Some New Divisor Cordial Graphs

A. Muthaiyan[#1] and P. Pugalenthi[*2]

[#*] *P.G. and Research Department of Mathematics, Govt. Arts College,*
Ariyalur – 621 713, Tamil Nadu, India.

Abstract **- In this paper, the duplication of an arbitrary vertex by a new edge of cycle C_n ($n \geq 3$), the duplication of an arbitrary edge by a new vertex of cycle C_n ($n \geq 3$), $< S_n^{(1)} : S_n^{(2)} : S_n^{(3)} >$, $< W_n^{(1)} : W_n^{(2)} : W_n^{(3)} >$ and the graph obtained by joining two copies of S_n by a path P_k ($n \geq 4$).**

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I. Introduction

All graphs in this paper are simple, finite, connected and undirected graphs. Let G = (V(G),E(G)) be a graph with p vertices and q edges. For standard terminology and notations related to graph theory we refer to Harary [3] while for number theory we refer to Burton [2]. Graph labeling, where the vertices and edges are assigned real values or subsets of a set are subject to certain conditions. For a dynamic survey on various graph labeling problems we refer to Gallian []. The concept of cordial labeling was introduced by Cahit [1]. After this many labeling schemes are also introduced with minor variations in cordial theme. The concept of divisor cordial labeling was introduced by Varatharajan et al.[9]. In this paper [9], they have proved that path, cycle, wheel, star, $K_{2,n}$ and $K_{3,n}$ are divisor cordial graphs. The divisor cordial labeling of full binary trees, $G * K_{2,n}$, $G * K_{3,n}$, $< K_{1,n}^{(1)}, K_{1,n}^{(2)} >$ and $< K_{1,n}^{(1)}, K_{1,n}^{(2)}, K_{1,n}^{(3)} >$ are reported by the same authors in [10]. Vaidya et.al. [7, 8] have proved that degree splitting graph of $B_{n,n}$, shadow graph of $B_{n,n}$, square graph of $B_{n,n}$, splitting graphs of star $K_{1,n}$, splitting graphs of bistar $B_{n,n}$, helm H_n, flower graph Fl_n, Gear graph G_n, switching of a vertex in cycle C_n, switching of a rim vertex in wheel W_n and switching of the apex vertex in helm H_n are divisor cordial graphs. The divisor cordial labeling of some cycle related graphs are reported by Maya et.al [5]. Lawrence Rozario raj et.al [6] have proved that $< S_n^{(1)} : S_n^{(2)} >$, $< W_n^{(1)} : W_n^{(2)} >$, the graph obtained by joining two copies of W_n by a path P_k ($n \geq 3$), $G_v \odot K_1$, where G_v denotes graph obtained by switching of any vertex v of C_n ($n \geq 4$) and Pl_n ($n \geq 5$) divisor cordial graphs. In this paper we had discussed divisor cordial labeling of the duplication of an arbitrary vertex by a new edge of cycle C_n ($n \geq 3$), the duplication of an arbitrary edge by a new vertex of cycle C_n ($n \geq 3$), $< S_n^{(1)} : S_n^{(2)} : S_n^{(3)} >$, $< W_n^{(1)} : W_n^{(2)} : W_n^{(3)} >$ and the graph obtained by joining two copies of S_n by a path P_k ($n \geq 4$). We will provide brief summary of definitions and other information which are necessary for the present investigations.

Definition :1.1

A mapping $f : V(G) \rightarrow \{0,1\}$ is called binary vertex labeling of G and f(v) is called the label of the vertex v of G under f .

Notation : 1.1

If for an edge e = uv, the induced edge labeling $f^* : E(G) \rightarrow \{0,1\}$ is given by $f^*(e) = | f(u) - f(v) |$. Then $v_f(i)$ = number of vertices of having label i under f and $e_f(i)$ = number of edges of having label i under f*.

Definition :1.2

A binary vertex labeling f of a graph G is called a cordial labeling if $| v_f(0) - v_f(1) | \leq 1$ and $| e_f(0) - e_f(1) | \leq 1$. A graph G is cordial if it admits cordial labeling.

Definition :1.3

Let a and b be two integers. If a divides b means that there is a positive integer k such that b = ka. It is denoted by $a | b$. If a does not divide b, then we denote $a \nmid b$.

