Numerical Investigation of first order linear Singular Systems using Leapfrog Method

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***Abstract*— In this paper numerical investigation of first order linear singular systems of time-invariant and time varying cases [6] using Leapfrog method is considered. The obtained discrete solutions using Leapfrog method are compared with the exact solutions of the first order linear singular systems of time-invariant and time varying cases and single-term Haar wavelet series (STHWS) method. Tables and graphs are presented to show the efficiency of this method. This Leapfrog method can be easily implemented in a digital computer and the solution can be obtained for any length of time.**

***Keywords*— Haar wavelets, Single-term Haar wavelet series, Leapfrog method, Ordinary differential equations, Singular systems.**

I. INTRODUCTION

Singular systems are being applied to solve a variety of problems involved in various disciplines of science and engineering. They are applied to analyse neurological events and catastrophic behaviour and they also provide a convenient form for the dynamical equations of large scale interconnected systems. Further, singular systems are found in many areas such as constrained mechanical systems, fluid dynamics, chemical reaction kinetics, simulation of electrical networks, electrical circuit theory, power systems, aerospace engineering, robotics, aircraft dynamics, neural networks, neural delay systems, network analysis, time series analysis, system modelling, social systems, economic systems, biological systems etc.[5, 7-13]

Wazwaz [14] published a paper on modified Runge-Kutta formula based on a variety of means of third order. Murugesan *et al.* [1 - 4] have analysed different second-order systems and multivariable linear systems via RK method based on centroidal mean, and also, they extended RK formulae based on variety of means to solve system of IVPs. In this paper, we apply the Leapfrog method for finding the numerical solution of first order linear singular systems of time-invariant and time varying cases with more accuracy.

II. FIRST ORDER LINEAR SINGULAR SYSTEM

In general a first order linear singular system of time-invariant case is represented in the following form

$$K\dot{x}(t) = Ax(t) + Bu(t) \tag{1}$$

with initial condition $x(0) = x_0$.

where K is an $n \times n$ singular matrix, A and B are $n \times n$ and $n \times p$ constant matrices respectively. $x(t)$ is an n-state vector and $u(t)$ is the p-input control vector.

A first order linear singular system of time-varying case is represented in the following form

$$K(t)\dot{x}(t) = A(t)x(t) + B(t)u(t) \tag{2}$$

with initial condition $x(0) = x_0$.

where $x(t)$ and $u(t)$ are defined as in (1) and $K(t)$ is an $n \times n$ singular matrix, $A(t)$ and $B(t)$ are $n \times n$ and $n \times p$ matrices respectively. The elements (not necessarily all the elements) of the matrices $K(t), A(t)$ and $B(t)$ are time dependent.

III. LEAPFROG METHOD

The most familiar and elementary method for approximating solutions of an initial value problem is Euler's Method. Euler's Method approximates the derivative in the form of $y' = f(t, y),\ y(t_0) = y_0,\ y \in R^d$ by a finite difference quotient $y'(t) \approx (y(t+h) - y(t))/h$. We shall usually discretize the independent variable in equal increments:

$$t_{n+1} = t_n + h,\ n = 0,1,\dots,t_0\,.$$

Henceforth we focus on the scalar case, N = 1. Rearranging the difference quotient gives us the corresponding approximate values of the dependent variable:

$$y_{n+1} = y_n + hf(t_n, y_n),\ n = 0,1,\dots,t_0$$

To obtain the leapfrog method, we discretize t_n as in $t_{n+1} = t_n + h,\ n = 0,1,\dots,t_0$, but we double the time interval, h, and write the midpoint approximation $y(t+h) - y(t) \approx hy'\left(t + \frac{h}{2}\right)$ in the form

$$y'(t+h) \approx (y(t+2h) - y(t))/h$$

and then discretize it as follows:

$$y_{n+1} = y_{n-1} + 2hf(t_n, y_n),\ n = 0,1,\dots,t_0$$

The leapfrog method is a linear m = 2-step method, with $a_0 = 0, a_1 = 1, b_{-1} = -1, b_0 = 2$ and $b_1 = 0$. It uses slopes evaluated at odd values of n to advance the values at points at even values of n, and vice versa, reminiscent of the children's game of the same name. For the same reason, there are multiple solutions of the leapfrog method with the same initial value $y = y_0$. This situation suggests a potential instability present in multistep methods, which must be addressed when we analyze them—two values, y_0 and y_1, are required to initialize solutions of $y_{n+1} = y_{n-1} + 2hf(t_n, y_n),\ n = 0,1,\dots,t_0$ uniquely, but the analytical problem $y' = f(t, y),\ y(t_0) = y_0,\ y \in R^d$ only provides one. Also for this reason, one-step methods are used to initialize multistep methods.

