMINARIES

Let M^n be n-dimensional C^∞ -manifold. If there exist a tensor field F of type (1, 1), a vector field ξ and a 1-form η in M^n satisfying

(1.1) $\quad \bar{\bar{X}} = X - \eta(X)\,\xi, \qquad \bar{X} = F(X), \qquad \eta(\xi) = 1$

then M^n is called an almost para contact manifold.

Let g be the Riemannian metric satisfying

(1.2) $\quad g(X, \xi) = \eta(X)$

(1.3) $\quad \eta(F, X) = 0, \qquad F\xi = 0, \qquad \text{rank } F = (n - 1)$

(1.4) $\quad g(FX, FY) = g(X, Y) - \eta(X)\eta(Y)$

Then the set (F, ξ, η, g) satisfying (1.1), (1.2), (1.3) and (1.4) is called an almost para-contact Riemannian structure. The manifold with such structure is called an almost p-contact Riemannian manifold [1].

If we define $'F(X, Y) = g(\bar{X}, Y)$, then in addition to the above relations the following are satisfied:

(1.5) $\quad 'F(X, Y) = {'F}(Y, X)$

(1.6) $\quad 'F(\bar{X}, \bar{Y}) = {'F}(X, Y)$

Let us consider an n-dimensional differentiable manifold M with a positive definite metric g which admits 1-forms η satisfying

(1.7) $\quad (\nabla_X \eta)(Y) - (\nabla_Y \eta)(X) = 0$

And

(1.8) $\quad (\nabla_X\nabla_Y \eta)(Z) = - g(X, Z)\eta(Y) - g(X, Y)\eta(Z) + 2\eta(X)\,\eta(Y)\eta(Z)$

Where, ∇ denote the covariant differentiation with respect to g. Moreover, if we put,

(1.9) $\quad \eta(X) = g(X, \xi), \qquad (\nabla_X\xi) = \bar{X}$

Then it can be easily verified that the manifold in consideration becomes an almost para-contact Riemannian manifold. Such a manifold is called p-Saskian manifolds [2].

For a p-Saskian manifold the following relations hold [4]:

(1.10) $R(X, Y)\xi = \eta(X)Y - \eta(Y)X$

(1.11) $R(\xi, X)Y = \eta(Y)X - g(X, Y)\xi$

(1.12) $R(\xi, X)\xi = X - \eta(X)\xi$

(1.13) $S(X, \xi) = -(n - 1)\eta(X)$

(1.14) $Q\xi = -(n - 1)\xi$

(1.15) $\eta(R(X, Y)U) = g(X, U)\eta(Y) - g(Y, U)\eta(X)$

(1.16) $\eta(R(X, Y)\xi) = 0$

(1.17) $\eta(R(\xi, X)Y) = \eta(X)\eta(Y) - g(X, Y)$

For any vector field X, Y, Z whare R and S are the curvature tensor and Ricci tensor and Q is the Ricci operator.

2. A P-SASAKIAN EINSTEIN MANIFOLD SATISFYING $R = -n(n - 1)$, $a + 2(n - 1)b \neq 0$

a p-Sasakian manifold M^n is said to be Einstein manifold, if its Ricci tensor S is of the form

(2.1) $S(X, Y) = kg(X, Y)$

where k is constant.

Putting $Y = \xi$ in (2.1), we get $S(X, \xi) = kg(X, \xi)$

Since $S(X, \xi) = -(n - 1)\eta(X)$ and $g(X, \xi) = \eta(X)$, we have

(2.2) $k = -(n - 1)$

From (2.1) and (2.2), we get

(2.3) $S(X, Y) = -(n - 1)g(X, Y)$

Contracting (2.3), we get,

(2.4) $QY = -(n - 1)Y$

Where $S(X, Y) = g(QX, Y)$.

Let (M^n, g) be n-dimensional Riemannian manifold, the Quasi-conformal curvature tensor $\tilde{C}$ is defined by [9].