Definition :1.4

Let G = (V(G), E(G)) be a simple graph and $f : \rightarrow \{1,2,...,|V(G)|\}$ be a bijection. For each edge uv, assign the label 1 if $f(u) | f(v)$ or $f(v) | f(u)$ and the label 0 otherwise. The function f is called a divisor cordial labeling if $|e_f(0) - e_f(1)| \leq 1$. A graph with a divisor cordial labeling is called a divisor cordial graph.

Definition :1.5

The shell S_n is the graph obtained by taking $n - 3$ concurrent chords in cycle C_n. The vertex at which all the chords are concurrent is called the apex vertex.

Definition :1.6

A wheel graph W_n is a graph with n+1 vertices, formed by connecting a single vertex to all the vertices of cycle C_n. It is denoted by $W_n = C_n + K_1$.

Definition :1.7

Duplication of a vertex v_k by a new edge $e = v'v''$ in a graph G produces a new graph G′ such that $N(v') = \{v_k, v''\}$ and $N(v'') = \{v_k, v'\}$.

Definition :1.8

Duplication of an edge $e = v_iv_{i+1}$ by a vertex v′ in a graph G produces a new graph G′ such that $N(v') = \{v_i, v_{i+1}\}$.

Definition :1.9

Consider two copies of graph G namely G_1 and G_2. Then the graph $G' = < G_1:G_2 >$ is a graph obtained by joining the apex vertices of G_1 and G_2 by a new vertex x.

Definition :1.10

Consider k copies of graph G namely $G_1, G_2, \ldots, G_k$. Then the graph $G' = < G_1:G_2:\ldots:G_k>$ is a graph obtained by joining the apex vertices of each G_{p-1} and G_p by a new vertex x_{p-1}, where $2 \le p \le k$.

II. MAIN THEOREMS

Theorem : 2.1

The graph obtained by duplication of an arbitrary vertex by a new edge in cycle C_n ($n\ge3$) is divisor cordial graph.

Proof.

Let C_n be cycle with n vertices $v_1, v_2, ..., v_n$ and n edges $e_1, e_2, \ldots, e_n$, where $n \ge 3$.
Without loss of generality we duplicate the vertex v_2 by an edge e_{n+1} with end vertices as v′ and v″.
Let the graph so obtained is G. Then $|V(G)| = n+2$ and $|E(G)| = n + 3$.
Define vertex labeling $f : V(G) \to \{1, 2, ..., n+2\}$ as follows

$f(v') = n+1$ and $f(v'') = n+2$

Label the vertices $v_1, v_2, ..., v_{n-1}$ and v_n in the following order.

$$\begin{array}{lllll} 1, & 2, & 2^2, & ..., & 2^{k_1}, \\ 3, & 3\times2 & 3\times2^2 & ..., & 3\times2^{k_2}, \\ 5, & 5\times2 & 5\times2^2 & ..., & 5\times2^{k_3}, \\ ... & ... & ... & ... & ..., \\ ... & ... & ... & ... & ... \end{array}$$

where $(2m-1)2^{k_m} \le n$ and $m \ge 1$, $k_m \ge 0$.

Also $(2m - 1)2^a$ divides $(2m - 1)2^b$ $(a < b)$ and $(2m-1)2^{k_i}$ does not divide 2m+1.

Interchange the labels of v_1 and v_2.

The consecutive adjacent vertices having labels odd and even numbers contribute 1 to each edge. Similarly, the consecutive the above labeling, the consecutive adjacent vertices of $v_2, v_3, ..., v_n$ having the labels even numbers and cons adjacent vertices having the labels odd numbers and consecutive adjacent vertices having labels even and odd numbers contribute 0 to each edge. Also $f(v_2)|f(v_1)$, $f(v_n)\nmid f(v_1)(f(v_1)\nmid f(v_n))$, $f(v_2)|f(v')$, $f(v_2)|f(v'')$ and $f(v')\nmid f(v'')$ $(f(v'')\nmid f(v'))$.

Thus, $e_f(0) = e_f(1) = \frac{n+3}{2}$, if n is odd.