IV. NUMERICAL EXAMPLES

In this section, the exact solutions and approximated solutions obtained by Leapfrog method and STHWS method. To show the efficiency of the Leapfrog method, we have considered the following problem taken from [6], with step size $t = 0.1$ along with the exact solutions.

The discrete solutions obtained by the two methods, Leapfrog method and the STHWS methods; the absolute errors between them are tabulated and are presented in Table 1 - 2. To distinguish the effect of the errors in accordance with the exact solutions, graphical representations are given for selected values of "*t*" and are presented in Fig. 1 to Fig. 5 for the following problem, using three dimensional effects.

Example 4.1

The first order linear singular system of time-invariant case with three variables of the form (1) is given by [6]

$$K = \begin{bmatrix} 0 & 1 & 4 \\ 0 & -2 & 0 \\ 0 & 1 & 1 \end{bmatrix}$$, A=I (an identify matrix of appropriate dimension) and $B = 0$

with initial condition $x(0) = [1/6 \quad 1 \quad -1/3]^T$,

and the exact solution is

$$x_1 = (\exp(-t/2))/6$$

$x_2 = \exp(-t/2)$

$x_3 = -(\exp(-t/2))/3$

Example 4.2

The first order linear singular system of time-varying case with two variables of the form (2) is given by [6]

$$K = \begin{bmatrix} 0 & 0 \\ 1 & t \end{bmatrix}, A = \begin{bmatrix} -1 & 1-t \\ 0 & -2 \end{bmatrix}, B = \begin{bmatrix} e^t & 1 \\ t^2 & 2 \end{bmatrix} \quad \text{and } u = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

with initial condition

$$x(0) = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

and the exact solution is

$x_1 = (1+t)e^t - t^3$

$x_2 = t^2 - e^t$

using Leapfrog method and STHWS method to solve the above problems, the absolute errors are evaluated and are presented in Table 1 and Table 2 with various time step size. Error graphs are presented Fig. 1 to Fig. 6 to highlight the efficiency of the method.

TABLE I

t	**Example 4.1**					
	STHWS Error			**Leapfrog Error**		
	x_1	x_2	x_3	x_1	x_2	x_3
0.1	1E-09	4E-09	3E-09	1E-11	4E-11	3E-11
0.2	2E-09	6E-09	5E-09	2E-11	6E-11	5E-11
0.3	3E-09	8E-09	7E-09	3E-11	8E-11	7E-11
0.4	4E-09	1E-08	9E-09	4E-11	1E-10	9E-11
0.5	5E-09	1.2E-08	1.1E-08	5E-11	1.2E-10	1.1E-10
0.6	6E-09	1.4E-08	1.3E-08	6E-11	1.4E-10	1.3E-10
0.7	7E-09	1.6E-08	1.5E-08	7E-11	1.6E-10	1.5E-10
0.8	8E-09	1.8E-08	1.7E-08	8E-11	1.8E-10	1.7E-10
0.9	9E-09	2E-08	1.9E-08	9E-11	2E-10	1.9E-10
1.0	1E-08	2.2E-08	2.1E-08	1E-10	2.2E-10	2.1E-10

TABLE III

t	**Example 4.2**			
	STHWS Error		**Leapfrog Error**	
	x_1	x_2	x_1	x_2
0.1	2E-09	7E-09	2E-11	7E-11
0.2	5E-09	9E-09	5E-11	9E-11
0.3	8E-09	1.1E-08	8E-11	1.1E-10
0.4	1.1E-08	1.3E-08	1.1E-10	1.3E-10
0.5	1.4E-08	1.5E-08	1.4E-10	1.5E-10
0.6	1.7E-08	1.7E-08	1.7E-10	1.7E-10
0.7	2E-08	1.9E-08	2E-10	1.9E-10
0.8	2.3E-08	2.1E-08	2.3E-10	2.1E-10
0.9	2.6E-08	2.3E-08	2.6E-10	2.3E-10
1.0	2.9E-08	2.5E-08	2.9E-10	2.5E-10