(2.5) $\tilde{C}(X, Y)Z = aR(X, Y)Z + b[S(Y, Z)X – S(X, Z)Y + g(Y, Z)QX - g(X, Z)QY] - \frac{r}{n}(\frac{a}{n-1} + 2b)[g(Y, Z)X - g(X, Z)Y]$

Using (2.3) and (2.4) in (2.5), we get

(2.6) $\tilde{C}(X, Y)Z = aR(X, Y)Z - [2(n - 1)b + \frac{r}{n}(\frac{a}{n-1} + 2b)][g(Y, Z)X - g(X, Z)Y]$

The endomorphism $X\wedge Y$ and $X\wedge_S Y$ and the homeomorphism $R(X, \xi)\tilde{C}$ and $\tilde{C}(X, \xi)R$ are defined by

(2.7) $(X \wedge Y)Z = g(Y, Z)X - g(X, Z)Y$

(2.8) $(X \wedge_S Y)Z = S(Y, Z)X - S(X, Z)Y$

(2.9) $(R(X, \xi)\cdot\tilde{C})(U, Z)W = R(X, \xi)\tilde{C}(U, Z)W - \tilde{C}(R(X, \xi)U, Z)W - \tilde{C}(U, R(X, \xi)Z)W - \tilde{C}(U, Z)R(X, \xi)W$

(2.10) $(\tilde{C}(X, \xi)\cdot R)(U, Z)W = \tilde{C}(X, \xi)R(U, Z)W - R(\tilde{C}(X, \xi)U, Z)W - R(U, Z(X, \xi)\tilde{C})W - R(U, Z)\tilde{C}(X, \xi)W$

respectively, where X, Y, Z are vector fields of M.

3. Main Results:

Theorem -1 An n-dimensional p-Sasakian Einstein manifold M with Quasi-conformal curvature tensor $\tilde{C}$, satisfying r=-n(n - 1), a +2(n -1)b $\neq$ 0 then we have

$$(R(X, \xi)\cdot\tilde{C}) = \tilde{C}(X, \xi). R.$$

Proof : Substituting U and W by ξ in (2.9) yields

(2.11) $(R(X, \xi).\tilde{C})(\xi, Z)\xi = R(X, \xi)\tilde{C}(\xi, Z)\xi - \tilde{C}(R(X, \xi)\xi, Z)\xi$

$$-\tilde{C}(\xi, R(X, \xi)Z)\xi - \tilde{C}(\xi, Z)R(X, \xi)\xi$$

From (2.6) we get by virtue of (1.2) and (1.12),

(2.12) $\tilde{C}(\xi, Z)\xi = (a +2(n-1)b)\left[1+\frac{r}{n(n-1)}\right][Y - \eta(Y)\xi]$

If r = -n(n – 1), provided a +2(n-1)b $\neq$ 0 then from (2.12), we have (2.13) $\tilde{C}(\xi, Z)\xi = 0$ and similarly

(2.14) $\tilde{C}(Z, \xi)\xi = 0$, for any vector field Z.

Thus we have,

(2.15) $(R(X, \xi).\tilde{C})(\xi, Z)\xi = -\tilde{C}(R(X, \xi)\xi, Z)\xi - \tilde{C}(\xi, Z)R(X, \xi)\xi$

Using (1.12), we have

$$\tilde{C}(R(X, \xi)\xi, Z)\xi = -\tilde{C}(X, Z)\xi$$

$$\tilde{C}(\xi, Z)R(X, \xi)\xi = -\tilde{C}(\xi, Z)X$$

Thus we have from (2.15)

(2.16) $(R(X, \xi).\tilde{C})(\xi, Z)\xi = \tilde{C}(X, Z)\xi + \tilde{C}(\xi, Z)X$

On the other hand

(2.17) $(\tilde{C}(X, \xi).R)(\xi, Z)\xi = \tilde{C}(X, \xi)R(\xi, Z)\xi - R(\tilde{C}(X, \xi)\xi, Z)\xi - R(\xi, Z(X, \xi)\tilde{C})\xi - R(\xi, Z)\tilde{C}(X, \xi)\xi$