$e_f(0) = \frac{n+2}{2}$ and $e_f(1) = \frac{n+4}{2}$, if n is even.

Therefore, $|e_f(0) - e_f(1)| \le 1$.

Hence G is divisor cordial graph.

Example : 2.1

The graph obtained by duplicating vertex by an edge in cycle C_8 and its divisor cordial labeling is given in Figure 2.1.

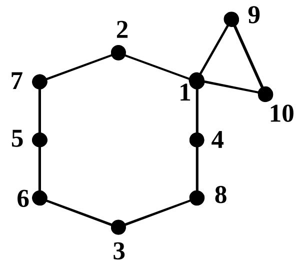

Figure 2.1

Theorem : 2.2

The graph obtained by duplication of an arbitrary edge by a new vertex in cycle C_n $(n \geq 3)$ is divisor cordial graph.

Proof.

Let C_n be cycle with n vertices $v_1, v_2, ..., v_n$ and n edges $e_1, e_2, …, e_n$, where $n \geq 3$. Without loss of generality we duplicate the edge v_1v_2 by a vertex v'. Let the graph so obtained is G. Then $|V(G)| = n+1$ and $|E(G)| = n + 2$.

Define vertex labeling $f : V(G) \to \{1, 2, ..., n+1\}$ as follows

Case 1 : n = 3

$f(v_1) = 2, f(v_2) = 3, f(v_3) = 1, f(v') = 4.$

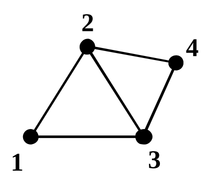

Figure 2.2

Thus, $e_f(0) = 0$ and $e_f(1) = 3$.

Therefore $|e_f(0) - e_f(1)| \leq 1$.

Hence G is divisor cordial graph, for n = 3.

Case 2 : $n \geq 4$

$f(v') = n+1$

Label the vertices $v_1, v_2, ..., v_{n-1}$ and v_n in the following order.

$$\begin{array}{lllll} 1, & 2, & 2^2, & ..., & 2^{k_1}, \\ 3, & 3\times 2 & 3\times 2^2 & ..., & 3\times 2^{k_2}, \\ 5, & 5\times 2 & 5\times 2^2 & ..., & 5\times 2^{k_3}, \\ ... & ... & ... & ... & ..., \\ ... & ... & ... & ... & ... \end{array}$$

where $(2m-1)2^{k_m} \leq n$ and $m \geq 1$, $k_m \geq 0$.

Also $(2m - 1)2^a$ divides $(2m - 1)2^b$ $(a < b)$ and $(2m-1)2^{k_i}$ does not divide 2m+1.

Interchange the labels of v_1 and v_2.

In the above labeling, the consecutive adjacent vertices of $v_2, v_3, ..., v_n$ having the labels even numbers and consecutive adjacent vertices having labels odd and even numbers contribute 1 to each edge. Similarly, the consecutive adjacent vertices having the labels odd numbers and consecutive adjacent vertices having labels even and odd numbers contribute 0 to each edge.

Also $f(v_2)|f(v_1)$, $f(v_n)\nmid f(v_1)(f(v_1)\nmid f(v_n))$, $f(v_2)|f(v')$, $f(v_1)|f(v')$ if n is odd and $f(v_1)\nmid f(v')$ if n is even.

Thus, $e_f(0) = \frac{n+1}{2}$ and $e_f(1) = \frac{n+3}{2}$, if n is odd.

$e_f(0) = e_f(1) = \frac{n+2}{2}$, if n is even.

Therefore, $|e_f(0) - e_f(1)| \leq 1$. Hence G is divisor cordial graph.

Example :2.2

The graph obtained by duplication of an edge by a vertex in C_5 and its divisor cordial labeling is shown in Figure 2.3.

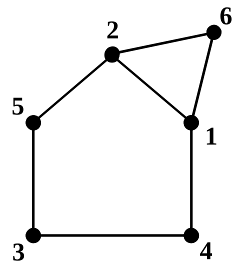

Figure 2.3

Theorem : 2.3

The graph G = $< S_n^{(1)} : S_n^{(2)} : S_n^{(3)} >$ is divisor cordial.

Proof.