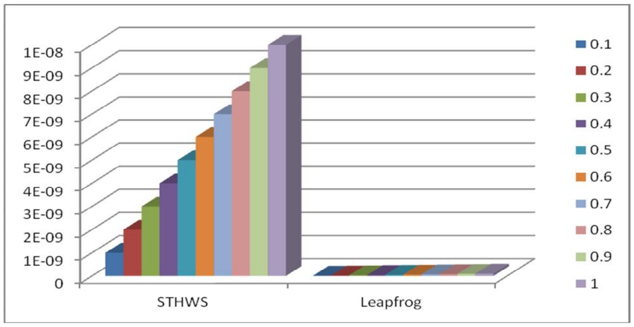

Fig. 1 Error estimation of Example 4.1 at x_1

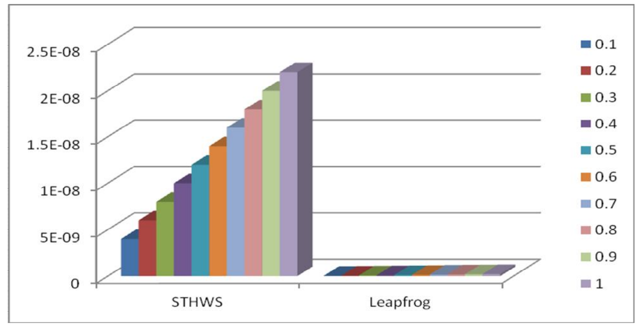

Fig. 2 Error estimation of Example 4.1 at x_2

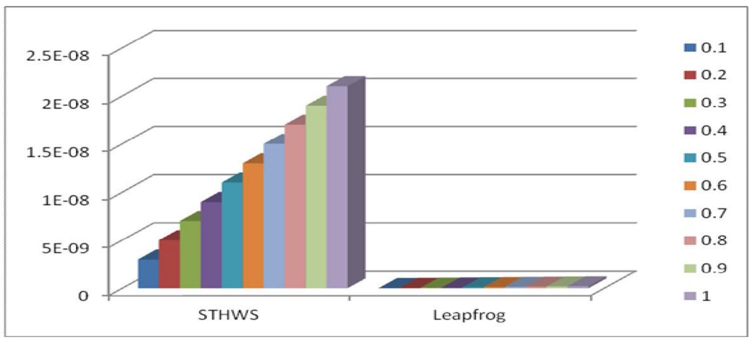

Fig. 3 Error estimation of Example 4.1 at x_3

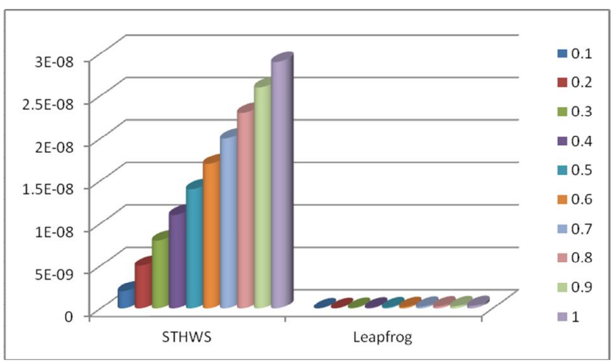

Fig. 4 Error estimation of Example 4.2 at x_1

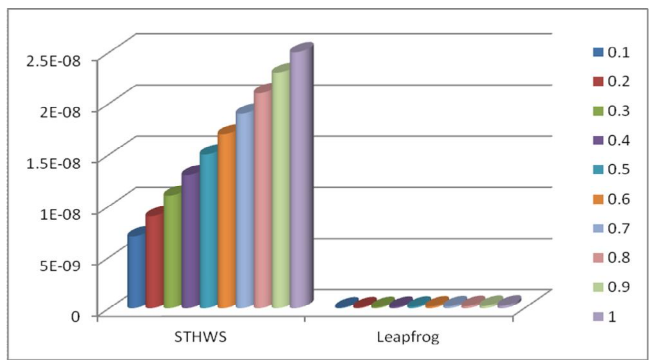

Fig. 6 Error estimation of Example 4.2 at x_2

V. CONCLUSIONS

A simple and easy method is introduced in this paper to obtain discrete solutions of first order linear singular systems of time-invariant and time varying cases using Leapfrog method. The efficiency and the accuracy of the Leapfrog method have been illustrated by suitable examples. The solutions obtained are compared well with the exact solutions and STHWS method. It has been observed that the solutions by our method show good agreement with the exact solutions. The present method is very convenient as it requires only simple computing systems, less computing time and less memory. The Leapfrog method is very simple and direct which provides the solutions for any length of time.

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