Using (1.12, (1.15) and (2.14), we obtain the following equations

$$(\tilde{C}(X, \xi)R(\xi, Z)\xi = \tilde{C}(X, \xi)Z$$

$$R(\tilde{C}(X, \xi)\xi, Z)\xi = 0$$

$$R(\xi, Z(X, \xi)\tilde{C})\xi = \tilde{C}(X, \xi)Z$$

$$R(\xi, Z)\tilde{C}(X, \xi)\xi = 0$$

Using these equation in (2.17), we have

(2.18) $(\tilde{C}(X, \xi).R)(\xi, Z)\xi = 0$

Thus our condition satisfies the following equation

$$(R(X, \xi).\tilde{C})(\xi, Z)\xi = 0$$

Therefore from (2.16), we have

$$\tilde{C}(X, Z)\xi + \tilde{C}(\xi, Z)X = 0$$

Using (1.2), (1.11), (1.12) and (2.6), we have

$$(a + 2(n-1)b)\left[1+\frac{r}{n(n-1)}\right][2\eta(X)Z - \eta(Z)X - g(X, Z)\xi] = 0$$

Which true for $r = -n(n-1)$, $a + 2(n-1)b \neq 0$.

Hence the theorem is proved.

Theorem-2 : An n-dimensional p-Sasakian Einstein manifold M with Quasi-conformal curvature tensor $\tilde{C}$, satisfying $r = -n(n-1)$, $a + 2(n-1)b \neq 0$ then we have

$$R(X, \xi).\tilde{C} = L\{(X \wedge \xi).C\},\ L \neq -1, \text{ where L is some function on M.}$$

Proof: We denote the expression in the bracket on the right hand side of (2.9) by A, and we calculate it. Thus

(2.19) $A = L((X\wedge\xi).\tilde{C})(\xi, Z)\xi = L\{((X\wedge\xi)\tilde{C})(\xi, Z)\xi - \tilde{C}((X\wedge\xi)\xi, Z)\xi - \tilde{C}(\xi, (X\wedge\xi)Z)\xi - \tilde{C}(\xi, Z)(X\wedge\xi)\xi\}$

Using (2.13), we have

$$(X\wedge\xi)\tilde{C}(\xi, Z)\xi = 0$$

$$\tilde{C}((X\wedge\xi)\xi, Z)\xi = \tilde{C}(X - \eta(X)\xi, Z)\xi$$

$$= \tilde{C}(X, Z)\xi - \eta(X)\tilde{C}(\xi, Z)\xi$$

$$= \tilde{C}(X, Z)\xi$$

$$\tilde{C}(\xi,(X\wedge\xi)Z)\xi = 0$$

$$\tilde{C}(\xi, Z)(X\wedge\xi)\xi\} = \tilde{C}(\xi, Z)X$$

From the above and using (2.16), we have

$$\tilde{C}(X, Z)\xi + \tilde{C}(\xi, Z)X = L\{-\tilde{C}(X, Z)\xi - \tilde{C}(\xi, Z)X\}$$

(2.20) $(1 + L)[\tilde{C}(X, Z)\xi + \tilde{C}(\xi, Z)X] = 0$

Using (1.2), (1.11), (1.12) and (2.6), we have

(2.21) $(1 + L)(a + 2(n-1)b)\left[1+\dfrac{r}{n(n-1)}\right][2\eta(X)Z - \eta(Z)X - g(X, Z)\xi] = 0$

Since $L \neq -1$.

Thus which true for $r = -n(n-1)$, $a + 2(n-1)b \neq 0$.

Hence the theorem is proved.

Theorem -3: An n-dimensional p-Sasakian Einstein manifold M with Quasi-conformal curvature tensor $\tilde{C}$, Satisfying $r = -n(n-1)$, $a + 2(n-1)b \neq 0$ then we have $R(X, \xi).\tilde{C} = f\{(X\wedge_s^n\xi).C\}$, $f = \dfrac{-1}{(1-n)^n}$, where f is some function on M.