Let $v_1^{(i)}, v_2^{(i)}, ..., v_n^{(i)}$ be the pendant vertices of $S_n^{(i)}$ and let $v_1^{(i)}$ be the apex vertex of $S_n^{(i)}$ for i = 1, 2, 3. The apex vertices $v_1^{(1)}$ and $v_1^{(2)}$ are joined by an edge as well as to a new vertex x_1 and the apex vertices $v_1^{(2)}$ and $v_1^{(3)}$ are joined by an edge as well as to a new vertex x_2. Let G be $< S_n^{(1)} : S_n^{(2)} : S_n^{(3)} >$. Then $|V(G)| = 3n+2$ and $|E(G)| = 6n - 5$.

Define vertex labeling $f : V(G) \to \{1, 2, ..., 3n+2\}$ as follows

$f(v_1^{(1)}) = 1$,

$f(v_1^{(2)}) = 2$,

$f(v_1^{(3)}) = 3$,

For n is odd.

$f(x_1) = 3n+1$,

$f(x_2) = 3n+2$,

$f(v_2^{(1)}) = 4$,

$f(v_3^{(1)}) = 6$,

$f(v_{2i+2}^{(1)}) = 6i - 1$ for $1 \le i \le \frac{n-3}{2}$

$f(v_{2i+3}^{(1)}) = 6i + 1$ for $1 \le i \le \frac{n-3}{2}$

$f(v_{2i}^{(2)}) = 6i + 2$ for $1 \le i \le \frac{n-1}{2}$

$f(v_{2i+1}^{(2)}) = 6i + 4$ for $1 \le i \le \frac{n-3}{2}$

$f(v_n^{(2)}) = 3n - 4$,

$f(v_i^{(3)}) = 3i + 6$, for $1 \le i \le n - 1$

$f(v_n^{(3)}) = 3n - 2$.

For n is even.

$f(x_1) = 3n+2$,

$f(x_2) = 3n+1$,

$f(v_2^{(1)}) = 4$,

$f(v_3^{(1)}) = 6$,

$f(v_{2i+2}^{(1)}) = 6i - 1$ for $1 \le i \le \frac{n-2}{2}$

$f(v_{2i+3}^{(1)}) = 6i + 1$ for $1 \le i \le \frac{n-4}{2}$

$f(v_{2i}^{(2)}) = 6i + 2$ for $1 \le i \le \frac{n-2}{2}$

$f(v_{2i+1}^{(2)}) = 6i + 4$ for $1 \le i \le \frac{n-2}{2}$

$f(v_n^{(2)}) = 3n - 5$,

$f(v_i^{(3)}) = 3i + 6$, for $1 \le i \le n - 1$

$f(v_n^{(3)}) = 3n - 1$.

In both case, $e_f(0) = 3n - 2$ and $e_f(1) = 3n - 3$.
Therefore $|e_f(0) - e_f(1)| \le 1$.
Hence G is divisor cordial.

Example :2.3

The graph $G = < S_n^{(1)} : S_n^{(2)} : S_n^{(3)} >$ and its divisor cordial labeling is given in Figure 2.4.

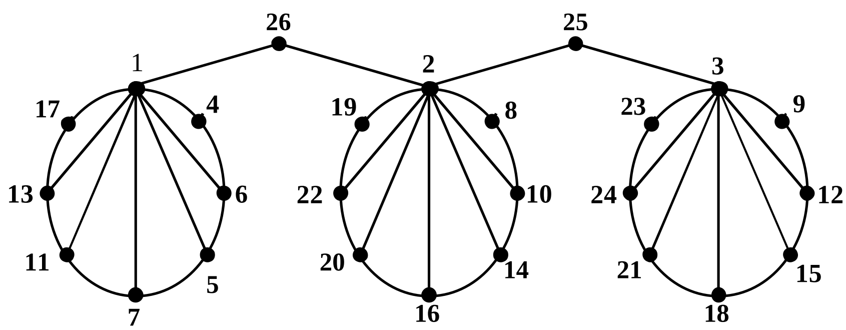

Figure 2.4

Theorem : 2.4

The graph $G = < W_n^{(1)} : W_n^{(2)} : W_n^{(3)} >$ is divisor cordial.

Proof.

Let $v_1^{(i)}, v_2^{(i)}, ..., v_n^{(i)}$ be the pendant vertices of $W_n^{(i)}$ and let c_i be the apex vertex of $W_n^{(i)}$ for i = 1, 2, 3. The apex vertices c_1 and c_2 are joined by an edge as well as to a new vertex x_1 and the apex vertices c_2 and c_3 are joined by an edge as well as to a new vertex x_2.

Let G be $< W_n^{(1)} : W_n^{(2)} : W_n^{(3)} >$.

Then $|V(G)| = 3n+5$ and $|E(G)| = 6n+4$.

Define vertex labeling $f : V(G) \to \{1, 2, ..., 3n+5\}$ as follows

$f(c_1) = 1$,
$f(c_2) = 2$,
$f(c_3) = 3$,

For n is even.

$f(x_1) = 3n+4$,
$f(x_2) = 6$,
$f(v_1^{(1)}) = 4$,

$f(v_{2i}^{(1)}) = 5 + 6(i - 1)$ for $1 \le i \le \frac{n}{2}$

$f(v_{2i+1}^{(1)}) = 7 + 6(i - 1)$ for $1 \le i \le \frac{n-2}{2}$

$f(v_{2i-1}^{(2)}) = 8 + 6(i - 1)$ for $1 \le i \le \frac{n}{2}$

$f(v_{2i}^{(2)}) = 10 + 6(i - 1)$ for $1 \le i \le \frac{n-2}{2}$

$f(v_n^{(2)}) = 3n+1$,

$f(v_i^{(3)}) = 9 + 3(i-1)$, for $1 \leq i \leq n-1$

$f(v_n^{(3)}) = 3n+5$.

For n is odd.

$f(x_1) = 3n+5$,

$f(x_2) = 6$,

$f(v_1^{(1)}) = 4$,

$f(v_{2i}^{(1)}) = 5 + 6(i-1)$ for $1 \leq i \leq \frac{n-1}{2}$

$f(v_{2i+1}^{(1)}) = 7 + 6(i-1)$ for $1 \leq i \leq \frac{n-1}{2}$

$f(v_{2i-1}^{(2)}) = 8 + 6(i-1)$ for $1 \leq i \leq \frac{n-1}{2}$

$f(v_{2i}^{(2)}) = 10 + 6(i-1)$ for $1 \leq i \leq \frac{n-1}{2}$

$f(v_n^{(2)}) = 3n+2$,

$f(v_i^{(3)}) = 9 + 3(i-1)$, for $1 \leq i \leq n-1$

$f(v_n^{(3)}) = 3n+4$.

In both case, $e_f(0) = e_f(1) = 3n+2$.

Therefore $|e_f(0) - e_f(1)| \leq 1$.

Hence G is divisor cordial.

Example :2.4

The graph $G = < W_6^{(1)} : W_6^{(2)} : W_6^{(3)} >$ and its divisor cordial labeling is given in Figure 2.5.

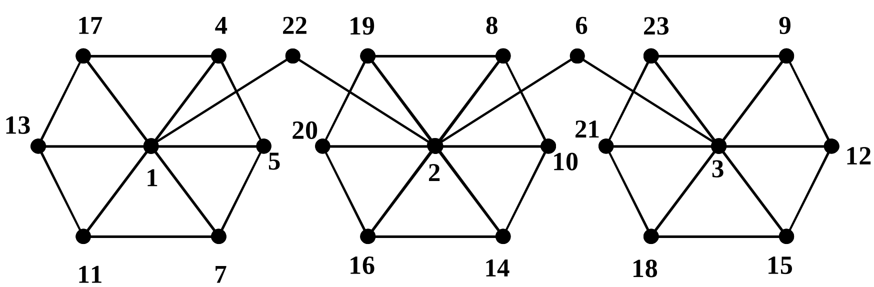

Figure 2.5

Theorem :2.5

The graph obtained by joining two copies of S_n by path P_k admits divisor cordial labeling where $n \geq 4$.

Proof.

Let G be the graph obtained by joining two copies of S_n by path P_k. Let $u_1, u_2, ..., u_n$ be the vertices of first copy of S_n and $v_1, v_2, ..., v_n$ be the vertices of second copy of S_n.