Proof: We denote the expression in the bracket on the right hand side of (2.9) by A, and we calculate it. Thus

(2.22) $(R(X, \xi).\tilde{C})(\xi, Z)\xi = f\{((X \wedge_s^n \xi).C)(\xi, Z)\xi\}$

Where $(X \wedge_s^n \xi) = S^n(Y, Z)X - S^n(X, Z)Y$, And $S^n(Y, Z) = g(Q^nX, Y)$

Then

$A = f\{((X \wedge_s^n \xi).\tilde{C})(\xi, Z)\xi\} = f\{(X \wedge_s^n \xi)\tilde{C}(\xi, Z)\xi - \tilde{C}((X \wedge_s^n \xi)\xi, Z)\xi$

$- \quad \tilde{C}(\xi, (X \wedge_s^n \xi)Z)\xi - \tilde{C}(\xi, Z)((X \wedge_s^n \xi)\xi\}$

Using (2.13), we have

$(X \wedge_s^n \xi)\tilde{C}(\xi, Z)\xi = 0$

$\tilde{C}((X \wedge_s^n \xi)\xi, Z)\xi = [(1-n)]^n\tilde{C}(X, Z)\xi$

$\tilde{C}(\xi, (X \wedge_s^n \xi)Z)\xi = 0$

$\tilde{C}(\xi, Z)((X \wedge_s^n \xi)\xi\} = [(1-n)]^n\tilde{C}(\xi, Z)X$

From the above and using (2.16), we have

$\tilde{C}(X, Z)\xi + \tilde{C}(\xi, Z)X = f\{-[(1-n)]^n\tilde{C}(X, Z)\xi - [(1-n)]^n\tilde{C}(\xi, Z)X\}$

$= - f[(1 - n)]^n\{\tilde{C}(X, Z)\xi + \tilde{C}(\xi, Z)X\}$

$(1 + f[+(1 - n)]^n)[\tilde{C}(X, Z)\xi + \tilde{C}(\xi, Z)X] = 0$

Using (2.21), we have

$$(1 + f[+(1 - n)]^n)(a + 2(n - 1)b)[1+\frac{r}{n(n-1)}][2\eta(X)Z - \eta(Z)X - g(X, Z)\xi] = 0$$

Since $f = \dfrac{-1}{(1 - n)^n}$.

Thus which true for $r = -n(n - 1)$, $a + 2(n - 1)b \neq 0$.

Hence the theorem is proved.

4. A p-Sasakian Einstein manifold satisfying (div $\tilde{C}$) (X, Y) Z = 0

We assume that

(4.1) $\qquad \operatorname{div} \tilde{C} = 0$

Where ‘div’ denotes the divergence.

Now differentiating (2.5) covariantly with respect to U, we get

(4.2) $$(D_u\tilde{C})(X, Y)Z = a(D_uR)(X, Y)Z + b[(D_uS)(Y, Z)X - (D_uS)(X, Z)Y - (n - 1) D_u\{g(Y, Z)X\} + (n - 1) D_u\{g(X, Z)Y\}] - \frac{1}{n}(\frac{a}{n-1} + 2b)(D_ur)[g(Y, Z)X - g(X, Z)Y].$$

contraction of (4.2) with respect to X, we get

(4.3) $$(\operatorname{div}\tilde{C})(X, Y)Z = b(n - 1)(D_uS)(Y, Z) - \frac{n-1}{n}(\frac{a}{n-1} + 2b)(g(Y, Z))(U_r)$$

From (2.3), We have

(4.4) $\qquad (D_uS)(Y, Z) = 0$

Using (4.1) and (4.4) in (4.3), we obtain

$$\frac{n-1}{n}\left(\frac{a}{n-1}+2b\right)(g(Y, Z))(U_r) = 0$$

Since $g(Y, Z) \neq 0$, then we have $U_r = 0$, $a + 2(n-1)b \neq 0$.

Which gives r is covariant constant.

Again if r is covariant constant i.e. $U_r = 0$, then from (4.3) and (4.4), we obtain

$$(\mathrm{div}\tilde{C})(X, Y)Z = 0.$$

Hence we can state the following theorem.

Definition: A manifold Mn is said to be Quasi-Conformally conservative if div $\tilde{C} = 0$ [8].

Theorem 4: A p-Sasakian Einstein manifold is Quasi-Conformally conservative if and only if the scalar r is covariant constant, $a + 2(n-1)b \neq 0$.

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