Let $w_1, w_2,...,w_k$ be the vertices of path P_k with $u_1 = w_1$ and $v_1 = w_k$.

Then $|V(G)| = 2n + k - 2$ and $|E(G)| = 4n + k - 7$.

Define vertex labeling $f : V(G) \rightarrow \{1, 2, ..., 2n+k-2\}$ as follows

Label the vertices $w_k, w_{k-1}, ..., w_3, w_2$ in the following order.

$$\begin{array}{lllll} 2, & 2^2, & 2^3, & ..., & 2^{k_1}, \\ 3, & 3\times 2 & 3\times 2^2 & ..., & 3\times 2^{k_2}, \\ 5, & 5\times 2 & 5\times 2^2 & ..., & 5\times 2^{k_3}, \\ ... & ... & ... & ... & ..., \\ ... & ... & ... & ... & ... \end{array}$$

where $(2m-1)2^{k_m} \leq k-1$ and $m \geq 1$, $k_m \geq 0$.

Also $(2m - 1)2^a$ divides $(2m - 1)2^b$ $(a < b)$ and $(2m-1)2^{k_i}$ does not divide $2m+1$.

In the above labeling, the consecutive adjacent vertices of w_k, w_{k-1}, ..., w_3, w_2 having the labels even numbers and consecutive adjacent vertices having labels odd and even numbers contribute 1 to each edge. Similarly, the consecutive adjacent vertices having the labels odd numbers and consecutive adjacent vertices having labels even and odd numbers contribute 0 to each edge and $f(w_1) | f(w_2)$.

For k is odd

$f(u_i) = k + 2(i-1)$, $2 \leq i \leq n-1$
$f(v_i) = k - 1 + 2(i-1)$, $2 \leq i \leq n-1$
$f(u_n) = f(u_{n-1}) + 1$,
$f(v_n) = f(v_{n-1}) + 3$.

For k is even

$f(u_i) = k - 1 + 2(i-1)$, $2 \leq i \leq n-1$
$f(v_i) = k + 2(i-1)$, $2 \leq i \leq n-1$
$f(u_n) = f(u_{n-1}) + 3$,
$f(v_n) = f(v_{n-1}) + 1$.

Thus, $e_f(0) = \dfrac{4n+k-6}{2}$ and $e_f(1) = \dfrac{4n+k-8}{2}$, if k is odd

$e_f(0) = e_f(1) = \dfrac{4n+k-7}{2}$, if k is even.

Hence $|e_f(0) - e_f(1)| \leq 1$.
Hence G is divisor cordial graph.

Example :2.5

The graph G obtained by joining two copies of S_6 by path P_6 and its divisor cordial labeling is given in Figure 2.6.

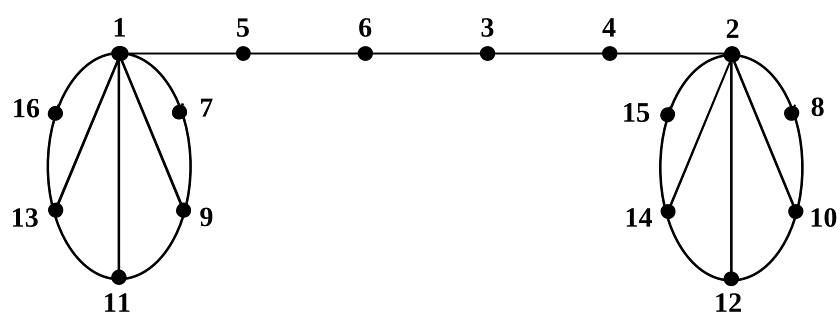

Figure 2.6

III. CONCLUSIONS

In this paper, we prove that the duplication of an arbitrary vertex by a new edge of cycle C_n $(n \geq 3)$, the duplication of an arbitrary edge by a new vertex of cycle C_n $(n \geq 3)$, $< S_n^{(1)} : S_n^{(2)} : S_n^{(3)} >$, $< W_n^{(1)} : W_n^{(2)} : W_n^{(3)} >$ and the graph obtained by joining two copies of S_n by a path P_k $(n \geq 4)$ are divisor cordial graph.

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